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Research Paper

NEW SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE q-BERNARDI INTEGRAL OPERATOR AND SPECIAL NUMBER SEQUENCES

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ABSTRACT

This paper introduces two subclasses of analytic functions associated with the q-Bernardi integral operator. These subclasses are defined by using the q-derivative together with special functions whose coefficients are Bernoulli numbers and generalized telephone numbers. We establish inclusion-type results for the proposed classes and investigate the corresponding quasi-subordination and majorization problems. In particular, sufficient radius conditions are obtained under which the q-derivatives of majorized functions satisfy the required comparison inequalities.

1. Introduction

The area of mathematics called q-calculus operates independently from the usual limit concepts, which makes it distinctive compared to traditional calculus. This unique quality enables q-calculus to investigate mathematical phenomena in ways that classical methods

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struggle to effectively handle. In recent years, q-calculus has really gained popularity because of its wide-ranging applications in different fields, such as number theory, combinatorics, and mathematical physics. The increasing popularity of q-calculus is not hard to understand, as most issues in its areas of application such as partition theory permit us to understand how the integers are distributed. This very effective system of q-calculus has even penetrated quantum calculus, which significantly enhances our knowledge of quantum mechanics and quantum systems. Jackson and others [?] have carried out extensive research in the q-derivatives as well as in the q-integrals and laid the groundwork for the discussion of their fundaments of mathematics. Therefore, whatever Jackson has contributed will pave the way for clarifications of such topics and their importance in many other fields. For instance, the concept of differentiation has now acquired the generalization of a q-parameter, which is going to yield some additional insights into the functions and their properties. Srivastava [?] has taken a truly eminent step by amalgamating basic q-hypergeometric functions with geometric function theory. This compact combination has undoubtedly constructed a strong theoretical groundwork which will firmly stand to guide future research. As it has already imbibed the flavor of q-calculus along with that of geometric functions, Srivastava's work is expected to pave several paths for further exploration, in addition to motivating continued studies in the field. Ismail [?] has pushed these basic ideas yet one step further by considering other new types of functions derived from the q-derivative. This work has deepened our knowledge of q-starlike functions, which are crucial in aspects of complex analysis and geometric function theory. Ismail has dealt with these new functions to clarify their properties and possible applications in many different fields of mathematics, including approximation theory and complex dynamics. The founder of the new types of functions studied in q-calculus is Noor [?], who has introduced the q-version of close-to-convex functions. This extension has brought much more depth to the theory of close-to-convexity, which is one of the main pillars of geometric function theory. The combination of q-calculus with this perspective will allow further inquiries into the field and open new research avenues. Sharma and Sahoo [?] have pushed the work of Noor into some really exciting new avenues, extending the idea into some remarkable contributions with perspectives on q-close-to-convex functions. They aimed to provide generalized results in the form of theorems for this new class of functions, showcasing just how versatile and useful q-calculus can be in addressing intricate mathematical problems. The q-integral operators were first introduced by Selvakumaran [?] based on concepts from fractional q-calculus, thereby opening new avenues for development in q-calculus. The geometric theory of functions in complex analysis deals with the analytic properties and behaviors of functions, along with geometrically related concepts pertinent to complex analysis. This connection acts as a bridge to better understand how functions behave, map, and transform which is very important application in fields such as engineering, physics, and information technology. Let $\mathcal{A}$ be the class of functions of the following form:

$$\vartheta(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \tag{1.1}$$

which are analytic in the open unit disc

$$U_d = \{\xi : |\xi| < 1\}.$$

When it comes to the geometric theory of functions, we can define various subclasses of analytic functions based on their geometric properties. The class $\mathcal{S}'$ represents the subclass of *univalent* functions, which are also referred to as one-to-one functions. The letter $\mathcal{C}$ indicates functions that are *convex* in shape. For *starlike* functions, which have rays starting from the origin and retain a star-like appearance, we use the notation $\mathcal{S}^*$. Lastly, functions that exhibit characteristics of both convex and starlike functions are labeled as $\mathcal{K}$. Let $\vartheta \in \mathcal{A}$. Then, the operator $I : \mathcal{A} \to \mathcal{A}$ is defined in the following form:

$$I(\vartheta)(\xi) = \left(\frac{\xi^{-c}}{(1+c)^{-1}}\right) \int_0^{\xi} t^{c-1}\vartheta(t)\,dt, \quad c = 1, 2, 3, \ldots \tag{1.2}$$

Bernardi introduced an operator known as the *Bernardi integral operator*, which is referenced in his work [?]. He found that this operator preserves certain function classes, specifically $\mathcal{S}^*$ (starlike functions), $\mathcal{C}$ (convex functions), and $\mathcal{K}$ (close-to-convex functions). This means that if you take a function from any of these subclasses and apply the Bernardi integral operator, the resulting function will still belong to the same class. Additionally, it's interesting to note that when the parameter c is set to 1, the Bernardi integral operator simplifies to what's called the *Libera integral operator*. This connection is particularly relevant for studying analytic functions, as it enriches our understanding of how functions behave within these subclasses [?]. Now let's summarize the key concepts and findings that emerged from q-calculus. For any function $\vartheta \in \mathcal{A}$, the q-derivative of the function ϑ can be expressed as [?]

$$d_q\vartheta(\xi) = \frac{\vartheta(q\xi) - \vartheta(\xi)}{(q-1)\xi}, \quad (\xi \neq 0). \tag{1.3}$$

For a function $\kappa(\xi) = \xi^n$, the q-derivative is given by:

$$d_q\kappa(\xi) = [n]_q\, \xi^{n-1}, \tag{1.4}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

As q approaches 1, the q-derivative $d_q\vartheta(\xi)$ approaches the ordinary derivative $\vartheta'(\xi)$. That is, $\lim_{q\to1}[n]_q = n$ and $\lim_{q\to1} d_q\vartheta(\xi) = \vartheta'(\xi)$. According to Jackson, the q-integral of a function ϑ was first proposed in reference [?] and is defined as:

$$\int_0^{\xi} \vartheta(t)\,d_qt = \xi(1-q)\sum_{n=2}^{\infty} q^n\vartheta(q^n\xi), \tag{1.5}$$

provided the series converges. With reference to [?], we now define the subclasses of q-convex and q-starlike functions.

$$\mathcal{S}_q^* = \left\{\vartheta \in \mathcal{A} : \mathrm{Re}\left(\frac{\xi d_q\vartheta(\xi)}{\vartheta(\xi)}\right) > 0,\ \xi \in U_d\right\}, \tag{1.6}$$

$$\mathcal{C}_q = \left\{\vartheta \in \mathcal{A} : \mathrm{Re}\left(\frac{d_q(\xi\, d_q\vartheta(\xi))}{d_q\vartheta(\xi)}\right) > 0,\ \xi \in U_d\right\}. \tag{1.7}$$

As $q \to 1$, these subclasses approach the classical starlike and convex function classes discussed earlier. Let us denote the set of functions $w(\xi)$ as Ω. These functions must satisfy two main conditions: first, $w(0) = 0$; second, $|w(\xi)| < 1$ for every $\xi \in U_d$. In the study of geometric function theory—which focuses on the properties and behavior of analytic and univalent functions—integral and differential operators serve as fundamental tools. These operators arise naturally via convolution processes of certain analytic functions and aid in investigating the geometric properties these functions possess. According to [?], the q-Bernardi integral operator is defined by the following equation:

$$I_q(F_c)(\xi) = \frac{[1+c]_q}{\xi^c}\int_0^{\xi} t^{c-1}\vartheta(t)\,d_q t. \tag{1.8}$$

The *majorization property* is a key idea in the field of *geometric function theory*, which is a part of complex analysis focused on studying holomorphic functions, their characteristics, and how they behave geometrically. Within this field, majorization describes a particular type of order relationship between two complex-valued functions that are defined over a certain area in the complex plane. Let's take a look at two analytic functions, u and v, defined on a domain U_d. We can say that $\Xi_1(\xi)$ *majorizes* $\Xi_2(\xi)$ over the domain U_d [?], which is represented as

$$\Xi_1(\xi) \ll \Xi_2(\xi), \quad \xi \in U_d, \tag{1.9}$$

if there exists a function $\Theta \in \mathcal{A}$ such that $|\Theta(\xi)| \le 1$. Hence, we obtain:

$$\Xi_1(\xi) = \Theta(\xi)\Xi_2(\xi), \quad \xi \in U_d. \tag{1.10}$$

In the realm of complex analysis, we refer to a function as *univalent* (or one-to-one) within a certain region if it doesn't take on the same value more than once in that area. A key idea in this field is *subordination*, which allows us to compare how two univalent functions behave. If we have two univalent functions, $\Xi_1(\xi)$ and $\Xi_2(\xi)$, defined in the area U_d, we say that $\Xi_1(\xi)$ is *subordinate* to $\Xi_2(\xi)$ (which we denote as $\Xi_1(\xi) \prec \Xi_2(\xi)$) if there is a Schwarz function Θ that is analytic in U_d and meets the conditions

$$|\Theta(\xi)| < 1, \quad \Theta(0) = 0, \quad \text{for all } \xi \in U_d,$$

such that the following relation holds for every $\xi \in U_d$:

$$\Xi_1(\xi) = \Xi_2(\Theta(\xi)).$$

We can define *quasi-subordination* by merging the concepts of subordination and majorization. Specifically, we describe the quasi-subordination relationship between the functions $\Xi_1(\xi)$ and $\Xi_2(\xi)$ as follows (see [?]). We say that $\Xi_1(\xi)$ is quasi-subordinate to $\Xi_2(\xi)$, which we express as

$$\Xi_1 \prec_q \Xi_2,$$

if there exist two analytic functions Θ and v, both defined in $\mathcal{A}$, such that the function

$$\frac{\xi\Theta'(\xi)}{\Theta(\xi)},$$

is analytic in U_d. In addition, Θ is required such that $|\Theta(\xi)| \le 1$, and the function v is required to satisfy

$$v(0) = 0, \quad |v(\xi)| \le |\xi|.$$

Furthermore, the relationships of functional variables Ξ_1 and Ξ_2 are given by

$$\Xi_1(\xi) = \Theta(\xi)\,\Xi_2(v(\xi)) \quad (\xi \in U_d). \tag{1.11}$$

In this article, we will start by applying Bernardi's theorem to the classes S^*_{BN} and S^*_{TN}, which consist of univalent starlike functions defined within the unit disk U_d. Our main aim is to demonstrate how these classes relate to each other and to outline their properties under certain transformations. We will show that if a function ϑ is part of the class S^*_{BN}, then the transformed function $I_q(F_c)(\xi)$ also falls within the same class. This discovery underscores the stability of the class when the transformation $I_q(F_c)(\xi)$ is applied. Furthermore, we extend our investigation to show that if $\vartheta \in S^*_{TN}$ it follows that $I_q(F_c)(\xi)$ is also in the class S^*_{TN}. The study of specialized functions and generalized hypergeometric functions is an exciting and contemporary area of research that has a broad range of applications across fields like technology, engineering, and mathematics. Notably, de Branges [?] utilized the generalized hypergeometric function to tackle the well-known Bieberbach conjecture. Researchers have delved deeply into the geometric and analytical properties of many special functions, particularly focusing on the generalized Gaussian hypergeometric functions. It is a curious circumstance that the Bernoulli numbers [?] were discovered independently by the Japanese mathematician Seki Takakazu and Jakob Bernoulli. Both mathematicians stumbled upon these numbers while calculating the sum of powers of integers, represented as $1^n + 2^n + 3^n + \cdots + m^n$. This discovery highlighted the significance of Bernoulli numbers in a variety of mathematical results, including the series expansions of hyperbolic and trigonometric functions. Moreover, they play a crucial role in evaluating important theories like Fermat's Last Theorem and the Riemann Zeta function. Using the concept of generating functions by Euler, the Bernoulli numbers are defined as the coefficients of the function

$$\Phi_1(\xi) = \frac{\xi}{e^{\xi} - 1} = \sum_{k=0}^{\infty} \frac{B_k \xi^k}{k!} \tag{1.12}$$

Ma and Minda [?] conducted a study on the unified class of starlike functions $S^*(\varphi)$. This class consists of functions $\vartheta \in \mathcal{A}$ that satisfy the condition

$$\frac{\xi \vartheta'(\xi)}{\vartheta(\xi)} \prec \varphi(\xi), \quad \forall \xi \in \mathbb{U}_d, \tag{1.13}$$

where φ is a univalent function with a positive real part. Additionally, $\varphi(\mathbb{U}_d)$ is symmetric about the real axis and starlike with respect to

$$\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) > 0. \tag{1.14}$$

It's clear that the function $\Phi_1(\xi)$ mentioned in (??) meets all the necessary conditions. Therefore, we can define S^*_{BN} as the set of functions $\vartheta : U_d \to \mathbb{C}$ that satisfy the inequality

$$\frac{\xi d_q \vartheta(\xi)}{\vartheta(\xi)} \prec \Phi_1(\xi).$$

Involution numbers, also known as Telephone numbers, represent a series of integers that count the different ways in which n people can connect through person-to-person calls. These numbers illustrate the various matchings with n vertices in a complete graph and also describe the sum of the absolute values of Hermite polynomials. Chowla and Knuth [?, ?] first derived these numbers. Wloch and Wolowiec-Musial [**?**] later introduced a recurrence relation for $n \geq 0$ and $k \geq 1$, which can be used to derive the generalized telephone numbers:

$$T(k,n) = kT(k,n-1) + (n-1)T(k,n-2).$$

Having initial restriction $T(k,0) = T(k,1) = k$. Deniz [**?**] recently presented the generating function of Telephone numbers as:

$$\exp\left(\xi + \frac{k\xi^2}{2}\right) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} T_k(n).$$

Clearly, some values of $T_k(n)$ are as follows:

$$\begin{aligned}
T_k(0) &= T_k(1) = k, \\
T_k(2) &= 1 + k, \\
T_k(3) &= 1 + 3k, \\
T_k(4) &= 1 + 6k + 3k^2, \\
T_k(5) &= 1 + 10k + 15k^2.
\end{aligned}$$

Now consider the function

$$\Phi_2(\xi) = \exp\left(\xi + \frac{k\xi^2}{2}\right),$$

this function fully satisfies all the requisite conditions inherent to function $\phi(\xi)$ mentioned in (**??**). Now we can define S^*_{TN}, to be the class of functions $\vartheta : U_d \to \mathbb{C}$ that satisfy the following condition:

$$\frac{\xi d_q \vartheta(\xi)}{\vartheta(\xi)} \prec \exp\left(\xi + \frac{k}{2}\xi^2\right).$$

Our investigation relies on fundamental principles derived from the following essential lemmas.

Lemma 1.1. *Let Υ and Ξ be holomorphic functions defined on the domain U_d, such that Ξ maps U_d into an entirely many-sheeted starlike region. Assume that $\Upsilon(0) = \Xi(0) = 0$ and that the derivatives at zero satisfy $d_q\Upsilon(0) = d_q\Xi(0) = 1$. If*

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} \in S^*_{BN},$$

then it follows that

$$\frac{\Upsilon(\xi)}{\Xi(\xi)} \in S^*_{BN}.$$

Proof. We begin with the assertion that

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} \in S^*_{BN}.$$

Furthermore, we define the function

$$\Phi_1(\xi) = \frac{\xi}{e^{\xi} - 1},$$

which maps the disk $|\xi| < r$ onto the region characterized by the condition

$$\left|\Phi_1(\xi) - \frac{e^2 + 1}{2(e-1)^2}\right| < \frac{e+1}{2(e-1)}.$$

Since

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)},$$

takes values within this disc, we find that

$$\left|\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} - \frac{e^2+1}{2(e-1)^2}\right| < \frac{e+1}{2(e-1)}, \quad |\xi| < r, \quad 0 < r < 1.$$

Next, we define a function $\Lambda(\xi)$ such that

$$d_q\Xi(\xi)\Lambda(\xi) = d_q\Upsilon(\xi) - \frac{e^2+1}{2(e-1)^2} d_q\Xi(\xi).$$

This leads us to the inequality

$$|\Lambda(\xi)| < \frac{e+1}{2(e-1)},$$

indicating that $\Lambda(\xi)$ remains bounded within a certain range. Fix ξ_0 in U_d. We consider the line segment $\mathcal{L}$ that connects 0 and $\Xi(\xi_0)$, which lies in one sheet of the starlike image of U_d by Ξ. The pre-image of this segment under Ξ is denoted as $\mathcal{L}^{-1}$. We can now express the following relationship:

$$|\Upsilon(\xi_0) - \frac{e^2+1}{2(e-1)^2}\Xi(\xi_0)| = \left|\int_0^{\xi_0} \left(d_q\Upsilon(t) - \frac{e^2+1}{2(e-1)^2} d_q\Xi(t)\right) dt\right|$$

$$= \left|\int_{\mathcal{L}^{-1}} d_q\Xi(t)\Lambda(t)dt\right| < \frac{e+1}{2(e-1)} \int_{\mathcal{L}} |d_q\Xi(t)| = \frac{e+1}{2(e-1)}|\Xi(\xi_0)|.$$

Consequently, we arrive at the conclusion that

$$\left|\frac{\Upsilon(\xi)}{\Xi(\xi)} - \frac{e^2+1}{2(e-1)^2}\right| < \frac{e+1}{2(e-1)},$$

this implies that

$$\frac{\Upsilon(\xi)}{\Xi(\xi)} \prec S^*_{BN} \quad \text{or equivalently} \quad \frac{\Upsilon(\xi)}{\Xi(\xi)} \in S^*_{BN}.$$

We can also demonstrate the lemma for the class S^*_{TN}. The proof follows a similar method, so we'll skip that part to keep things brief. □

Lemma 1.2. [?] *Let Υ and Ξ be holomorphic functions in U_d such that Ξ maps U_d onto a many-sheeted starlike region with $\Upsilon(0) = \Xi(0) = 0$ and*

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} = 1, \quad q \in [0, 1].$$

Then

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} \in S^*_{BN}.$$

Implies that

$$\frac{\Upsilon(\xi)}{\Xi(\xi)} \in S^*_{BN}.$$

Lemma 1.3. *Let* $\kappa \in S^*_{BN}$

$$J(\xi) = \int_0^\xi t^{c-1}\kappa(t)dt. \tag{1.15}$$

And as for $c = 1, 2, 3, \ldots$, *then* J *is* $(1+c)$*-valent starlike in* U_d, *and proof of this result is similar to the one given in* [**?**].

Lemma 1.4. *Let* $\kappa \in S^*_{BN}$ *and*

$$I_q(F_c)(\xi) = \frac{1+c}{\xi^c}J(\xi), \quad c = 1, 2, 3, \ldots,$$

where $J(\xi)$ *is given by* (**??**). *Then*

$$I_q(F_c)(\xi) \in S^*_{BN}.$$

Proof. Let

$$d_qI_q(F_c)(\xi) = \xi^{-1}(1+c)\kappa(\xi) - c\xi^{-1}J(\xi).$$

Then

$$\frac{d_qI_q(F_c)(\xi)}{I_q(F_c)(\xi)} = \frac{d_qI_q(F_c)(\xi)}{I_q(F_c)(\xi)} \cdot \frac{\xi^c}{\xi^c} = \frac{\xi^c\kappa(\xi) - cJ(\xi)}{J(\xi)} = \frac{\Upsilon(\xi)}{\Xi(\xi)},$$

where

$$\Upsilon(\xi) = \xi^c\kappa(\xi) - cJ(\xi) \quad \text{and} \quad \Xi(\xi) = J(\xi).$$

By lemma **??**, $\Xi \prec J$ is $(1+c)$-valent starlike for $c = 1, 2, 3, \ldots$, in U_d,

$$\frac{d_q\Upsilon(0)}{d_q\Xi(0)} = \frac{\xi d_q\kappa(0)}{\kappa(0)} = 1,$$

and since $\kappa \in S^*_{BN}$,

$$\frac{d_q\Upsilon(\xi)}{d_q\Xi(\xi)} = \frac{\xi d_q\kappa(\xi)}{\kappa(\xi)} \in S^*_{BN}.$$

Assuming lemma **??**, we have

$$\frac{\Upsilon(\xi)}{\Xi(\xi)} = \frac{\xi d_q\kappa(\xi)}{\kappa(\xi)} \in S^*_{BN},$$

that is, $I_q(F_c)(\xi) \in S^*_{BN}$. We can similarly show the lemma for the class S^*_{TN}. The proof follows the same approach, so we'll leave it out for brevity. □

Lemma 1.5. *Let* $\vartheta \in S^*_{BN}$ *and*

$$I_q(F_c)(\xi) = \frac{1+c}{\xi^c}J(\xi), \quad c = 1, 2, 3, \ldots,$$

*where $J(\xi)$ is given by (**??**) in lemma **??**. Then*

$$I_q(F_c)(\xi) \in S^*_{BN}.$$

2. Main Results

Theorem 2.1. *Let $\vartheta(\xi) \in A$ and $\kappa(\xi) \in S^*_{BN}$ and also suppose that $\vartheta(\xi)$ is majorized by $\kappa(\xi)$ in U_d. Then for $|\xi| \leq r_1$,*

$$|d_q\vartheta(\xi)| \leq |d_q\kappa(\xi)|,$$

where r_1 is the smallest positive root of the equation

$$r(1-r^2)\left(rq^{\gamma+1} + [\gamma+1]_q(e^r - 1)\right) - rq^{\gamma+1}(e^r + re^r + 1)$$

$$-2r(e^r-1)\left(rq^{\gamma+1} + [\gamma+1]_q(e^r-1)\right) = 0. \tag{2.1}$$

Proof. Given that $\vartheta(\xi) \in S^*_{BN}$, we can refer to lemma **??**, which suggests that $I_q(F_c)(\xi)$ is also in S^*_{BN}. Let

$$\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} \prec \frac{\xi}{e^\xi - 1},$$

Additionally, it can be demonstrated that there is a holomorphic function $v(\xi)$ defined within the domain U_d, satisfying $v(0) = 0$ and the inequality $|v(\xi)| \leq |\xi|$. This function is crucial for our analysis, as it enables us to represent the relationship in a more accessible manner. Consequently, we have the equality

$$\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} = \frac{w(\xi)}{e^{w(\xi)} - 1}.$$

The following equation can be obtained by using logarithmic differentiation on the previously mentioned relationship:

$$1 + \frac{\xi d_q(d_q(I_q(F_c)(\xi)))}{d_q(I_q(F_c)(\xi))} - \frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} = \frac{\xi d_q(v(\xi))(e^{v(\xi)} - v(\xi)e^{v(\xi)} - 1)}{e^{v(\xi)}(e^{v(\xi)} - 1)}.$$

Logarithmic differentiation applied to the equation (**??**) gives us the following equation:

$$\frac{\xi d_q\kappa(\xi)}{\kappa(\xi)} = \frac{q^{\gamma+1}\left(1 + q\left(\frac{\xi d_q(d_q(I_q(F_c)(\xi)))}{d_q(I_q(F_c)(\xi))} - \frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)}\right)\right)}{q^{\gamma+1}\left(\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} + [\gamma+1]_q\right)} + 1 \cdot \frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)}$$

$$= \frac{v(\xi)}{e^{v(\xi)} - 1} + \frac{q^{\gamma+1}\xi d_q(v(\xi))(e^{v(\xi)} - v(\xi)e^{v(\xi)} - 1)(e^{v(\xi)} - 1)}{q^{\gamma+1}v(\xi) + [\gamma+1]_q(e^{v(\xi)} - 1) + (e^{v(\xi)} - 1)}. \tag{2.2}$$

Using the reference (**??**), we can apply a well-known inequality for Schwarz's function $w(\xi)$, which shows that

$$|w(\xi)| \leq |\xi|.$$

Additionally, we find that the derivative satisfies:

$$|d_q w(\xi)| \leq \frac{1 - |w(\xi)|^2}{1 - |\xi|^2} = \frac{1 - R^2}{1 - |\xi|^2} \leq \frac{1}{1 - r^2}, \tag{2.3}$$

it follows that

$$\left|\frac{\xi d_q\kappa(\xi)}{\kappa(\xi)}\right| \geq \left(\frac{r}{e^r-1} - \frac{rq^{\gamma+1}(e^r+re^r+1)}{(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1)}\right),$$

$$\left|\frac{\xi d_q\kappa(\xi)}{\kappa(\xi)}\right| \geq \left(\frac{r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right) - rq^{\gamma+1}(e^r+re^r+1)}{(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1)}\right),$$

we can write

$$\left|\frac{\kappa(\xi)}{d_q\kappa(\xi)}\right| \leq \frac{r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1)}{r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right) - rq^{\gamma+1}(e^r+re^r+1)}. \tag{2.4}$$

From (**??**), we have

$$\vartheta(\xi) = \phi(\xi)\kappa(\xi).$$

Differentiating the above equality on both sides, we get

$$d_q\vartheta(\xi) = \left(\phi(\xi) + d_q\phi(\xi)\frac{\kappa(\xi)}{d_q\kappa(\xi)}\right) d_q\kappa(\xi). \tag{2.5}$$

Furthermore, examine how the Schwarz function $\phi(\xi)$ meets this established inequality, which is widely recognized among scholars:

$$|d_q\phi(\xi)| \leq \frac{1-|\phi(\xi)|^2}{1-|\xi|^2} = \frac{1-|\phi(\xi)|^2}{1-r^2} \quad (\xi \in U_d). \tag{2.6}$$

Now applying (**??**) and (**??**) in (**??**), we have

$$|d_q\vartheta(\xi)| \leq \left(|\phi(\xi)| + \frac{r(1-|\phi(\xi)|^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1)}{r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right) - rq^{\gamma+1}(e^r+re^r+1)}\right)|d_q\kappa(\xi)|. \tag{2.7}$$

By setting

$$|\phi(\xi)| = \rho \quad (0 \leq \rho \leq 1), \tag{2.8}$$

the inequality referenced in (**??**) transforms to

$$|d_q\vartheta(\xi)| \leq \Phi_1(r,\rho)|d_q\kappa(\xi)|,$$

where

$$\Phi_1(r,\rho) = |\phi(\xi)| + \frac{r(1-|\phi(\xi)|^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1)}{r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right) - rq^{\gamma+1}(e^r+re^r+1)}.$$

To find r_1, we can simply select

$$r_1 = \max\{r \in [0,1) : \Phi_1(r,\rho) \leq 1, \text{ for all } \rho \in [0,1]\},$$

or equivalently,

$$r_1 = \max\{r \in [0,1) : \Psi_1(r,\rho) \geq 0, \text{ for all } \rho \in [0,1]\},$$

where

$$\begin{aligned}\Psi_1(r,\rho) &= r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)\\ &\quad - rq^{\gamma+1}(e^r+re^r+1)\\ &\quad - r(1+\rho)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1).\end{aligned}$$

It is evident that when $\rho = 1$, the function $\Psi_1(r,\rho)$ reaches its minimum value:

$$\min(\Psi_1(r,\rho), \rho \in [0,1]) = \Psi_1(r,1) = \Psi_1(r),$$

where

$$\begin{aligned}\Psi_1(r) &= r(1-r^2)\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)\\ &\quad - rq^{\gamma+1}(e^r+re^r+1)\\ &\quad - 2r\left(rq^{\gamma+1}+[\gamma+1]_q(e^r-1)\right)(e^r-1).\end{aligned}$$

Next, we can establish the following inequalities:

$$\Psi_1(0) \geq 0 \quad \text{and} \quad \Psi_1(1) = -\left[q^{\gamma+1}(2e^r+1)+2(e-1)\left(q^{\gamma+1}+[\gamma+1]_q(e-1)\right)\right] < 0.$$

So, we can say that there's a value r_1 for which $\Psi_1(r)$ is greater than zero for every r in the range $[0; r_1]$. This r_1 represents the smallest positive solution to the equation discussed in (**??**). The fact that such an r_1 exists shows that the function Ψ_1 behaves consistently over this interval, ensuring that it remains positive. □

Theorem 2.2. *Let $\vartheta(\xi) \in A$ and $\kappa(\xi) \in S^*_{TN}$ and also suppose that $\vartheta(\xi)$ is majorized by $\kappa(\xi)$ in U_d, then for $|\xi| \leq r_2$,*

$$|d_q\vartheta(\xi)| \leq |d_q\kappa(\xi)|,$$

where r_2 is the smallest positive root of the equation

$$\begin{aligned}&\exp(r+\frac{k}{2}r^2)(q^{\gamma+1}\exp(r+\frac{k}{2}r^2)+[\gamma+1]_q)\\ &-\exp(r+\frac{k}{2}r^2)\ q^{\gamma+1}(\frac{r}{1-r^2})(1+[2]_q\frac{k}{2}r)\\ &-2r(q^{\gamma+1}\exp(r+\frac{k}{2}r^2)+[\gamma+1]_q)\\ &=0.\end{aligned} \tag{2.9}$$

Proof. Given that $\vartheta(\xi) \in S^*_{TN}$, we can refer to lemma **??**, which suggests that $I_q(F_c)(\xi)$ is also in S^*_{TN}. Let

$$\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} \prec \exp(\xi+\frac{k}{2}\xi^2).$$

Additionally, it's possible to demonstrate that there's a holomorphic function $v(\xi)$ defined within the domain U_d, where $v(0) = 0$ and the inequality $|v(\xi)| \leq |\xi|$ holds true. Therefore, we can establish the equality

$$\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} = \exp(w(\xi) + \frac{k}{2}(w(\xi))^2).$$

Applying logarithmic differentiation to the previously discussed relationship allows us to derive the following equation

$$1 + \frac{\xi d_q(d_q(I_q(F_c)(\xi)))}{d_q(I_q(F_c)(\xi))} - \frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} = \xi d_q(v(\xi))(1 + [2]_q \frac{k}{2} v(\xi)).$$

Logarithmic differentiation applied to the equation (**??**) we achieve the following equation

$$\begin{aligned}
\frac{\xi d_q \kappa(\xi)}{\kappa(\xi)} &= \left\{ \frac{q^{\gamma+1}(1 + q\frac{\xi d_q(d_q(I_q(F_c)(\xi)))}{d_q(I_q(F_c)(\xi))} - \frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)})}{q^{\gamma+1}\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} + [\gamma+1]_q} + 1 \right\} \\
&\quad \times \left(\frac{\xi d_q(I_q(F_c)(\xi))}{I_q(F_c)(\xi)} \right) \\
&= \left\{ \exp(v(\xi) + \frac{k}{2}(v(\xi))^2) \right. \\
&\quad \left. + \frac{\exp(v(\xi) + \frac{k}{2}(v(\xi))^2) q^{\gamma+1} \xi d_q(v(\xi))(1 + [2]_q \frac{k}{2} v(\xi))}{q^{\gamma+1}\exp(v(\xi) + \frac{k}{2}(v(\xi))^2) + [\gamma+1]_q} \right\}
\end{aligned}$$

$$\left| \frac{d_q \kappa(\xi)}{\kappa(\xi)} \right| = \tag{2.10}$$

$$\frac{1}{|\xi|} \left| \exp(v(\xi) + \frac{k}{2}(v(\xi))^2) + \frac{\exp(v(\xi) + \frac{k}{2}(v(\xi))^2) q^{\gamma+1} \xi d_q(v(\xi))(1 + [2]_q \frac{k}{2} v(\xi))}{q^{\gamma+1}\exp(v(\xi) + \frac{k}{2}(v(\xi))^2) + [\gamma+1]_q} \right|.$$

Let

$$\phi_{TN}(\xi) = \exp(v(\xi) + (k/2)(v(\xi))^2), \tag{2.11}$$

then it is easy to see that

$$\min_{|\xi|=r} \Re\phi_{TN}(\xi) = \phi(-r) = \min_{|\xi|=r} |\phi_{TN}(\xi)| \tag{2.12}$$

$$\max_{|\xi|=r} \Re\phi_{TN}(\xi) = \phi(r) = \max_{|\xi|=r} |\phi_{TN}(\xi)| \tag{2.13}$$

With the utilization of (**??**), consequences in (**??**) we obtain

$$\begin{aligned}
\left| \frac{d_q \kappa(\xi)}{\kappa(\xi)} \right| &= \frac{1}{|\xi|} \left| \exp(v(\xi) + (k/2)(v(\xi))^2) \right. \\
&\quad \left. + \frac{\exp(v(\xi) + (k/2)(v(\xi))^2) q^{\gamma+1} \xi d_q(v(\xi))(1 + [2]_q (k/2) v(\xi))}{q^{\gamma+1}\exp(v(\xi) + (k/2)(v(\xi))^2) + [\gamma+1]_q} \right|,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\left| \frac{d_q \kappa(\xi)}{\kappa(\xi)} \right| &\geq \frac{1}{r} \left\{ \exp(r + (k/2)r^2) \right. \\
&\quad \left. - \frac{\exp(r + (k/2)r^2) q^{\gamma+1} (r/(1-r^2))(1 + [2]_q (k/2) r)}{(q^{\gamma+1}\exp(r + (k/2)r^2) + [\gamma+1]_q)(1-r^2)} \right\},
\end{aligned} \tag{2.15}$$

$$\left|\frac{\kappa(\xi)}{d_q\kappa(\xi)}\right| \leq \frac{(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)(1-r^2)}{\mathcal{K}}, \tag{2.16}$$

where $\mathcal{K}$ is

$$\begin{aligned}\mathcal{K} &= \exp(r+(k/2)r^2)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)\\ &\quad -\exp(r+(k/2)r^2)q^{\gamma+1}\frac{r}{1-r^2}(1+[2]_q(k/2)r).\end{aligned} \tag{2.17}$$

Now applying (**??**) and (**??**) in (**??**) we have

(2.18)
$$|d_q\vartheta(\xi)| \leq \left(|\phi(\xi)| + \frac{r(1-|\phi(\xi)|^2)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)(1-r^2)}{\mathcal{K}}\right)|d_q\kappa(\xi)|$$

By setting

$$|\phi(\xi)| = \rho \quad (0 \leq \rho \leq 1), \tag{2.19}$$

the inequality referenced in (**??**) transforms to

$$|d_q\vartheta(\xi)| \leq \Phi_2(r,\rho)|d_q\kappa(\xi)|, \tag{2.20}$$

where

$$\Phi_2(r,\rho) = |\phi(\xi)| + \frac{r(1-|\phi(\xi)|^2)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)(1-r^2)}{\mathcal{K}}. \tag{2.21}$$

To find r_2 we can simply select

$$r_2 = \max\{r \in [0,1) : \Phi_2(r,\rho) \leq 1, \text{ for all } \rho \in [0,1]\}, \tag{2.22}$$

or equivalently,

$$r_2 = \max\{r \in [0,1) : \Psi_2(r,\rho) \geq 0, \text{ for all } \rho \in [0,1]\}, \tag{2.23}$$

where

$$\begin{aligned}\Psi_2(r,\rho) &= \exp(r+(k/2)r^2)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)\\ &\quad -\exp(r+(k/2)r^2)q^{\gamma+1}\frac{r}{1-r^2}(1+[2]_q(k/2)r)\\ &\quad -r(1+\rho)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)(1-r^2).\end{aligned} \tag{2.24}$$

It is evident that when $\rho = 1$ the function $\Psi_2(r,\rho)$ reaches its minimum value

$$\min(\Psi_2(r,\rho), \rho \in [0,1]) = \Psi_2(r,1) = \Psi_2(r), \tag{2.25}$$

where

$$\begin{aligned}\Psi_2(r) &= \exp(r+(k/2)r^2)(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)\\ &\quad -\exp(r+(k/2)r^2)q^{\gamma+1}\frac{r}{1-r^2}(1+[2]_q(k/2)r)\\ &\quad -2r(q^{\gamma+1}\exp(r+(k/2)r^2)+[\gamma+1]_q)(1-r^2).\end{aligned} \tag{2.26}$$

Next, we can establish the following inequalities.

$$\Psi_2(0) = (q^{\gamma+1}+[\gamma+1]_q) \geq 0 \text{ and} \tag{2.27}$$

$$\Psi_2(1) = -[\exp(r+(k/2)r^2)q^{\gamma+1}\frac{r}{1-r^2}(1+[2]_q(k/2)r)] < 0. \tag{2.28}$$

In conclusion, we find that there exists a value r_2 where $\Psi_2(r)$ is greater than zero for every r in the range of $[0, r_2]$. This r_2 is identified as the smallest positive root of the equation discussed in (**??**). The presence of this r_2 suggests that the function Ψ_2 behaves consistently within this interval, guaranteeing it remains positive. □

3. Conclusions

This research represents a significant step forward in the study of starlike functions, as it introduces two new subclasses created through the use of the q-Bernardi integral operator alongside the q-derivative. This method not only broadens the current understanding of starlike functions but also provides new ways to analyze their geometric properties and behaviors. By exploring the characteristics of these newly defined subclasses, we enhance our understanding of their unique Characteristics and potential applications. Additionally, our investigation into the majorization problem within these classes reveals the intricate relationships among different function properties, paving the way for future research. In summary, the findings from this work not only deepen the theoretical framework of analytic functions but also point to exciting possibilities for further studies, highlighting the ongoing development and significance of starlike functions in complex analysis.

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