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Research Paper

DIGRAPH CONSTRUCTIONS BASED ON INVERTER SUBSETS IN EQ-ALGEBRAS

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ABSTRACT

This paper introduces the notions of (meet, extended, symmetrical) EQ-algebras and valued indicator subsets regard to EQ-algebras. We present a new class of EQ-algebras and prove that every finite chain and countable set can be structured as an extended EQ-algebra. We try to establish a connection between the class of digraphs and the class of EQ-algebras. To this, we define the concepts of inverter digraphs and valued indicator digraphs based on EQ-algebras, which are founded on the corresponding notions of inverter subsets and valued indicator subsets. We investigate the relationship between these two types of digraphs, providing conditions under which they are isomorphic and showing that valued indicator digraphs are spanning subgraphs of inverter digraphs under certain constraints. Finally, we demonstrate a method to extract EQ-algebras from (weakly) strong digraphs.

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1. Introduction

Algebraic logic, which emerged from George Boole's seminal work in the 19th century, provides a powerful algebraic framework for studying formal logical systems. By interpreting logical connectives as algebraic operations, this field has established deep connections to order theory, universal algebra, and non-classical logics. Lattice-based structures such as residuated lattices, MV-algebras, and BL-algebras have proven particularly valuable for modeling fuzzy and many-valued logics, forming the algebraic backbone of modern approximate reasoning systems. Among these, EQ-algebras were introduced by Novák and De Baets in 2009 [11] as a novel generalization of residuated lattices, specifically designed to handle fuzzy equality as a primitive operation. This makes them uniquely suited for similarity-based reasoning, where the notion of approximate equivalence plays a more fundamental role than implication. Consequently, EQ-algebras have found applications in fuzzy type theory, linguistic modeling, and systems dealing with uncertainty and partial truth [9, 10, 8]. However, despite their logical significance, the structural and combinatorial aspects of EQ-algebras remain underexplored compared to other algebraic structures of logic, leaving a gap in our understanding of their internal organization and external representations. Concurrently, graph theory has evolved into an indispensable tool for modeling relational structures across mathematics and computer science. Directed graphs, in particular, capture dependencies, flows, and hierarchies in systems ranging from social networks to computational pipelines. The fruitful interplay between algebra and graph theory is well-documented: algebraic properties often illuminate graph invariants, while graph constructions reveal hidden algebraic features. Notable examples include zero-divisor graphs in ring theory [12, 13, 14], commuting graphs in group theory [7], fuzzy, coding and annihilator-ideal graphs in lattice theory [4, 5, 6]. These graph-theoretic perspectives not only offer intuitive visual representations but also enable the application of combinatorial methods to algebraic classification and analysis. Despite this rich interplay, no systematic bridge has been built between EQ-algebras and directed graphs. This absence is both a limitation and an opportunity: a restriction because it restricts our ability to analyze EQ-algebras using powerful graph-theoretic tools, and an opportunity because establishing such a connection could unlock new insights in both fields. Our work is motivated by the need to develop this bridge to create a bidirectional translation between the algebraic logic of EQ-algebras and the combinatorial language of digraphs. From an algebraic standpoint, representing EQ-algebras as graphs can reveal structural patterns through connectivity, completeness, and distance metrics, potentially leading to new classification schemes. Logically, such representations can support the development of visual reasoning systems and semantic networks where nodes represent concepts and edges encode fuzzy equivalence or entailment. Practically, many real-world problems in knowledge representation, network analysis, and decision support systems involve both relational (graph-like) and logical (algebra-like) dimensions; a unified framework that connects EQ-algebras to digraphs could enable more robust modeling and analysis of such complex systems.

In this paper, we consider this need by introducing and studying extended EQ-algebras, showing that every finite chain and hence every countable set can be endowed with this structure, thereby broadening the landscape of EQ-algebras. We then define, for each element of an EQ-algebra, its inverter set, the collection of elements that fully imply it, and investigate its fundamental properties. Building on this, we construct two novel directed graphs: the *inverter digraph*, where adjacency reflects a non-trivial relationship between inverter sets, and the 1-*indicator digraph*, where edges correspond to the validity of the implication between elements. We establish conditions under which these graphs are related for instance, when the 1-indicator digraph forms a spanning subgraph of the inverter digraph and prove an isomorphism theorem for symmetrical EQ-algebras. Conversely, we demonstrate how to build an EQ-algebra from any weakly connected digraph, creating what we term *graph-EQ-algebras*, and show that for such algebras the inverter digraph is complete with diameter one. We also derive combinatorial formulas for degree sums and relate graph diameters, providing concrete links between algebraic operations and graph parameters. These contributions not only advance the structural theory of EQ-algebras but also provide a systematic graph-theoretic toolkit for their analysis. Potential applications span automated reasoning (where graph algorithms can decide algebraic properties), knowledge representation (via fuzzy conceptual graphs), and network science (where algebraic constraints can be interpreted through inverter and indicator digraphs). The paper is structured as follows: Section 2 recalls preliminary definitions and results. Section 3 introduces extended EQ-algebras and establishes their ubiquity. Section 4 develops the theory of inverter subsets. Section 5 defines the two digraphs, studies their interrelations, and shows how to construct EQ-algebras from graphs. Finally, Section 6 concludes with remarks on future directions and applications.

2. **Preliminaries**

In what follow, we remember some concepts such that need to our work.

Definition 2.1. [1] An ordered pair $D = (V = \{v_i\}_i, E = \{e_i\}_i)$ is called a digraph, if $V \neq \emptyset$ and $E \subseteq V \times V$, which V is the set of all vertices and E is the set of all arces (directed edge) of D, respectively. In a digraph, edges have a direction from a tail to a head and for any $v \in V, deg^+(v) = |\{e \in E \mid tail(e) = v\}|$ and $deg^-(v) = |\{e \in E \mid head(e) = v\}|$ and a digraph without of loops is called a simple digraph.

Definition 2.2. [2] An algebra $(X, \wedge, \circleddash, *, 1)$ equiped with the binary operations $\wedge, \circleddash, *$ is a (commutative) EQ-*algebra*, if for all $x, y, z, t \in X$:

$(EQ1)$ $(X, \wedge, 1)$ is a $\wedge$-semilattice $(\forall\ x \in X, x \leq_X 1, x \leq_X y \Leftrightarrow x \wedge y = x)$,
$(EQ2)$ $(X, \circleddash, 1)$ is a (commutative) monoid and $\circleddash$ is isotone,
$(EQ3)$ $x * x = 1$,
$(EQ4)$ $((x \wedge y) * z) \circleddash (t * x) \leq_X (z * (t \wedge y))$,
$(EQ5)$ $(x * y) \circleddash (z * t) \leq_X (x * z) * (y * t)$,
$(EQ6)$ $(x \wedge y \wedge z) * x \leq_X (x \wedge y) * x$,
$(EQ7)$ $(x \wedge y) * x \leq_X (x \wedge y \wedge z) * (x \wedge z)$,
$(EQ8)$ $x \circleddash y \leq_X x * y$.

In [3], El-Zekey et. al proved that the axiom $(EQ7)$, follows from another axioms, so we can remove it from the above axioms. Based [3], for all $x, y \in X$, put $x \rightarrowtail y = (x \wedge y) * x$, $\nabla(x) = x * 0$(if X also contains a bottom element 0), $\overset{c}{x} = x * 1$ and the EQ-algebra X is called *good*, if for all $x \in X$, $\overset{c}{x} = x$. Also the EQ-algebra X is called *separated*, if for all $x, y \in X$, $x * y = 1$ implies $x = y$. Also we say for any $x \in X, x \leq_X y \Leftrightarrow x \vee y = y$ related to definition of $\vee$ and apply in the following results.

Theorem 2.3. [3] *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Then for all $a, b, c \in X$:*

(1) $a \rightarrowtail (b \wedge c) \leq_X a \rightarrowtail b$,
(2) $a \rightarrowtail b \leq_X (a \wedge c) \rightarrowtail b$,
(3) $a \rightarrowtail d \leq_X (b \rightarrowtail a) \rightarrowtail (b \rightarrowtail d)$,
(4) $b \rightarrowtail a \leq_X (a \rightarrowtail d) \rightarrowtail (b \rightarrowtail d)$,
(5) $a \rightarrowtail b \leq_X (a \wedge c) \rightarrowtail (b \wedge c)$,
(6) $a \rightarrowtail b = a \rightarrowtail (a \wedge b)$,
(7) $b \leq_X \overset{c}{b} \leq_X a \rightarrowtail b$,
(8) $a * b \leq_X a \rightarrowtail b$ *and* $a \rightarrowtail a = 1$,
(9) $a \rightarrowtail b \leq_X \nabla(b) \rightarrowtail \nabla(a)$,
(10) $a \rightarrowtail b = (a \vee b) \rightarrowtail b$,
(11) $a \rightarrowtail b = (a \vee c) \rightarrowtail (b \vee c)$.

Theorem 2.4. [3] *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Then for all $a, b, c \in X$:*

(1) $(a \rightarrowtail b) \ominus (b \rightarrowtail c) \leq_X a \rightarrowtail c$,
(2) $(a \rightarrowtail b) \ominus (a \rightarrowtail c) \leq_X a \rightarrowtail b \wedge c$,
(3) $a \ominus b \leq_X a \wedge b \leq_X a, b$,
(4) $a * b = 1$ *and* $c * d = 1$ *imply* $(a \wedge c) * (b \wedge d) = 1$,
(5) $a * b = 1$ *implies* $(a \wedge c) * (b \wedge c) = 1$ *and* $(a * c) * (b * c) = 1$,
(6) $a * b = 1$ *implies* $\overset{c}{a} * \overset{c}{b} = 1$,
(7) *if* $a \leq_X b$, *then* $\overset{c}{a} \leq_X \overset{c}{b}$, $a \rightarrowtail b = 1$, $a * b = b \rightarrowtail a$, $c \rightarrowtail a \leq_X c \rightarrowtail b$ *and* $b \rightarrowtail c \leq_X a \rightarrowtail c$.

Theorem 2.5. [3] *Let $(X, \wedge, \ominus, *, 1)$ be a good EQ-algebra. Then for all $a, b, c \in X$:*

(1) $a \rightarrowtail (b \rightarrowtail c) = b \rightarrowtail (a \rightarrowtail c)$,
(2) $a \rightarrowtail (b \rightarrowtail c) \leq_X a \ominus b \rightarrowtail c$.

3. Construction of extended EQ-algebras

In this section, we costruct a new class of EQ-algebras, which we need to continue in regard to our aims. Ley X be a nonvoid-set and $\mathcal{C}$ be an object of any arbitrary category. We say X is an **extended**-$\mathcal{C}$, if there exists a family of finitary operations (n-array) $\{O_i\}_{i \in I}$ on X such that $(X, \{O_i\}_{i \in I}) \in \mathcal{C}$.

Let $X = \{x_1, x_2, \cdots, x_n\}$. We define $\phi : \mathbb{N}_n \longrightarrow X$ by $\phi(i) = x_i$. It is easy to see that ϕ is a bijection. Define a relation $\leq_X$ on X by $x_i \leq_X x_j$ if and only if $i \leqslant j$. If $x_{max\{i\}} = x_n$, then x_n is the top element of X, therefore $(X, \leq_X, x_n)$ is a chain and so have the following lemma.

Lemma 3.1. *Any finite countable set is an extended-chain.*

Let $(X, \leq_X, 1)$ be a finite chain with top element 1 and $x, y \in X$. Define 2-arraies "$\wedge$", "$\ominus$" and "$*$" on X via $x \wedge y = x$ iff $x \leq_X y$, $x \ominus y = x \wedge y$ and $x * y = \begin{cases} 1, & \text{if } x = y, \\ x \wedge y, & \text{if } x \neq y, \end{cases}$, respectively. By these manipulations, one can see that $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra. So, any finite chain is an extended-EQ-algebra. Since the extended EQ-algebra $(X, \wedge, \ominus, *, 1)$ is dependend to the operation $\wedge$, from now on, will call it by meet-EQ-algebra.

Example 3.2. [2] Let $X = \{a, b, c, d, e, 1\}$. Then $(X, \wedge, \ominus, *, 1)$ is a meet-EQ-algebra, based on Tables 1, 2, 3 and Figure 1.

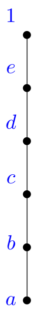

FIGURE 1. $(X, \leq)$

$\ominus$	a	b	c	d	e	1
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	e
1	a	b	c	d	e	1

TABLE 1. $(X, \ominus)$

$*$	a	b	c	d	e	1
a	1	a	a	a	a	a
b	a	1	b	b	b	b
c	a	b	1	c	c	c
d	a	b	c	1	d	d
e	a	b	c	d	1	e
1	a	b	c	d	e	1

TABLE 2. $(X, *)$

$\rightarrowtail$	a	b	c	d	e	1
a	1	1	1	1	1	1
b	a	1	1	1	1	1
c	a	b	1	1	1	1
d	a	b	c	1	1	1
e	a	b	c	d	1	1
1	a	b	c	d	e	1

TABLE 3. $(X, \rightarrowtail)$

4. Inverter set in EQ-algebras

In this part, we define the notion of inverter sets within EQ-algebra and investigate of their poperties.

Definition 4.1. Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Then for $z \in X$, define $\mathcal{I}n(z) = \{x \in X \mid x \rightarrowtail z = 1\}$ as $(\rightarrowtail)$-inverter of z in X.

It is clear that in any EQ-algebra $(X, \wedge, \ominus, *, 1), |\mathcal{I}n(z)| \geq 1$, because of $z \in \mathcal{I}n(z)$. We better explain this definition in the below two examples.

Example 4.2. [2] Let $X = \{x, y, z, t, 1\}$ and $x \leq_X y \leq_X z \leq_X t \leq_X 1$. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra that is not separated, as Tables 4 and 5.

$\ominus$	x	y	z	t	1
x	x	x	x	x	x
y	x	x	x	x	y
z	x	x	x	x	z
t	x	x	x	t	t
1	x	y	z	t	1

TABLE 4. $(X, \ominus)$

$*$	x	y	z	t	1
x	1	y	x	x	x
y	y	1	y	y	y
z	x	y	1	z	1
t	x	y	z	1	1
1	x	y	z	t	1

TABLE 5. $(X, *)$

$\rightarrowtail$	x	y	z	t	1
x	1	1	1	1	1
y	y	1	1	1	1
z	x	y	1	1	1
t	x	y	z	1	1
1	x	y	1	1	1

TABLE 6. $(X, \rightarrowtail)$

Computations show that

$$\mathcal{I}n(x) = \{x\}, \mathcal{I}n(y) = \{x, y\}, \mathcal{I}n(z) = \{x, y, z, 1\},$$
$$\text{and } \mathcal{I}n(t) = \mathcal{I}n(1) = X.$$

Theorem 4.3. *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra and $x, y, z \in X$. Then*

(i) $\mathcal{I}n(1) = X$.
(ii) $y \in \mathcal{I}n(x \rightarrowtail y)$.
(iii) *If* $x \leq_X y$, *then* $x \in \mathcal{I}n(y)$.
(iv) *If* $x \leq_X y$, *then* $\mathcal{I}n(x) \subseteq \mathcal{I}n(y)$.
(v) *If* $x \leq_X y$ *and* $z \in \mathcal{I}n(x)$, *then* $z \in \mathcal{I}n(y)$.
(vi) *If* $x \leq_X y$ *and* $y \in \mathcal{I}n(z)$, *then* $x \in \mathcal{I}n(z)$.

(vii) *If* $x \leq_X y$, *then* $x^c \in \mathcal{I}n(\overset{c}{y})$.

Proof. Imeddiate using Theorem 2.3 and Theorem 2.4. □

Example 4.4. Consider the EQ-algebra $(X, \wedge, \ominus, *, 1)$ in Example 4.2, and $z, t \in X$. Then $1 \in \mathcal{I}n(z)$, while $1 \not\leq_X z, \mathcal{I}n(1) = \mathcal{I}n(t)$, while $1 \not\leq_X t$ and $1 = \overset{c}{t} \in \mathcal{I}n(\overset{c}{z}) = \mathcal{I}n(\rightarrowtail, 1)$, while $t \not\leq_X z$. Hence the converse of Theorem 4.3, does not hold.

Corollary 4.5. *Let* $(X, \wedge, \ominus, *, 1)$ *be an* EQ*-algebra. Then for all* $x, y, z \in X$:

(i) $x \in \mathcal{I}n(x \vee y)$.
(ii) $x \wedge y \in \mathcal{I}n(x) \cap \mathcal{I}n(y)$.
(iii) $x \rightarrowtail (y \wedge z) \in \mathcal{I}n(x \rightarrowtail y)$.
(iv) $x \rightarrowtail y \in \mathcal{I}n((x \wedge z) \rightarrowtail y)$.
(v) $x * y \in \mathcal{I}n(x \rightarrowtail y)$.
(vi) $x \in \mathcal{I}n(\overset{c}{x})$.

Theorem 4.6. *Let* $(X, \wedge, \ominus, *, 1)$ *be an* EQ*-algebra. Then for all* $x, y, z \in X$:

(i) *If* $x \in \mathcal{I}n(y)$, *then* $\nabla(y) \in \mathcal{I}n(\nabla(x))$.
(ii) *If* $x \in \mathcal{I}n(z)$, *then* $(y \rightarrowtail x) \in \mathcal{I}n(y \rightarrowtail z)$.
(iii) *If* $y \in \mathcal{I}n(x)$, *then* $(x \rightarrowtail z) \in \mathcal{I}n(y \rightarrowtail z)$.
(iv) *If* $x \in \mathcal{I}n(y)$, *then* $(x \wedge z) \in \mathcal{I}n(y \wedge z)$.
(v) *If* $x \in \mathcal{I}n(y)$, *then* $(x * z) \in \mathcal{I}n(y * z)$.

Proof. Immediate based Theorem 2.3. □

The next example shows that the converse of Theorem 4.6, does not hold.

Example 4.7. [3] Let $X = \{0, a, b, c, 1\}$ and $0 \leq_X a \leq_X b \leq_X c \leq_X 1$. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra by Tables 7 and 8, which $x \wedge y = x$ iff $x \leq_X y$.

$\ominus$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	b	b
c	0	0	0	c	c
1	0	a	b	c	1

TABLE 7. $(X, \ominus)$

$*$	0	a	b	c	1
0	1	a	0	0	0
a	a	1	a	a	a
b	0	a	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

TABLE 8. $(X, *)$

$\rightarrowtail$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

TABLE 9. $(X, \rightarrowtail)$

Computations show that

$$\nabla(b) \in \mathcal{I}n(\nabla(c), \text{ while } c \notin \mathcal{I}n(b), 0 \rightarrowtail b \in \mathcal{I}n(0 \rightarrowtail a), \text{ while } b \notin \mathcal{I}n(a),$$
$$0 \rightarrowtail b \in \mathcal{I}n(a \rightarrowtail b), \text{ while } a \notin \mathcal{I}n(0), b \rightarrowtail a \in \mathcal{I}n(a \rightarrowtail a), \text{ while } b \notin \mathcal{I}n(a),$$
$$\text{and } a \wedge 0 \in \mathcal{I}n(0 \wedge 0), \text{ while } a \notin \mathcal{I}n(0).$$

Theorem 4.8. *Suppose* $(X, \wedge, \ominus, *, 1)$ *be an* EQ*-algebra. Then for all* $x, y, z, t \in X$:

(i) *If* $x \in \mathcal{I}n(y)$ *and* $y \in \mathcal{I}n(z)$, *then* $x \in \mathcal{I}n(z)$.
(ii) *If* $x \in \mathcal{I}n(y \wedge z)$, *then* $x \in \mathcal{I}n(y)$ *and* $x \in \mathcal{I}n(z)$.
(iii) *If* $x \in \mathcal{I}n(y)$ *and* $z \in \mathcal{I}n(t)$, *then* $(x \wedge z) \in \mathcal{I}n(y \wedge t)$.

Proof. Immediate applying Theorem 2.3. $\square$

The following example shows that the inverse of Theorem 4.8, does not hold.

Example 4.9. Assume $X = \{0, a, b, c, d, 1\}$. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra, based on Figure 2, Tables 10, 11 and 12. The multiplication, in this case, gives 0 everywhere except for $d \ominus d = c$.

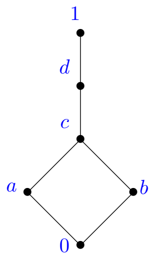

FIGURE 2 $(X, \leq)$

$\ominus$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	c	d
1	0	a	b	c	d	1

TABLE 10. $(X, \ominus)$

$*$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

TABLE 11. $(X, *)$

$\rightarrowtail$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

TABLE 12. $(X, \rightarrowtail)$

Computations show that

$$b \in \mathcal{I}n(d) \text{ and } 0 \in \mathcal{I}n(d), \text{ while } b \notin \mathcal{I}n(0),$$
$$a \wedge b \in \mathcal{I}n(c \wedge 0) \text{ and } a \in \mathcal{I}n(c), \text{ while } b \notin \mathcal{I}n(0).$$

Theorem 4.10. *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Then for all $x, y, z \in X$:*

(i) $x \in \mathcal{I}n(y)$ *if and only if* $x \in \mathcal{I}n(x \wedge y)$.
(ii) $x \in \mathcal{I}n(y)$ *if and only if* $x \vee y \in \mathcal{I}n(y)$.
(iii) $x \in \mathcal{I}n(y)$ *if and only if* $x \vee z \in \mathcal{I}n(y \vee z)$.

Proof. Immediate using Theorem 2.3. $\square$

Theorem 4.11. *Assume $(X, \wedge, \ominus, *, 1)$ is a good EQ-algebra and $x, y, z \in X$. Then*

(i) $\mathcal{I}n(y) \subseteq \mathcal{I}n(x \rightarrowtail y)$.
(ii) If $x \in \mathcal{I}n(y \rightarrowtail z)$, *then* $x \ominus y \in \mathcal{I}n(z)$.
(iii) $x \in \mathcal{I}n(y \rightarrowtail z)$ *if and only if* $y \in \mathcal{I}n(x \rightarrowtail z)$.

Proof. (i) Let $z \in \mathcal{I}n(y)$. Thus $z \rightarrowtail y = 1$. Applying Theorem 2.5, $z \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail (z \rightarrowtail y) = x \rightarrowtail 1 = 1$ and so $z \rightarrowtail (x \rightarrowtail y) = 1$. Hence $z \in \mathcal{I}n(x \rightarrowtail y)$ and so $\mathcal{I}n(y) \subseteq \mathcal{I}n(x \rightarrowtail y)$.

(ii) Since $x \in \mathcal{I}n(y \rightarrowtail z)$, so $x \rightarrowtail (y \rightarrowtail z) = 1$. Applying Theorem 2.5, $x \ominus y \rightarrowtail z = 1$. Hence $x \ominus y \in \mathcal{I}n(z)$.

(iii) By using Theorem 2.5, $x \rightarrowtail (y \rightarrowtail z) = 1$ if and only if $y \rightarrowtail (x \rightarrowtail z) = 1$. $\square$

Example 4.12. Consider the good EQ-algebra $(X, \wedge, \ominus, *, 1)$ in Example 4.9. One can see that the converse of Theorem 4.11, (i) and (ii), are not necessarily true, since $\{0, a, b, c, d\} = \mathcal{I}n(a \rightarrowtail b) \nsubseteq \mathcal{I}n(b) = \{0, b\}, d \ominus d \in \mathcal{I}n(c)$, while $d \notin \mathcal{I}n(d \rightarrowtail c)$.

5. Graph based on inverter subsets in EQ-algebras

In this part, we explore the concept of an inverter graph derived from the subsets of EQ-algebras. We examine their properties, where the inverter subsets of EQ-algebras act as the vertices of the associated graphs. Edges between any two vertices are established if their respective inverter subsets cannot be combined using the union or implication operations.

Definition 5.1. Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra and $x, y \in X$. Then $\Gamma(X) = (V = V(\Gamma(X)) = X\setminus\{1\}, E = E(\Gamma(X))$ is called an inverter graph of X, which for any two distinct vertices x and y, $(x, y) \in E$ if and only if $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y)$ and $(y, x) \in E$ if and only if $\mathcal{I}n(y) \cup \mathcal{I}n(x) \neq \mathcal{I}n(y \rightarrowtail x)$. We emphasize that $\mathcal{I}n(x) \cup \mathcal{I}n(y) = \mathcal{I}n(x \rightarrowtail y)$, means that $(x, y) \notin E$ or there is not any arc from vertex x to vertex y.

It is obvious that $\Gamma(X) = (V = V(\Gamma(X)) = X \setminus \{1\}, E = E(\Gamma(X)))$ has not loop and is not a simple graph, since for any $x, y \in V$, necessarily, $\mathcal{I}n(x \rightarrowtail y) \neq \mathcal{I}n(y \rightarrowtail x)$.

Example 5.2. [2] Suppose $X = \{0, a, b, c, d, 1\}$ and consider the lattice $(X, \leq_X)$ in Figure 3. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra as Tables 13 and 14. Now, we obtain the inverter graph $\Gamma(X)$ as shown in Figure 4.

$\ominus$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Table 13. $(X, \ominus)$

$*$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Table 14. $(X, *)$

$\rightarrowtail$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	d	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Table 15. $(X, \rightarrowtail)$

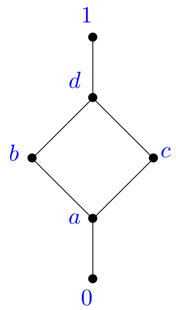

Figure 3. $(X, \leq_X)$

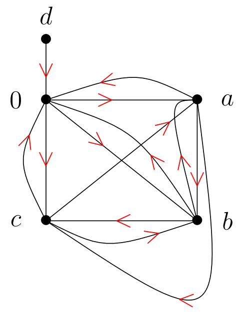

Figure 4. Inverter graph $\Gamma(X)$

Computations show that

$$\mathcal{I}n(0) = \{0\}, \mathcal{I}n(a) = \{0, a\}, \mathcal{I}n(b) = \{0, a, b\}, \mathcal{I}n(c) = \{0, a, c\}$$
$$\text{and } \mathcal{I}n(d) = \mathcal{I}n(1) = X.$$

Proposition 5.3. *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra and $x, y \in X$.*

(i) *If $(x, y) \notin E(\Gamma(X))$, then $\mathcal{I}n(x) \subseteq \mathcal{I}n(x \rightarrowtail y)$.*

(*ii*) *If* $(x, y), (y, x) \notin E(\Gamma(X))$, *then* $\mathcal{I}n(y \rightarrowtail x) = \mathcal{I}n(x \rightarrowtail y)$.

Proof. (i) Assume $x, y \in X$. As $(x, y) \notin E(\Gamma(X))$, we get that $\mathcal{I}n(x) \cup \mathcal{I}n(y) = \mathcal{I}n(x \rightarrowtail y)$ and so $\mathcal{I}n(x) \subseteq \mathcal{I}n(x \rightarrowtail y)$.

(ii) By item (i), $\mathcal{I}n(x) \cup \mathcal{I}n(y) = \mathcal{I}n(x \rightarrowtail y)$ and $\mathcal{I}n(x) \cup \mathcal{I}n(y) = \mathcal{I}n(y \rightarrowtail x)$. Thus $\mathcal{I}n(y \rightarrowtail x) = \mathcal{I}n(x \rightarrowtail y)$. □

From [1], for a digraph $D = (V, E)$, recall that, a directed (x, y)-path(a dipath from x to y) is a sequence $P : (x = v_0, e_1, v_1, ..., v_{l-1}, e_l, v_l = y)$, which $v_i \in V, e_i \in E$ are distinct (v_{i-1} and v_i are the tail and head of e_i, respectively). A digraph $D = (V, E)$ is (weakly) connected, if its underlying graph is connected and it is strongly connected if for any x, y, there exist two directed (x, y)-path and (y, x)-path.

Corollary 5.4. *Let* $(X, \wedge, \ominus, *, 1)$ *be an* EQ*-algebra.*

(*i*) *If for all* $x, y \in X$, $\mathcal{I}n(x) \nsubseteq \mathcal{I}n(x \rightarrowtail y)$, *then* $\Gamma(X)$ *is a weakly connected graph.*
(*ii*) *If for all* $x, y \in X$, $\mathcal{I}n(x) \nsubseteq \mathcal{I}n(x \rightarrowtail y)$ *and* $\mathcal{I}n(y) \nsubseteq \mathcal{I}n(y \rightarrowtail x)$, *then* $\Gamma(X)$ *is a strongly connected graph.*

Example 5.5. Consider the EQ-algebra $(X, \wedge, \ominus, *, 1)$ in Example 4.2. Then $\Gamma(X)$ is a weakly connected graph, given by Figure 5.

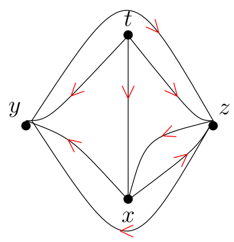

FIGURE 5. Weakly connected graph $\Gamma(X)$

A complete digraph is a simple digraph in which every pair of vertices is connected by a symmetric pair of directed arcs.

Theorem 5.6. *Assume* $(X, \wedge, \ominus, *, 1)$ *is a meet-*EQ*-algebra. Then* $\Gamma(X)$ *is a complete digraph.*

Proof. Assume $x, y \in X$. Since $(X, \wedge, \ominus, *, 1)$ is a meet-EQ-algebra, we get that $\mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(\rightarrowtail, 1)$ or $\mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(\rightarrowtail, x \wedge y)$. If $\mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(\rightarrowtail, 1)$, then $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y) \neq \mathcal{I}n(1) = X$ and it follows that $(x, y), (y, x) \in E$. Assume that $\mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(\rightarrowtail, x \wedge y)$. Now, X is a chain, so $x \leq y$ or $y \leq x$. If $y \leq x$, then $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(y) = \mathcal{I}n(x \rightarrowtail y)$ and so $(x, y), (y, x) \in E$. If $x \leq y$, then $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x) = \mathcal{I}n(y \rightarrowtail x)$ and so $(x, y), (y, x) \in E$. Hence $\Gamma(X)$ is a complete digraph. □

Example 5.7. Consider $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. By definition, $(\mathbb{Z}_4, \wedge, \ominus, *, 1)$ is a meet-EQ-algebra and $\Gamma(\mathbb{Z}_4)$ is a complete digraph, given by Figure 7.

$\ominus$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{2}$
$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

TABLE 16. $(\mathbb{Z}_4, \ominus)$

$*$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{3}$	$\bar{2}$
$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

TABLE 17. $(\mathbb{Z}_4, *)$

$\rightarrowtail$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$
$\bar{1}$	$\bar{0}$	$\bar{3}$	$\bar{3}$	$\bar{3}$
$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{3}$	$\bar{3}$
$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

TABLE 18. $(\mathbb{Z}_4, \rightarrowtail)$

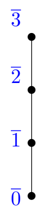

FIGURE 6. $(\mathbb{Z}_4, \leq)$

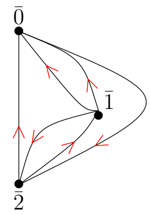

FIGURE 7. Complete digraph $\Gamma(\mathbb{Z}_4)$

In any digraph D, a cycle is a directed (x, x)-path. An acyclic digraphis a directed graph that does not contain any cycles.

Theorem 5.8. *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra and $\Gamma(X)$ be a weakly connected digraph.*

(i) If for all $x, y, z \in X$, $\mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z) \neq \mathcal{I}n(z \rightarrowtail x) \cup \mathcal{I}n(y)$, then $\Gamma(X)$ is an acyclic digraph.

(ii) If for all $x, y, z \in X$, $\mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z) \neq \mathcal{I}n(x \rightarrowtail z) \cup \mathcal{I}n(y)$, then $\Gamma(X)$ is an acyclic digraph.

Proof. $(i), (ii)$ Suppose $x, y, z \in X$. By Corollary 5.4, $\mathcal{I}n(x) \neq \mathcal{I}n(x \rightarrowtail y)$, because $\Gamma(X)$ is a connected digraph. Claim that, $\Gamma(X)$ is an acyclic digraph. Assume that $x, y, z \in X$, and consider the subgraphs G_1 and G_2 as Figures 8 and 9. In digraph $G_1 = (V_1, E_1)$ Figure

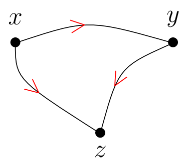

FIGURE 8. Digraph G_1

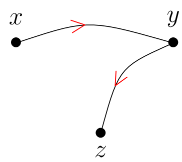

FIGURE 9. Digraph G_2

8, because $(x, y) \in E_1, (y, z) \in E_1$ and $(x, z) \in E_1$, we gain that $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y), \mathcal{I}n(y) \cup \mathcal{I}n(z) \neq \mathcal{I}n(y \rightarrowtail z)$ and $\mathcal{I}n(x) \cup \mathcal{I}n(z) = \mathcal{I}n(z \rightarrowtail x)$. Hence $\mathcal{I}n(y) \cup \mathcal{I}n(x) \cup \mathcal{I}n(z) \neq \mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z)$ and so $\mathcal{I}n(y) \cup \mathcal{I}n(z \rightarrowtail x) \neq \mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z)$. In a similar way, in digraph $G_2 = (V_2, E_2)$, we get $\mathcal{I}n(y) \cup \mathcal{I}n(z \rightarrowtail x) \neq \mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z)$. Now apply Proposition 5.3, so we get $\mathcal{I}n(y) \cup \mathcal{I}n(x \rightarrowtail z) \neq \mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z)$. □

Remark 5.9. In meet-EQ-algebras, one can see that $\mathcal{I}n(x \rightarrowtail y) \cup \mathcal{I}n(y \rightarrowtail z) = \mathcal{I}n(x \rightarrowtail z)$.

Example 5.10. Let $X = \{x, y, z, t, 1\}$ and $x \leq_X y \leq_X z \leq_X t \leq_X 1$. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra as Tables 19 and 20.

$\ominus$	x	y	z	t	1
x	x	x	x	x	x
y	x	x	x	x	y
z	x	x	x	x	z
t	x	x	x	t	t
1	x	y	z	t	1

TABLE 19. $(X, \ominus)$

$*$	x	y	z	t	1
x	1	y	x	x	x
y	y	1	y	y	y
z	x	y	1	1	1
t	x	y	z	1	1
1	x	y	z	t	1

TABLE 20. $(X, *)$

$\rightarrowtail$	x	y	z	t	1
x	1	1	1	1	1
y	y	1	1	1	1
z	x	y	1	1	1
t	x	y	1	1	1
1	x	y	1	1	1

TABLE 21. $(X, \rightarrowtail)$

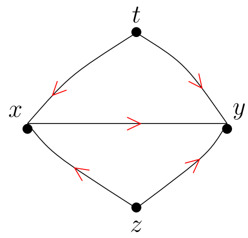

FIGURE 10. Acyclic digraph $\Gamma(X)$

Computations show that $\mathcal{I}n(x) = \{x\}, \mathcal{I}n(y) = \{x, y\}, \mathcal{I}n(z) = \mathcal{I}n(t) = \mathcal{I}n(1) = X$ and $\Gamma(X)$ is an acyclic digraph as shown in Figure 10.

Corollary 5.11. *Assume $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra. If for all $x, y \in X, \mathcal{I}n(x \rightarrowtail y) \neq \mathcal{I}n(y \rightarrowtail x)$, then $\Gamma(X)$ is a strongly connected digraph.*

Proof. Assume $x, y \in X$. Then $\Gamma(X)$ is a strongly connected digraph, if for any $x, y \in X, (x, y) \in E$ and $(y, x) \in E$ iff $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y)$ and $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(y \rightarrowtail x)$. Hence $\Gamma(X)$ is a strongly connected digraph, if for all $x, y \in X, \mathcal{I}n(x \rightarrowtail y) \neq \mathcal{I}n(y \rightarrowtail x)$. $\square$

Example 5.12. Let $X = \{a, b, c, d, 1\}$ and $a \leq_X b \leq_X c \leq_X d \leq_X 1$. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra, based on the Tables 22, 23 and $\Gamma(X)$ is a strongly connected digraph as shown in Figure 11.

$\ominus$	a	b	c	d	1
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	c	c
d	a	b	c	d	d
1	a	b	c	d	1

TABLE 22. $(X, \ominus)$

$*$	a	b	c	d	1
a	1	a	a	a	a
b	a	1	b	b	b
c	a	b	1	c	c
d	a	b	c	1	d
1	a	b	c	d	1

TABLE 23. $(X, *)$

$\rightarrowtail$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	1	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

TABLE 24. $(X, \rightarrowtail)$

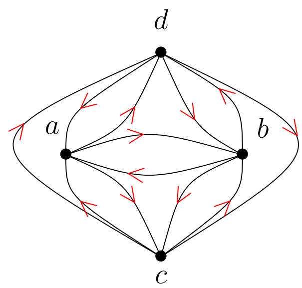

FIGURE 11. Strongly connected digraph $\Gamma(X)$

5.1. 1-Indicator graph based on EQ-algebras.

In this subsection, we introduce the concept of 1-Indicator graph in EQ-algebras and check their attributes. Indeed, we present conditions based on which 1-Indicator graph is a spanning subgraph of inverter graph. we also construct an EQ-algebra whose related 1-Indicator graph and inverter graph are isomorphic.

Definition 5.13. Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Describe the 1-indicator graph $UG(X)$ of X to be the simple directed graph with vertices $U(X) = \{a \in X \setminus \{1\} \mid \exists\, b \in X \text{ such that } b \rightarrowtail a = 1\}$, and for every separate vertices a, b, a is joined to b by an arc iff $a \rightarrowtail b = 1$. We will denote $UG(X) = (U(X), E(X))$, by the 1-indicator digraph of X.

For any EQ-algebra, $(X, \wedge, \ominus, *, 1)$, which $|X| \geq 3$, we obtain that $U(X) = X \setminus \{1\}$, because for all $a \in X$, $0 \rightarrowtail a = 1$ and so $U(X) \neq \emptyset$.

Example 5.14. (i) Let $X = \{0, a, b, c, d, e, f, 1\}$ and considre the lattice in Figure 12. Then $(X, \wedge, \ominus, *, 1)$ is an EQ-algebra as Tables 25 and 26.

$\ominus$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	d	f	1

TABLE 25. $(X, \ominus)$

$*$	0	a	b	c	d	e	f	1
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

TABLE 26. $(X, *)$

$\rightarrowtail$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1	1
b	f	f	1	1	1	1	1	1
c	d	f	e	1	1	1	1	1
d	c	c	c	c	1	1	1	1
e	a	a	c	c	f	1	f	1
f	b	0	b	c	e	e	1	1
1	0	a	b	c	d	e	f	1

TABLE 27. $(X, \rightarrowtail)$

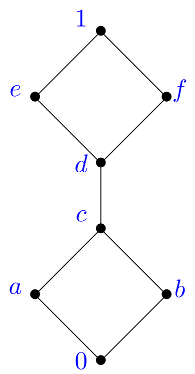

FIGURE 12. $(X, \leq_X)$

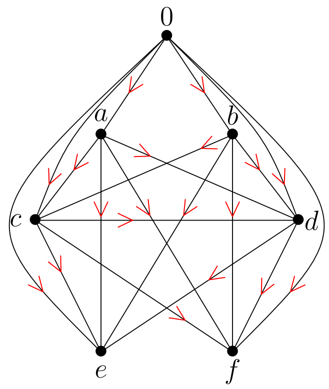

FIGURE 13. 1-indicator graph $UG(X)$

(ii) Consider the EQ-algebra $(X, \wedge, \ominus, *, 1)$ of Example 5.2. One can see that $U(X) = \{0, a, b, c, d\}$ and have the 1-indicator graph $UG(X)$ in Figure 14.

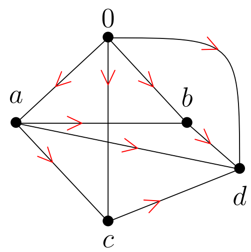

FIGURE 14. Inverter graph $\Gamma(X) = (V, E)$

Theorem 5.15. *Let $(X, \wedge, \ominus, *, 1)$ be an EQ-algebra. Then for all $x \in X \setminus \{1\}$, $\mathcal{I}n(x) \neq \mathcal{I}n(1)$ iff $UG(X)$ is a spanning subgraph of $\Gamma(X)$.*

Proof. Let $x \in X \setminus \{1\}$ and $\mathcal{I}n(x) \neq \mathcal{I}n(1)$. Clearly, $V(\Gamma(X)) = V(UG(X)) = X \setminus \{1\}$. Now, we consider the following cases:

Case 1: Assume that $(x, y) \in E(UG(X))$ and $(y, x) \notin E(UG(X))$. Hence $x \rightarrowtail y = 1$ and $y \rightarrowtail x \neq 1$ or y is not adjacent to x. Hence $x \in \mathcal{I}n(y)$ but $y \notin \mathcal{I}n(x)$. By Theorem 4.3, $\mathcal{I}n(y) \subseteq \mathcal{I}n(x \rightarrowtail y)$ and by assumption, $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(1) = X$. Thus $(x, y) \in E(\Gamma(X))$.

Case 2: Assume that $(x,y),(y,x) \in E(UG(X))$. Hence $x \rightarrowtail y = 1$ and $y \rightarrowtail x = 1$. Accordingly $x \in \mathcal{I}n(y)$ and $y \in \mathcal{I}n(x)$. By Theorem 4.3, $\mathcal{I}n(y) \subseteq \mathcal{I}n(x \rightarrowtail y)$ and $\mathcal{I}n(x) \subseteq \mathcal{I}n(y \rightarrowtail x)$. as well as $\mathcal{I}n(x) \neq \mathcal{I}n(1)$ and $\mathcal{I}n(y) \neq \mathcal{I}n(1)$. Then $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(1) = X$ and $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(y \rightarrowtail x) = \mathcal{I}n(1) = X$. Thus $(x,y),(x,y) \in E(\Gamma(X))$.
Now, let $UG(X)$ is a spanning subgraph of $\Gamma(X)$ and $x,y \in X$. Suppose x,y are neighbor in $UG(X)$, then $x \rightarrowtail y = 1$. Also x,y are neighbor in $\Gamma(X)$, then $\mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(1) = X$. Because $y \leq_X x \rightarrowtail y$, using Theorem 4.3, get that $\mathcal{I}n(y) \subseteq \mathcal{I}n(x \rightarrowtail y)$, must be $\mathcal{I}n(x) \neq \mathcal{I}n(x \rightarrowtail y)$. □

Example 5.16. Consider the EQ-algebra of Example 4.9. One can see that the $UG(X)$ in Figure 15 is a spanning subgraph of $\Gamma(X)$ in Figure 16.

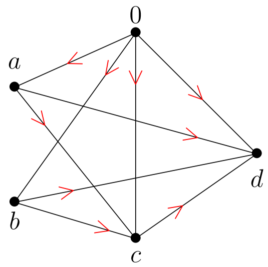

Figure 15. 1-indicator graph $UG(X)$

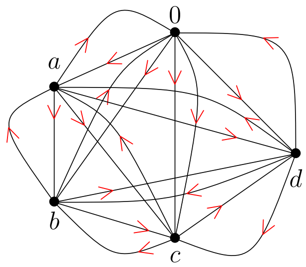

Figure 16. Inverter graph $\Gamma(X) = (V,E)$

Copmputations show that $\mathcal{I}n(0) = \{0\}, \mathcal{I}n(a) = \{0,a\}, \mathcal{I}n(b) = \{0,b\}, \mathcal{I}n(c) = \{0,a,b,c\}, \mathcal{I}n(d) = \{0,a,b,c,d\}$ and $\mathcal{I}n(1) = X$. Beacuase $a \rightarrowtail d = 1$ and $d \rightarrowtail a \neq 1$, get $(a,d) \in E(UG(X))$ but $(d,a) \notin E(UG(X))$. Also $a \in \mathcal{I}n(d)$ but $d \notin \mathcal{I}n(a)$. Clearly $\mathcal{I}n(d) \subseteq \mathcal{I}n(a \rightarrowtail d) = \mathcal{I}n(1)$. Since $\{0,a,b,c,d\} = \mathcal{I}n(a) \cup \mathcal{I}n(d) \neq \mathcal{I}n(a \rightarrowtail d) = \mathcal{I}n(1)$, thus $(a,d) \in E(\Gamma(X))$. Similarly, it can be shown for the rest of the edges of $UG(X)$.

In what follows, we show that some connected graphs generate EQ-algebras. A path graph is a graph $P_n = (V_n, E)$ whose vertices can be listed respectively $V_n = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and for any $1 \leq i \leq n-1, \{v_i, v_{i+1}\} \in E$ as Figure 17.

Figure 17. Path Graph P_n

Theorem 5.17. *Any path graph is an extended-EQ-algebra.*

Proof. Consider the path graph $P_n = (V_n, E)$ as Figure 17, whose $V_n = \{v_1, v_2, \ldots, v_{n-1}, v_n = 1\}$ and for any $1 \leq i \leq n-1, \{v_i, v_{i+1}\} \in E$. For any $1 \leq i \neq j \leq n$, define the binary operations $v_i \ominus v_j = v_i \wedge v_j = v_{\min\{i,j\}}$ and $v_i * v_j = \begin{cases} v_n, & \text{if } i = j, \\ v_{\min\{i,j\}}, & o.w, \end{cases}$. By some manipulations, one can see that $(P_n, \wedge, \ominus, *, v_n)$ is an EQ-algebra. □

Example 5.18. Consider the path graph $P_5 = (V_5, E)$ by $V_5 = \{a,b,c,d,1\}$ as Figure 18. Then $(P_5, \wedge, *, *, 1)$ is an EQ-algebra as Tables 28 and 29.

$*$	a	b	c	d	1
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	c	c
d	a	b	c	d	d
1	a	b	c	d	1

TABLE 28. $(P_5, *)$

$*$	a	b	c	d	1
a	1	a	a	a	a
b	a	1	b	b	b
c	a	b	1	c	c
d	a	b	c	1	d
1	a	b	c	d	1

TABLE 29. $(P_5, *)$

$\rightarrowtail$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	1	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

TABLE 30. $(P_5, \rightarrowtail)$

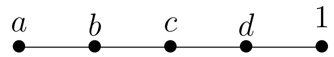

FIGURE 18. Path graph P_5

Corollary 5.19. *Any weakly connected graph can be an EQ-algebra.*

Proof. Suppose $G = (V, E)$ be a weakly connected graph and $|V| = n$. Then for any $x, y \in V$, there is a path $P(x, y)$ of x into y. Thus we can extract a path graph $P_n = (V_n, E)$ as a subgraph of $G = (V, E)$, so using Theorem 5.17, $G = (V, E)$ can be an EQ-algebra. $\square$

From now on, for any EQ-algebra $(X, \wedge, \ominus, *, 1)$ which is construced from any graph, we will call it by graph-EQ-algebra.

Theorem 5.20. *Let $(X, \wedge, \ominus, *, 1)$ be a graph-EQ-algebra and $|X| = n$. Then*

(i) $\displaystyle\sum_{v_i \in V(UG(X))} deg^+(v_i) = \sum_{v_i \in V(UG(X))} deg^-(v_i)$.

(ii) $\displaystyle\sum_{v_i \in V(\Gamma(X))} deg^+(v_i) = \sum_{v_i \in V(\Gamma(X))} deg^-(v_i)$.

Proof. (i) Clearly, $|V(UG(X)))| = |X \setminus \{1\}| = n - 1$. Since for all $1 \leq i < j \leq n-1$, $v_i, v_j \in V(UG(X))$, $v_i \rightarrowtail v_j = 1$, so $deg^+(v_i) = n-(i+1)$. Also since for all $1 \leq s < i \leq n-1$, $v_s \rightarrowtail v_i = 1$, so $deg^-(v_i) = i - 1$. It follows that $\displaystyle\sum_{v_i \in V(UG(X))} deg^+(v_i) = \sum_{i=1}^{n-1}(n - (i+1)) = \frac{(n-1)(n-2)}{2}$ and $\displaystyle\sum_{v_i \in V(UG(X))} deg^-(v_i) = \sum_{i=1}^{n-1}(i-1) = \frac{(n-1)(n-2)}{2}$.

(ii) Clearly, $|V(\Gamma(X))| = |X \setminus \{1\}| = n - 1$. Assume that for all $1 \leq i < j \leq n-1$, $v_i, v_j \in V(\Gamma(X))$. Hence $v_i \rightarrowtail v_j = v_n$ and $v_j \rightarrowtail v_i = v_i$. Therefore

$$\mathcal{I}n(v_j) = \mathcal{I}n(v_i) \cup \mathcal{I}n(v_j) \neq \mathcal{I}n(v_i \rightarrowtail v_j) = \mathcal{I}n(v_n) = X \text{ and}$$
$$\mathcal{I}n(v_j) = \mathcal{I}n(v_i) \cup \mathcal{I}n(v_j) \neq \mathcal{I}n(v_j \rightarrowtail v_i) = \mathcal{I}n(v_i).$$

Thus $(v_i, v_j) \in E(\Gamma(X)$ and $(v_j, v_i) \in E(\Gamma(X))$. Then $\Gamma(X)$ is a complete digraph and so $deg^+(v_i)(v_i) = deg^-(v_i)(v_i) = n-2$. It follows that $\displaystyle\sum_{v_i \in V(\Gamma(X))} deg^+(v_i) = \sum_{v_i \in V(\Gamma(X))} deg^-(v_i) = (n-1)(n-2)$. $\square$

Example 5.21. (i) Consider the graph-EQ-algebra as shown in Figure 22. By Theorem 5.17, $(X, \wedge, \ominus, *, f)$ is an EQ-algebra on Tables 31, 32, and by Theorem 5.20, we get the 1-indicator graph $UG(X)$ as pictured in Figure 20.

$*$	a	b	c	d	e	1
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	e
1	a	b	c	d	e	1

TABLE 31. $(X, *)$

$*$	a	b	c	d	e	1
a	1	a	a	a	a	a
b	a	1	b	b	b	b
c	a	b	1	c	c	c
d	a	b	c	1	d	d
e	a	b	c	d	1	e
1	a	b	c	d	e	1

TABLE 32. $(X, *)$

$\rightarrowtail$	a	b	c	d	e	1
a	1	1	1	1	1	1
b	a	1	1	1	1	1
c	a	b	1	1	1	1
d	a	b	c	1	1	1
e	a	b	c	d	1	1
1	a	b	c	d	e	1

TABLE 33. $(X, \rightarrowtail)$

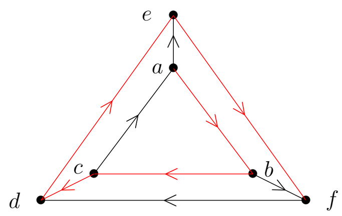

FIGURE 19. Graph-EQ-algebra

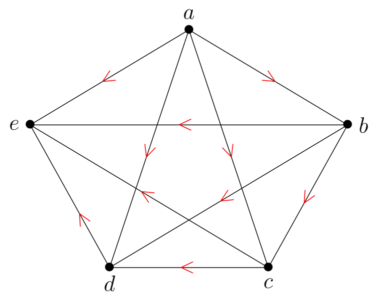

FIGURE 20. 1-Indicator Graph $UG(X)$

Computations show that

$$deg^+(a) = 4, deg^-(a) = 0, deg^+(b) = 3, deg^-(b) = 1, deg^+(c) = 2, deg^-(c) = 2,$$
$$deg^+(d) = 1, deg^-(d) = 3, deg^+(e) = 0 \text{ and } deg^-(e) = 4,$$
$$\sum_{v_i \in V(UG(X))} deg^+(v_i) = \sum_{v_i \in V(UG(X))} deg^-(v_i) = 10$$

(ii) Consider the graph-EQ-algebra as shown in Figure 22. Then by Theorem 5.20, we get an inverter graph $\Gamma(X)$ in Figure 21.

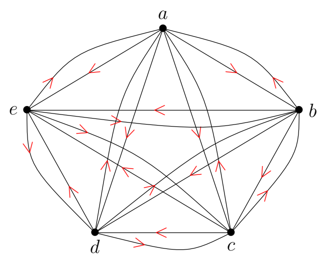

FIGURE 21. Inverter Graph $\Gamma(X)$.

for any $v_i, deg^+(v_i) = 4, deg^-(v_i) = 4$ and so $\sum_{v_i \in V(UG(X))} deg^+(v_i) = \sum_{v_i \in V(UG(X))} deg^-(v_i) =$ 20.

Corollary 5.22. *Let $(X, \wedge, \ominus, *, 1)$ be a graph-EQ-algebra and $|X| = n$. Then $diam(\Gamma(X)) = diam(UG(X))$.*

We want to find the some conditions such that two graphs $UG(X)$ and $\Gamma(X)$ be isomorphic. In this regard we have to construct special EQ-algebras such that these conditions are stabilished. Let $(X, \leq_X, 0, 1)$ be a finite chain with top element 1. Define operations "$\wedge$", "$\circleddash$" and "$*$" on X by $x \wedge y = x$, iff $x \leq_X y$, $x \circleddash y = 0$ and $x * y = \begin{cases} 1, & \text{if } x = y, \\ \max\{x, y\}, & \textit{otherwise}, \end{cases}$.

Theorem 5.23. *The structure $(X, \wedge, \circleddash, *, 1)$ is an extended EQ-algebra.*

Proof. By some manipulations, we can see that $(X, \wedge, \circleddash, *, 1)$ is an EQ-algebra. $\square$

From now on, we will call the EQ-algebra $(X, \wedge, \circleddash, *, 1)$ in Theorem 5.23 by symmetrical-EQ-algebra.

Theorem 5.24. *Let $(X, \wedge, \circleddash, *, 1)$ be a symmetrical-EQ-algebra and $|X| = n$. Then $UG(X) \cong \Gamma(X)$.*

Proof. Since $|X| = n$, $V(\Gamma(X)) = V(UG(X)) = X \setminus \{1\}$. Assume that $x, y \in X \setminus \{1\}$ and $x \leq y$. Hence $x \rightarrowtail y = 1$ and so $(x, y) \in E(UG(X))$ and $(y, x) \notin E(UG(X))$, because of $y \rightarrowtail x = y \wedge x * y = y$. In addition, $\mathcal{I}n(y) = \mathcal{I}n(x) \cup \mathcal{I}n(y) \neq \mathcal{I}n(x \rightarrowtail y) = \mathcal{I}n(1) = X$ and $\mathcal{I}n(y) = \mathcal{I}n(x) \cup \mathcal{I}n(y) = \mathcal{I}n(y \rightarrowtail x) = \mathcal{I}n(y)$. Hence $(x, y) \in E(\Gamma(X))$ and $(y, x) \notin E(\Gamma(X))$. Now define a map $\varphi : UG(X) \to \Gamma(X)$ by $\varphi(v) = v$. Since φ is a bijection, by above describtions, for any $x, y \in X \setminus \{1\}$, it is clear that $(x, y) \in E(UG(X))$ if and only if $(x, y) \in E(\Gamma(X))$. Thus $UG(X) \cong \Gamma(X)$. $\square$

Example 5.25. Assume $X = \{x, y, z, t, 1\}$ and $x \leq_X y \leq_X z \leq_X t \leq_X 1$. If $(X, \wedge, \circleddash, *, 1)$ is a symmetrical-EQ-algebra based Theorem 5.23 and Tables 34 and 35, then the inverter digraph $\Gamma(X)$ and the 1-indicator digraph $UG(X)$ constructed from this algebra are isomorphic. In fact, both digraphs have the identical graphical representation, shown in Figure 22.

$\circleddash$	x	y	z	t	1
x	x	x	x	x	x
y	x	x	x	x	y
z	x	x	x	x	z
t	x	x	x	x	t
1	x	y	z	t	1

Table 34. $(X, \circleddash)$

$*$	x	y	z	t	1
x	1	y	z	t	x
y	y	1	z	t	y
z	z	z	1	t	z
t	t	t	t	1	t
1	x	y	z	t	1

Table 35. $(X, *)$

$\rightarrowtail$	x	y	z	t	1
x	1	1	1	1	1
y	y	1	1	1	1
z	z	z	1	1	1
t	t	t	t	1	1
1	x	y	z	t	1

Table 36. $(X, \rightarrowtail)$

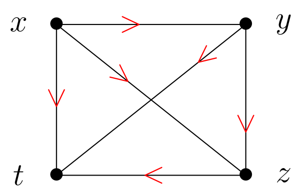

Figure 22. Digraph

Computations show that

$$\mathcal{I}n(x) = \{x\}, \mathcal{I}n(y) = \{x, y\}, \mathcal{I}n(z) = \{x, y, z\}, \mathcal{I}n(t) = X - \{1\} \text{ and } \mathcal{I}n(1) = X$$

5.2. **Categorical interpretation.** In this subsection, in our constructions naturally, we suggest a categorical tool. Indeed, we can formalize the relationship between EQ-algebras and digraphs as follows.

Let **EQAlg** denote the category whose objects are EQ-algebras and whose morphisms are EQ-algebra homomorphisms (preserving $\wedge$, $\ominus$, $*$, and 1). Let **Digraph** denote the category of directed graphs (allowing loops) with graph homomorphisms. Define the *inverter functor* $\Gamma : \mathbf{EQAlg} \to \mathbf{Digraph}$ by:

(1) For an EQ-algebra X, $\Gamma(X)$ is the inverter digraph defined in Section 5.
(2) For an EQ-algebra homomorphism $f : X \to Y$, define $\Gamma(f) : \Gamma(X) \to \Gamma(Y)$ on vertices by $v \mapsto f(v)$ (restricted to $X \setminus \{1_X\}$), and on arcs by $(x, y) \mapsto (f(x), f(y))$ provided the corresponding inverter set condition is preserved.

Also, define the *graph-EQ functor* $\mathcal{E} : \mathbf{Digraph} \to \mathbf{EQAlg}$ by:

(1) For a digraph $D = (V, E)$, $\mathcal{E}(D)$ is the graph-EQ-algebra constructed from D as in Theorem 5.17.
(2) For a graph homomorphism $\varphi : D \to D'$, define $\mathcal{E}(\varphi) : \mathcal{E}(D) \to \mathcal{E}(D')$ as the induced EQ-algebra homomorphism that maps vertices according to φ and extends linearly.

Now, have the following result.

Proposition 5.26. *The assignments Γ and $\mathcal{E}$ are functors. Moreover, $\Gamma \circ \mathcal{E}$ sends a digraph D to a complete digraph (by Theorem 5.20), while $\mathcal{E} \circ \Gamma$ sends an EQ-algebra X to a potentially larger EQ-algebra whose inverter digraph is isomorphic to $\Gamma(X)$.*

6. Conclusions

This study has established a comprehensive bridge between EQ-algebras and directed graph theory through several key contributions. We introduced and formalized new algebraic structures including meet, extended, and symmetrical EQ-algebras, alongside the novel concepts of inverter subsets and valued indicator subsets. A fundamental achievement was proving that every finite chain and countable set can be structured as an extended EQ-algebra, significantly expanding the known classes of these algebraic systems. The core theoretical advancement lies in creating a dual correspondence between digraphs and EQ-algebras, enabling bidirectional translations between these mathematical domains. We developed innovative graph constructions through inverter digraphs and valued indicator digraphs derived from EQ-algebraic foundations. Our detailed analysis revealed precise conditions for isomorphism between these digraph families and established that valued indicator digraphs form spanning subgraphs of inverter digraphs under specific constraints. The principal advantage of this framework is its capacity to solve graph-theoretic problems using algebraic methods and vice versa, enriching both fields simultaneously. However, limitations include the failure of several converse theorems, as demonstrated by our counterexamples, and structural dependencies on specific operations that restrict immediate generalization. Future research will explore applications in fuzzy logic systems and semantic network modeling, where this correspondence can provide new analytical tools. We aim to develop more relaxed conditions to broaden applicability across diverse algebraic classes. Investigating computational aspects for large-scale algebra-to-graph conversions presents another critical direction. Further work will also examine categorical relationships between these structures and extend the framework to other non-classical logics. This research ultimately creates a robust foundation for continued interdisciplinary exploration between algebraic logic and graph theory, with promising applications in knowledge representation and complex system analysis.

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