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Symmetries of Vaidya spacetimes

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Abstract. In this paper, we study Killing and 2-Killing vector fields on Vaidya spacetimes. We determine all Killing symmetries of the metric and obtain several classes of 2-Killing vector fields under the assumption of a single non-zero component.

Keywords: Vaidya spacetimes, Killing and 2-Killing vector fields.

1. Introduction

In 1951, Vaidya introduced a solution to Einstein's field equations describing a spherically symmetric star that radiates energy and loses mass over time [10]. This solution, now known as the Vaidya spacetime, is an extension of the Schwarzschild metric to dynamical situations. It describes radiating black holes and is widely used to study the formation and evolution of horizons in non-stationary spacetimes.

Over the years, many researchers have studied the Vaidya solution and its extensions. For instance, Bonnor and Vaidya [6] examined charged radiating stars, non-spherical generalizations [13], and applications in modified gravity

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theories [3]. These investigations revealed interesting phenomena, including energy fluxes, evolving apparent horizons, and the effects of radiation on spacetime symmetries, showing that even relatively simple metrics can have rich geometric and physical structures [4, 5, 8, 11].

A standard way to study spacetime symmetries is to use Killing vector fields. A vector field X on a pseudo-Riemannian manifold (M, g) is called Killing if

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}_X$ denotes the Lie derivative along X. Killing vectors preserve the metric, leading to conserved quantities along geodesics and uncovering symmetries of the spacetime [1, 2, 9]. Static spacetimes such as Schwarzschild admit a relatively large set of Killing vectors.

Oprea [7] introduced the concept of 2-Killing vector fields, which generalizes the notion of Killing vector fields. A vector field X is called 2-Killing if

$$\mathcal{L}_X \mathcal{L}_X g = 0.$$

Every Killing vector field is also 2-Killing, but the converse is not necessarily true. Thus, 2-Killing vector fields can detect hidden symmetries that are not invisible by Killing symmetries. They may provide a new view of the geometric structure of spacetime.

Recently, Vertogradov and Kudryavtsev studied the generalized Vaidya spacetime. In their model the mass depends on both time v and radius r, i.e., $M = M(v, r)$. They derived the field equations, examined the horizons, and found the surface gravity. They also showed that for certain mass functions, the metric has a homothetic Killing field [12].

In this paper, we consider the standard Vaidya spacetime where the mass depends only on the advanced time, $m = m(v)$. We solve the Killing equations and find all Killing vector fields of the metric. For $m'(v) \neq 0$, only the three rotational Killing vectors exist, while in the static case $m'(v) = 0$ (Schwarzschild spacetime), an additional timelike Killing vector appears.

Although the Vaidya metric has been widely studied, its Killing and 2-Killing vector fields are not fully classified. Most earlier works focused on static or stationary cases [13], while radiating cases are more complicated due to the time-dependent mass. In this work, we extend the analysis to 2-Killing vector fields under the single-component assumption, finding conditions on the mass function for additional symmetries.

2. Killing Vector Fields of the Vaidya Metric

The Vaidya spacetime represents a spherically symmetric radiating star and is described by the metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.1}$$

where $m(v)$ is a function of the advanced time v, representing the time-dependent mass of the star. When $m'(v) = 0$, the metric reduces to the Schwarzschild solution, while for $m'(v) \neq 0$ it describes a radiating (non-stationary) black hole.

We consider the general vector field

$$X = X^v(v,r,\theta,\phi)\,\partial_v + X^r(v,r,\theta,\phi)\,\partial_r + X^\theta(v,r,\theta,\phi)\,\partial_\theta + X^\phi(v,r,\theta,\phi)\,\partial_\phi.$$

The symmetries of the Vaidya spacetime are described by its Killing vector fields. In this section, we explicitly solve the Killing equations in standard coordinates for both the dynamic case ($m'(v) \neq 0$) and the static limit ($m'(v) = 0$).

The Killing equations $(\mathcal{L}_X g)_{\mu\nu} = 0$ reduce to the following ten PDEs:

$$\frac{m'(v)}{r}\,X^v - \frac{m(v)}{r^2}\,X^r - \Big(1 - \frac{2m(v)}{r}\Big)\partial_v X^v + \partial_v X^r = 0, \tag{2.2}$$

$$\partial_r X^r + \partial_v X^v - \Big(1 - \frac{2m(v)}{r}\Big)\partial_r X^v = 0, \tag{2.3}$$

$$r^2\,\partial_v X^\theta - \Big(1 - \frac{2m(v)}{r}\Big)\partial_\theta X^v + \partial_\theta X^r = 0, \tag{2.4}$$

$$r^2\sin^2\theta\,\partial_v X^\phi - \Big(1 - \frac{2m(v)}{r}\Big)\partial_\phi X^v + \partial_\phi X^r = 0, \tag{2.5}$$

$$\partial_r X^v = 0, \tag{2.6}$$

$$r^2\,\partial_r X^\theta + \partial_\theta X^v = 0, \tag{2.7}$$

$$r^2\sin^2\theta\,\partial_r X^\phi + \partial_\phi X^v = 0, \tag{2.8}$$

$$X^r + r\,\partial_\theta X^\theta = 0, \tag{2.9}$$

$$\sin^2\theta\,\partial_\theta X^\phi + \partial_\phi X^\theta = 0, \tag{2.10}$$

$$X^r\sin\theta + r\,\cos\theta X^\theta + r\sin\theta\partial_\phi X^\phi = 0. \tag{2.11}$$

Integrating (2.6) with respect to r implies

$$X^v = A_1(v,\theta,\phi). \tag{2.12}$$

Setting (2.12) in (2.3), it follows

$$\partial_r X^r + \partial_v A_1(v,\theta,\phi) = 0, \tag{2.13}$$

which gives us

$$X^r = -r\partial_v A_1(v,\theta,\phi) + A_2(v,\theta,\phi).$$

Setting the above equation in (2.2), we get

$$r^3\partial_v^2 A_1(v,\theta,\phi) - r^2(\partial_v A_2(v,\theta,\phi) - \partial_v A_1(v,\theta,\phi)) - r(A_1(v,\theta,\phi)m'(v) \tag{2.14}$$
$$+ 3m(v)\partial_v A_1(v,\theta,\phi)) + A_2(v,\theta,\phi)m(v) = 0,$$

which gives

$$A_1(v, \theta, \phi) = B_1(\theta, \phi), \quad A_2(v, \theta, \phi) = 0. \tag{2.15}$$

Using the above equations in (2.14), we have

$$B_1(\theta, \phi) m'(v) = 0.$$

According to the last equation, we consider two cases $B_1(\theta, \phi) = 0$ or $m'(v) = 0$.

Case A: Let $B_1(\theta, \phi) = 0$. From (2.12) and (2.15) we have

$$X^v = 0.$$

Thus (2.4), (2.7) and (2.9) imply

$$X^\theta = E_1(\phi).$$

Using (2.5) and (2.8), we have

$$X^\phi = B_2(\theta, \phi).$$

Hence, from (2.10) and (2.11), we have the following system of partial differential equations:

$$\sin^2\theta \partial_\theta B_2(\theta, \phi) + E_1'(\phi) = 0, \tag{2.16}$$

$$E_1(\phi)\cos\theta + \sin\theta \partial_\phi B_2(\theta, \phi) = 0. \tag{2.17}$$

Integrating (2.17) with respect to ϕ, it follows

$$B_2(\theta, \phi) = -\cot\theta \int E_1(\phi) d\phi + C(\theta). \tag{2.18}$$

According to the above equation, we get

$$\partial_\theta B_2(\theta, \phi) = \csc^2\theta \int E_1(\phi) d\phi + C'(\theta).$$

Substitute the last equation in (2.16), we have

$$\int E_1(\phi) d\phi + \sin^2\theta C'(\theta) + E_1'(\phi) = 0,$$

hence, it follows

$$\int E_1(\phi) d\phi + E_1'(\phi) = k = -\sin^2\theta C'(\theta), \tag{2.19}$$

where k is constant. From the second equation of (2.19), we obtain

$$C(\theta) = k\cot\theta + c_1, \tag{2.20}$$

where c_1 is a constant. Let $F(\phi) = \int E_1(\phi) d\phi$. Then $F'(\phi) = E_1(\phi)$ and the first equation of (2.19) becomes:

$$F''(\phi) + F(\phi) = k,$$

which has the following solution:

$$F(\phi) = c_2 \sin\phi + c_3 \cos\phi + k, \tag{2.21}$$

where c_2 and c_3 are constants. Thus it follows

$$E_1(\phi) = F'(\phi) = c_2 \cos\phi - c_3 \sin\phi.$$

Substitute (2.20) and (2.21) in (2.18), we have

$$B_2(\theta,\phi) = c_1 - \cot\theta\big(c_2 \sin\phi + c_3 \cos\phi\big).$$

Therefore, we have

$$X^v = X^r = 0, \quad X^\theta = c_2 \cos\phi - c_3 \sin\phi, \quad X^\phi = c_1 - \cot\theta\,(c_2 \sin\phi + c_3 \cos\phi).$$

Case B: Let $m'(v) = 0$. So, we have $m(v) = c$, where c is constant. In this case, (2.9) implies

$$X^\theta = C_1(v,r,\phi)\,.$$

According to (2.4), we get

$$r^2\partial_v C_1(v,r,\phi) + \partial_\theta B_1(\theta,\phi)\,(-1 + \frac{2c}{r}) = 0, \tag{2.22}$$

which gives us

$$C_1(v,r,\phi) = -\frac{(2c-r)v}{r^3}\partial_\theta B_1(\theta,\phi) + f_1(r,\phi).$$

Using the above equation in (2.7) and differentiating it with respect to v, it follows

$$B_1(\theta,\phi) = E_2(\phi).$$

Hence (2.2)-(2.11) reduce to

$$r^2\partial_v X^\phi \sin^2\theta + E_2'(\phi)(-1 + \frac{2c}{r}) = 0, \tag{2.23}$$

$$\partial_r f_1(r,\phi) r^2 = 0, \tag{2.24}$$

$$r^2\partial_r X^\phi \sin^2\theta + E_2'(\phi) = 0, \tag{2.25}$$

$$\partial_\theta X^\phi \sin^2\theta + \partial_\phi f_1(r,\phi) = 0, \tag{2.26}$$

$$\partial_\phi X^\phi \sin\theta + f_1(r,\phi)\cos\theta = 0. \tag{2.27}$$

Differentiating (2.25) with respect to v gives us

$$\partial_v\partial_r X^\phi = 0.$$

Using the above equation and differentiating (2.23) with respect to r, we get

$$r\sin^2\theta\,\partial_v X^\phi - \frac{c}{r^2}E_2'(\phi) = 0.$$

According to the above equation we find $E_2(\phi) = c_1$, where c_1 is constant. Hence we have $B_1(\theta,\phi) = c_1$. Applying this and from (2.12) and (2.15), it follows $X^v = c_1$. Also, the above equation and (2.25) lead to $\partial_v X^\phi = \partial_r X^\phi = 0$. Thus we have

$$X^\phi = B_2(\theta, \phi). \tag{2.28}$$

On the other hand (2.24) gives

$$f_1(r, \phi) = E_3(\phi).$$

So, (2.23)-(2.27) reduce to

$$\partial_\theta B_2(\theta, \phi) \sin^2\theta + E_3'(\phi) = 0,$$
$$\partial_\phi B_2(\theta, \phi) \sin\theta + E_3(\phi)\cos\theta = 0.$$

Similar Case A, we obtain

$$X^v = c_1, \quad X^r = 0, \quad X^\theta = c_3\cos\phi - c_4\sin\phi, \quad X^\phi = c_2 - \cot\theta\,(c_3\sin\phi + c_4\cos\phi).$$

Based on the calculations and discussions presented above, the Killing vectors for the Vaidya metric can be summarized in the following theorem:

Theorem 2.1. *Consider the standard Vaidya metric as given in (2.1), with a mass function $m(v)$. Then:*

(1) *If $m'(v) \neq 0$, the spacetime admits only the three rotational Killing vectors generating the $SO(3)$ algebra, as*

$$\xi^1 = -\partial_\phi, \quad \xi^2 = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \quad \xi^3 = -\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi.$$

(2) *If $m'(v) = 0$, the spacetime admits the following four Killing vector fields:*

$$\xi^0 = -\partial_v, \quad \xi^1 = -\partial_\phi, \quad \xi^2 = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \quad \xi^3 = -\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi.$$

Here, ξ^0 is the timelike Killing vector, and ξ^1, ξ^2, ξ^3 are the rotational Killing vectors generating the $SO(3)$ algebra.

3. **2-Killing Vector Fields on Vaidya Spacetime**

In this section, we study 2-Killing vector fields in the Vaidya spacetime. A 2-Killing vector field is a generalization of a usual Killing vector field which given by

$$\pounds_X \pounds_X g = 0, \tag{3.1}$$

where $\pounds_X$ is the Lie derivative along X. Every Killing vector is also a 2-Killing vector, but not every 2-Killing vector is a Killing vector. Proper 2-Killing vectors describe higher-order symmetries of the spacetime. Using the general formula for the second Lie derivative, we have

$$(\pounds_X \pounds_X g)(Y, Z) = X\big((\pounds_X g)(Y, Z)\big) - (\pounds_X g)([X, Y], Z) - (\pounds_X g)(Y, [X, Z]),$$

which in local coordinates can be written as

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{ij} =& X^m(\partial_m X^n \partial_n g_{ij} + X^n \partial_m \partial_n g_{ij} + g_{jn}\partial_m \partial_i X^n \\
&+ g_{in}\partial_m \partial_j X^n + 2\partial_m g_{ni}\partial_j X^n) + 2\partial_m g_{nj}\partial_i X^n \\
&+ g_{im}\partial_j X^n \partial_n X^m + g_{jm}\partial_i X^n \partial_n X^m + 2g_{mn}\partial_i X^m \partial_j X^n.
\end{aligned} \tag{3.2}
$$

We now apply this formulation to the Vaidya metric. Using the Vaidya metric (2.1) in (3.2), we obtain a coupled system of nonlinear partial differential equations for the components of the vector field

$$
X = X^v\,\partial_v + X^r\,\partial_r + X^\theta\,\partial_\theta + X^\phi\,\partial_\phi, \tag{3.3}
$$

where X^v, X^r, X^θ, and X^ϕ are smooth functions of the coordinates v, r, θ, ϕ. Substituting this vector field in (3.2) leads to the following component expressions:

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{vv} =& \frac{1}{r^3}\{2X^\theta\, r^2(2m(v) - r)\,\partial_v\partial_\theta X^v - 2X^\phi\, r^2(r - 2m(v))\,\partial_v\partial_\phi X^v \\
&- 2r^2X^v(r-2m(v))\,\partial_v^2 X^v - 2r^2X^r(r-2m(v))\,\partial_v\partial_r X^v + 2X^\theta\, r^3\,\partial_v\partial_\theta X^r \\
&+ 2X^\phi\, r^3\,\partial_v\partial_\phi X^r + 2r^3X^v\,\partial_v^2 X^r + 2r^3X^r\,\partial_v\partial_r X^r - 4r^2(r-2m(v))\,(\partial_v X^v)^2 \\
&+ (6r^3\partial_v X^r + 10r^2m'(v)X^v - 8m(v)rX^r)\partial_v X^v - 2r\big(r(r-2m(v))\partial_r X^v - r^2\partial_r X^r \\
&+ m(v)X^v\big)\partial_v X^r + 2r^5(\partial_v X^\theta)^2 - 2r^2\big((r-2m(v))\partial_\theta X^v - r\partial_\theta X^r\big)\partial_v X^\theta \\
&+ 2r^5\sin^2\theta(\partial_v X^\phi)^2 - 2r^2\big((r-2m(v))\partial_\phi X^v - r\partial_\phi X^r\big)\partial_v X^\phi + 2m'(v)\,\partial_\theta X^v X^\theta r^2 \\
&+ 2m'(v)\,\partial_\phi X^v X^\phi r^2 + 2(X^v)^2\, m''(v) r^2 + 2X^r m'(v)\partial_r X^v r^2 - 2m(v)\partial_\theta X^r X^\theta r \\
&- 2m(v)\partial_\phi X^r X^\phi r - 2m(v)X^r\partial_r X^r r - 4X^vX^r m'(v) r + 4m(v)(X^r)^2\},
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{vr} =& \frac{1}{r^2}\{((2m(v) - r)X^v - rX^r)\,r\,\partial_v\partial_r X^v - r(r-2m(v))X^\theta\,\partial_r\partial_\theta X^v \\
&- r(r-2m(v))X^\phi\,\partial_r\partial_\phi X^v - r(r-2m(v))X^r\,\partial_r^2 X^v + r^2X^\theta\,\partial_v\partial_\theta X^v \\
&+ r^2X^\theta\,\partial_r\partial_\theta X^r + r^2X^\phi\,\partial_v\partial_\phi X^v + r^2X^\phi\,\partial_\phi\partial_r X^r + r^2X^v\,\partial_v^2 X^v \\
&+ r^2X^v\,\partial_v\partial_r X^r + r^2X^r\,\partial_r^2 X^r + \big[-3r(r-2m(v))\partial_v X^v - r(r-2m(v))\partial_r X^r \\
&+ 4r^2\partial_v X^r + 4rX^v m'(v) - 4X^r m(v)\big]\partial_r X^v + r\Big[r(\partial_v X^v)^2 + 2r\partial_v X^v\partial_r X^r \\
&+ r(\partial_r X^r)^2 + ((-r+2m(v))\partial_\theta X^v + 2r^3\partial_v X^\theta + r\partial_\theta X^r)\partial_r X^\theta + (-r+2m(v))\partial_\phi X^v \\
&+ 2r^3\sin^2\theta\partial_v X^\phi + r\partial_\phi X^r)\partial_r X^\phi + r(\partial_v X^\theta\partial_\theta X^v + \partial_v X^\phi\partial_\phi X^v)\Big]\},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{v\theta} = \frac{1}{r^2}\{ & r(r-2m(v))(-X^\phi\,\partial_\phi\partial_\theta X^v - X^v\,\partial_v\partial_\theta X^v - X^r\,\partial_r\partial_\theta X^v \\
& - X^\theta\ \partial_r^2 X^v) + r^2 X^\phi\,\partial_\phi\partial_\theta X^r + r^2 X^v\,\partial_v\partial_\theta X^r + r^2 X^r\,\partial_r\partial_\theta X^r \\
& + r^4 X^\theta\,\partial_v\partial_\theta X^\theta + r^4 X^\phi\,\partial_v\partial_\phi X^\theta + r^2 X^\theta\,\partial_r^2 X^\theta + r^4 X^v\,\partial_v^2 X^\theta + r^4 X^r\,\partial_v\partial_r X^\theta \\
& + \Big[- r(r-2m(v))(\partial_\theta X^\theta + 3\partial_v X^v) + 3r^2\partial_v X^r + 4rX^v m'(v) - 4X^r m(v)\Big]\partial_\theta X^v \\
& + r\Big[(\partial_\theta X^\theta r + 2r\partial_v X^v + (-r+2m(v))\partial_r X^v + r\partial_r X^r)\partial_\theta X^r \\
& \quad + 3r^3\partial_v X^\theta\partial_\theta X^\theta + ((-r+2m(v))\partial_\phi X^v + 2r^3\sin^2\theta\partial_v X^\phi + r\partial_\phi X^r)\partial_\theta X^\phi \\
& \quad + r^2\big(r\partial_v X^v\partial_v X^\theta + 4X^r\partial_v X^\theta + r(\partial_v X^r\partial_r X^\theta + \partial_v X^\phi\partial_\phi X^\theta)\big)\Big]\},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{v\phi} = \frac{1}{r^2}\{ & -X^\theta\, r(r-2m(v))\,\partial_\theta\partial_\phi X^v - r(r-2m(v))X^\phi\,\partial_\phi^2 X^v \\
& - rX^v(r-2m(v))\,\partial_v\partial_\phi X^v - rX^r(r-2m(v))\,\partial_r\partial_\phi X^v + r^2 X^\theta\,\partial_\theta\partial_\phi X^r \\
& + r^2 X^\phi\,\partial_\phi^2 X^r + r^2 X^v\,\partial_v\partial_\phi X^r + r^2 X^r\,\partial_r\partial_\phi X^r + r^4\sin^2\theta\, X^\theta\,\partial_v\partial_\theta X^\phi \\
& + r^4\sin^2\theta\, X^\phi\,\partial_v\partial_\phi X^\phi + r^4\sin^2\theta\, X^v\,\partial_v\partial_v X^\phi + r^4\sin^2\theta\, X^r\,\partial_v\partial_r X^\phi \\
& + \big[- r(r-2m(v))\partial_\phi X^\phi - 3r(r-2m(v))\partial_v X^v + 3r^2\partial_v X^r + 4rX^v m'(v) \\
& - 4X^r m(v)\big]\partial_\phi X^v + 4r\Big[\frac{1}{2}(\frac{r}{2}\partial_\phi X^\phi + r\partial_v X^v + (\frac{-r}{2} + m(v))\partial_r X^v \\
& + \frac{r}{2}\partial_r X^r)\partial_\phi X^r + \frac{1}{2}((\frac{-r}{2} + m(v))\partial_\theta X^v + r(r^2\partial_v X^\theta + \frac{\partial_\theta X^r}{2})\partial_\phi X^\theta \\
& + r^2\sin\theta\big(\frac{3}{4}\, r\,\partial_v X^\phi\partial_\phi X^\phi\sin\theta + \frac{r}{4}\ \partial_v X^v\partial_v X^\phi\sin\theta + (X^\theta r\cos\theta \\
& + X^r\sin\theta)\partial_v X^\phi + \frac{r}{4}\ \sin\theta(\partial_v X^r\partial_r X^\phi + \partial_v X^\theta\partial_\theta X^\phi)\big)\Big]\},
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{rr} = \frac{1}{r}\{ & 2X^\theta\,\partial_\theta\partial_r X^v\, r + 2X^\phi\,\partial_\phi\partial_r X^v\, r + 2X^v\,\partial_v\partial_r X^v\, r \\
& + 2X^r\,\partial_r^2 X^v\, r + (-2r + 4m(v))\,(\partial_r X^v)^2\ + 2r\,(\partial_v X^v\ + 3\,\partial_r X^r)\,\partial_r X^v \\
& + 2r\,[(\partial_r X^\theta)^2\, r^2 + \partial_r X^\theta\,\partial_\theta X^v + \partial_r X^\phi\,(\partial_r X^\phi\, r^2\sin^2\theta + \partial_\phi X^v)]\},
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{r\theta} = \frac{1}{r}\{ & r\,\partial_\theta\partial_\phi X^v\, X^\phi + r\,\partial_v\partial_\theta X^v\, X^v + r\,\partial_r\partial_\theta X^v\, X^r + r^3\,\partial_r\partial_\theta X^\theta\, X^\theta \\
& + r^3\,\partial_\phi\partial_r X^\theta\, X^\phi + r\,\partial_r^2 X^v\, X^\theta + r^3\,\partial_v\partial_r X^\theta\, X^v + r^3\,\partial_r^2 X^\theta\, X^r \\
& + ((-2r + 4m(v))\,\partial_r X^v + r(\partial_v X^v + 2\partial_r X^r + \partial_\theta X^\theta))\,\partial_\theta X^v \\
& + r((\partial_v X^\theta\, r^2 + 3\partial_\theta X^r)\,\partial_r X^v + r(\partial_r X^r r + 3\partial_\theta X^\theta r + 4X^r)\,\partial_r X^\theta \\
& + (2\partial_r X^\phi\, r^2\sin^2\theta + \partial_\phi X^v)\,\partial_\theta X^\phi + \partial_r X^\phi\,\partial_\phi X^\theta\, r^2)\},
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{r\phi} = {} & \partial_\theta\partial_\phi X^v\, X^\theta + \partial^2_\phi X^v\, X^\phi + \partial_v\partial_\phi X^v\, X^v + \partial_r\partial_\phi X^v\, X^r \\
& + (-2\partial_r X^v + \partial_v X^v + 2\partial_r X^r + \partial_\phi X^\phi)\partial_\phi X^v + 3\partial_\phi X^r \partial_r X^v \\
& + \sin\theta\partial_r X^\phi\, r(4X^\theta r\cos\theta + 4X^r\sin\theta) + (2\partial_r X^\theta r^2 + \partial_\theta X^v)\partial_\phi X^\theta \\
& + \frac{4}{r}m(v)\partial_r X^v \partial_\phi X^v + r^2\sin^2\theta[\partial_r\partial_\theta X^\phi\, X^\theta + \partial_r\partial_\phi X^\phi\, X^\phi \\
& + \partial_v\partial_r X^\phi\, X^v + \partial^2_r X^\phi\, X^r + \partial_v X^\phi\, \partial_r X^v + \partial_r X^\phi(\partial_r X^r + 3\partial_\phi X^\phi) + \partial_r X^\theta\partial_\theta X^\phi],
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{\theta\theta} = {} & 2r^2[\sin^2\theta\,(\partial_\theta X^\phi)^2 + \partial_\theta X^v\partial_v X^\theta + \partial_r X^\theta\partial_\theta X^r + \partial_\theta X^\phi\partial_\phi X^\theta \\
& + X^r\partial_r\partial_\theta X^\theta + 2\,(\partial_\theta X^\theta)^2 + X^\phi\partial_\phi\partial_\theta X^\theta + X^\theta\partial^2_r X^\theta + X^v\partial_v\partial_\theta X^\theta] \\
& + 2r[\,X^\theta\partial_\theta X^r + 4\,X^r\partial_\theta X^\theta + \partial_r X^r\, X^r + X^\phi\partial_\phi X^r + X^v\partial_v X^r] \\
& + \frac{4}{r}m(v)(\partial_\theta X^v)^2 + 2[-(\partial_\theta X^v)^2 + 2\,\partial_\theta X^v\partial_\theta X^r + (X^r)^2],
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{\theta\phi} = r^2\Bigg\{ & X^\theta\,\partial_\phi\partial_\theta X^\theta + \sin^2\theta\,(X^\phi\partial_\phi\partial_\theta X^\phi + X^v\partial_v\partial_\theta X^\phi\,) + X^\phi\,\partial^2_\phi X^\theta \\
& + \sin^2\theta\, X^r\partial_r\partial_\theta X^\phi + X^v\partial_v\partial_\phi X^\theta + X^r\partial_\phi\partial_r X^\theta + \sin^2\theta\, X^\theta\partial^2_r X^\phi\,\} + \frac{1}{r}\{\Big((-2r \\
& + 4m(v))\,\partial_\phi X^v + r(r^2\sin^2\theta\,\partial_v X^\phi + 2\,\partial_\phi X^r)\Big)\partial_\theta X^v + 4r\Big[r\sin\theta\Big(X^\theta r\cos\theta \\
& + \frac{3r}{4}\sin\theta\,\partial_\phi X^\phi + \frac{r}{4}\sin\theta\,\partial_\theta X^\theta + X^r\sin\theta\Big)\partial_\theta X^\phi + \Big(\frac{r^2}{4}\,\partial_v X^\theta + \frac{1}{2}\,\partial_\theta X^r\Big)\partial_\phi X^v \\
& + \frac{r}{4}\Big((r\,\partial_\phi X^\phi + 3r\,\partial_\theta X^\theta + 4X^r)\partial_\phi X^\theta + r(\sin^2\theta\,\partial_r X^\phi\,\partial_\theta X^r + \partial_\phi X^r\,\partial_r X^\theta)\Big)\Big]\Bigg\},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
(\pounds_X \pounds_X g)_{\phi\phi} = 2\Bigg\{ & r^2\sin^2\theta\Big[2(\partial_\phi X^\phi)^2 + \partial_\phi X^r\partial_r X^\phi + \partial_v X^\phi\partial_\phi X^v \\
& + X^r\partial_\phi\partial_r X^\phi + X^\phi\partial^2_\phi X^\phi + X^\theta\partial_\phi\partial_\theta X^\phi + X^v\partial_v\partial_\phi X^\phi \\
& + \partial_\theta X^\phi\partial_\phi X^\theta\Big] + r^2\sin\theta\cos\theta\Big[4X^\theta\partial_\phi X^\phi + X^v\partial_v X^\theta + X^r\partial_r X^\theta \\
& + X^\phi\partial_\phi X^\theta + X^\theta\partial_\theta X^\theta\Big] - (\partial_\phi X^v)^2 + 2\partial_\phi X^v\partial_\phi X^r + \sin^2\theta\,(X^r)^2 \\
& + r^2(\partial_\phi X^\theta)^2 + r\sin^2\theta\Big[4(\partial_\phi X^\phi)X^r + X^r\partial_r X^r + X^\phi\partial_\phi X^r + X^v\partial_v X^r \\
& + \partial_\theta X^r\, X^\theta\Big] + \frac{2}{r}m(v)(\partial_\phi X^v)^2 + (2\cos^2\theta - 1)\,r^2(X^\theta)^2 + r\sin 2\theta X^r X^\theta\Bigg\}.
\end{aligned}
\tag{3.13}
$$

Finding all 2-Killing vector fields in the Vaidya spacetime requires solving a complex system of partial differential equations. To simplify the problem, we focus on vector fields with only one nonzero component along a coordinate direction. Specifically, we consider vector fields of the form

$$X = X^v\,\partial_v + X^r\,\partial_r + X^\theta\,\partial_\theta + X^\phi\,\partial_\phi,$$

and examine the cases where only one component $X^\mu(v, r, \theta, \phi)$ is nonzero. This gives four separate classes:

$$X = X^v\,\partial_v, \quad X = X^r\,\partial_r, \quad X = X^\theta\,\partial_\theta, \quad X = X^\phi\,\partial_\phi.$$

For each case, we substitute the vector field into the 2-Killing equations (3.2) to obtain a simpler system that can be solved independently. This approach allows us to find explicit forms of 2-Killing vector fields and study their properties in the Vaidya spacetime.

Class I: Assume the vector field has only the v-component nonzero:

$$X = X^v(v, r, \theta, \phi)\,\partial_v.$$

In this case, the 2-Killing equations (3.4)–(3.13) reduce to a simpler system of coupled partial differential equations for $X^v(v, r, \theta, \phi)$:

$$(2m(v) - r)\left(X^v\,\partial_v^2 X^v + 2(\partial_v X^v)^2\right) + X^v\left(5m'(v)\,\partial_v X^v + m''(v) X^v\right) = 0, \tag{3.14}$$

$$\begin{aligned}&(2m(v) - r)\left(X^v\,\partial_v\partial_r X^v + 3\,\partial_r X^v\,\partial_v X^v\right) + r(\partial_v X^v)^2\\ &+ X^v\left(r\,\partial_v^2 X^v + 4m'(v)\,\partial_r X^v\right) = 0,\end{aligned} \tag{3.15}$$

$$(2m(v) - r)\left(X^v\,\partial_v\partial_\theta X^v + 3\,\partial_v X^v\,\partial_\theta X^v\right) + 4m'(v)\,X^v\,\partial_\theta X^v = 0, \tag{3.16}$$

$$(2m(v) - r)\left(X^v\,\partial_v\partial_\phi X^v + 3\,\partial_v X^v\,\partial_\phi X^v\right) + 4m'(v)\,X^v\,\partial_\phi X^v = 0, \tag{3.17}$$

$$2r\left(X^v\,\partial_v\partial_r X^v + \partial_r X^v\,\partial_v X^v\right) + (-2r + 4m(v))\,(\partial_r X^v)^2 = 0, \tag{3.18}$$

$$rX^v\,\partial_v\partial_\theta X^v + \partial_\theta X^v\left[(-2r + 4m(v))\,\partial_r X^v + r\,\partial_v X^v\right] = 0, \tag{3.19}$$

$$rX^v\,\partial_v\partial_\phi X^v + \partial_\phi X^v\left[(-2r + 4m(v))\,\partial_r X^v + r\,\partial_v X^v\right] = 0, \tag{3.20}$$

$$(\partial_\theta X^v)^2(r - 2m(v)) = (\partial_\phi X^v)^2(r - 2m(v)) = 0, \tag{3.21}$$

$$(r - 2m(v))\,\partial_\theta X^v\,\partial_\phi X^v = 0. \tag{3.22}$$

Using (3.21), we find that X^v is independent of the angular coordinates:

$$X^v = \mathcal{A}(v, r).$$

Substituting this form into (3.14)–(3.22), the system reduces to the following coupled partial differential equations for $\mathcal{A}(v,r)$:

$$(2m(v)-r)\big(\mathcal{A}(v,r)\,\partial_v^2\mathcal{A}(v,r)+2(\partial_v\mathcal{A}(v,r))^2\big) \tag{3.23}$$

$$+\,\mathcal{A}(v,r)(5m'(v)\,\partial_v\mathcal{A}(v,r)+m''(v)\mathcal{A}(v,r))=0,$$

$$(2m(v)-r)(\mathcal{A}(v,r)\,\partial_v\partial_r\mathcal{A}(v,r)+3\,\partial_r\mathcal{A}(v,r)\,\partial_v\mathcal{A}(v,r)) \tag{3.24}$$

$$+\,r(\partial_v\mathcal{A}(v,r))^2+\mathcal{A}(v,r)(r\,\partial_v^2\mathcal{A}(v,r)+4m'(v)\,\partial_r\mathcal{A}(v,r))=0,$$

$$r(\mathcal{A}(v,r)\,\partial_v\partial_r\mathcal{A}(v,r)+\partial_r\mathcal{A}(v,r)\,\partial_v\mathcal{A}(v,r)) \tag{3.25}$$

$$+\,(-r+2m(v))(\partial_r\mathcal{A}(v,r))^2=0.$$

To simplify the analysis of this nonlinear system, we consider different possible functional forms for $\mathcal{A}(v,r)$. Each class corresponds to a specific physical or geometrical assumption about the symmetry properties of the spacetime.

Case A: $\mathcal{A}(v,r):=F(v)$. In this case, the vector field depends only on the advanced time coordinate v, representing a purely time-dependent symmetry along the null direction. Substituting $F(v)$ into the system (3.23)–(3.25) yields the following ordinary differential equations:

$$F(v)\Big(F(v)m''(v)+5F'(v)m'(v)+2F''(v)m(v)-F''(v)r\Big) \tag{3.26}$$

$$+\,2(F'(v))^2(2m(v)-r)=0,$$

$$(F'(v))^2+F(v)F''(v)=0. \tag{3.27}$$

Differentiating (3.26) with respect to r yields

$$F(v)F''(v)+2(F'(v))^2=0.$$

Comparing the last equation with (3.27), we find $F'(v)=0$. Hence $F(v)=c_1$, where c_1 is a constant. Substituting this result into (3.26), we obtain $c_1^2m''(v)=0$. Thus the mass function is linear:

$$m(v)=c_2v+c_3,$$

where c_2 and c_3 are constants.

Case B: $\mathcal{A}(v,r):=G(r)$. Here, the vector field depends only on the radial coordinate r. Substituting $G(r)$ into the reduced system (3.23)–(3.25) yields the following relations:

$$G(r)^2\,m''(v)=0,\qquad (G'(r))^2\,(-r+2m(v))=0,\qquad 4G(r)G'(r)\,m'(v)=0.$$

From these equations, we obtain the nontrivial admissible solution

$$G(r)=c_1,\qquad m(v)=c_2v+c_3,$$

where c_1, c_2, and c_3 are constants.

Case C: $\mathcal{A}(v,r) := F(v)G(r)$. In this case, system (3.23)–(3.25) yields the following relations:

$$\frac{G(r)}{r}\Big(4F(v)^2 m'(v)G'(r) + F(v)G(r)F''(v)\,r + 8F(v)m(v)F'(v)G'(r) \\ - 4F(v)F'(v)G'(r)\,r + G(r)(F'(v))^2 r\Big) = 0, \tag{3.28}$$

$$2G(r)^2\Big(F(v)^2 m''(v) + 2F(v)F''(v)m(v) - F(v)F''(v)\,r \\ + 5F(v)F'(v)m'(v) + 4m(v)(F'(v))^2 - 2(F'(v))^2 r\Big) = 0, \tag{3.29}$$

$$2F(v)G'(r)\Big(2F(v)G'(r)m(v) - F(v)G'(r)\,r + 2F'(v)G(r)\,r\Big) = 0. \tag{3.30}$$

We now analyze these equations for nontrivial solutions $F \not\equiv 0$, $G \not\equiv 0$. (3.30) gives

$$G'(r)\Big(2G'(r)m(v) - G'(r)\,r + 2\frac{F'(v)}{F(v)}G(r)\,r\Big) = 0.$$

This implies $G'(r) = 0$, and hence we have

$$G(r) = c_1,$$

where c_1 is constant. Applying the above equation, (3.28) and (3.29) reduce to

$$\begin{cases} F(v)F''(v) + (F'(v))^2 = 0, \\ F(v)F''(v) + 2(F'(v))^2 = 0, \end{cases}$$

which gives us $F'(v) = 0$. Thus $F(v) = c_2$, where c_2 is constant. So, from (3.29), it follows

$$2c_1^2 c_2^2 m''(v) = 0,$$

hence mass function is linear, i.e.,

$$m(v) = c_3 v + c_4.$$

Theorem 3.1. *In the Vaidya spacetime, let*

$$X = X^v(v,r,\theta,\phi)\,\partial_v.$$

Then the 2-Killing equations (3.23)–(3.25) admit the nontrivial solution

$$X^v = c_1, \qquad m(v) = c_2 v + c_3,$$

where c_1, c_2, c_3 are constants.

Class II: We now assume that only the r-component of the vector field is nonzero:

$$X = X^r(v,r,\theta,\phi)\,\partial_r.$$

In this case, the 2-Killing equations (3.4)–(3.13) reduce to the following system of partial differential equations:

$$\frac{2m(v)}{r^2}\, X^r\, \partial_r X^r - \frac{4m(v)}{r^3}(X^r)^2 - 2X^r\, \partial_v \partial_r X^r - 2\,(\partial_v X^r)(\partial_r X^r) = 0, \tag{3.31}$$

$$X^r\, \partial_r^2 X^r + (\partial_r X^r)^2 = 0, \tag{3.32}$$

$$X^r\, \partial_r \partial_\theta X^r + (\partial_\theta X^r)(\partial_r X^r) = 0, \tag{3.33}$$

$$X^r\, \partial_r \partial_\phi X^r + (\partial_\phi X^r)(\partial_r X^r) = 0, \tag{3.34}$$

$$r\, X^r\, \partial_r X^r + X^r = 0. \tag{3.35}$$

From (3.35), it immediately follows that either $X^r = 0$ or $\partial_r X^r = -\frac{1}{r}$. Substituting these possibilities into the remaining equations shows that only the trivial solution is admissible.

Theorem 3.2. *In the Vaidya spacetime, for the vector field $X = X^r(v, r, \theta, \phi)\, \partial_r$, the 2-Killing equations (3.31)–(3.35) admit only the trivial solution:*

$$X^r(v, r, \theta, \phi) = 0, \qquad m(v) = m(v).$$

Hence, no nontrivial 2-Killing vector field exists in this case.

Class III: Assume that $X = X^\theta(v, r, \theta, \phi)\, \partial_\theta$. In this case, the 2-Killing equations reduce to

$$\partial_v X^\theta = \partial_r X^\theta = 0, \tag{3.36}$$

$$(\partial_v X^\theta)(\partial_r X^\theta) = 0, \tag{3.37}$$

$$X^\theta\, \partial_v \partial_\theta X^\theta + 3(\partial_v X^\theta)(\partial_\theta X^\theta) = 0, \tag{3.38}$$

$$(\partial_v X^\theta)(\partial_\phi X^\theta) = (\partial_r X^\theta)(\partial_\phi X^\theta) = 0, \tag{3.39}$$

$$X^\theta\, \partial_r \partial_\theta X^\theta + 3(\partial_r X^\theta)(\partial_\theta X^\theta) = 0, \tag{3.40}$$

$$X^\theta\, \partial_r^2 X^\theta + 2(\partial_\theta X^\theta)^2 = 0, \tag{3.41}$$

$$X^\theta\, \partial_\phi \partial_\theta X^\theta + 3(\partial_\phi X^\theta)(\partial_\theta X^\theta) = 0, \tag{3.42}$$

$$(\partial_\phi X^\theta)^2 + (2\cos^2\theta - 1)(X^\theta)^2 + X^\theta(\partial_\theta X^\theta)\sin\theta\cos\theta = 0. \tag{3.43}$$

From (3.36), we have that X^θ is independent of v and r. Substituting this into equations (3.38)–(3.43)) shows that all terms vanish only if $X^\theta = 0$.

Theorem 3.3. *For the vector field $X = X^\theta(v, r, \theta, \phi)\, \partial_\theta$, the 2-Killing equations (3.36)–(3.43) admit only the trivial solution:*

$$X^\theta(v, r, \theta, \phi) = 0.$$

No nontrivial 2-Killing vector field exists in this case.

Class IV: Let

$$X = X^{\phi}(v, r, \theta, \phi)\, \partial_{\phi}.$$

Then, the 2-Killing equations reduce to

$$\partial_v X^{\phi} = \partial_r X^{\phi} = \partial_{\theta} X^{\phi} = 0, \tag{3.44}$$

$$(\partial_v X^{\phi})(\partial_r X^{\phi}) = (\partial_v X^{\phi})(\partial_{\theta} X^{\phi}) = 0, \tag{3.45}$$

$$X^{\phi}\, \partial_v \partial_{\phi} X^{\phi} + 3(\partial_v X^{\phi})(\partial_{\phi} X^{\phi}) = 0, \tag{3.46}$$

$$(\partial_r X^{\phi})(\partial_{\theta} X^{\phi}) = 0, \tag{3.47}$$

$$X^{\phi}\, \partial_r \partial_{\phi} X^{\phi} + 3(\partial_r X^{\phi})(\partial_{\phi} X^{\phi}) = 0, \tag{3.48}$$

$$X^{\phi}\, \partial_{\phi} \partial_{\theta} X^{\phi} + 3(\partial_{\phi} X^{\phi})(\partial_{\theta} X^{\phi}) = 0, \tag{3.49}$$

$$X^{\phi}\, \partial_{\phi}^2 X^{\phi} + 2(\partial_{\phi} X^{\phi})^2 = 0. \tag{3.50}$$

Applying (3.44), it immediately follows that X^{ϕ} is independent of v, r, and θ, i.e., $X^{\phi} = X^{\phi}(\phi)$. Hence (3.50) reduces to a ordinary differential equation, which has the following solution:

$$X^{\phi}(\phi) = \big(3c_1\phi + 3c_2\big)^{1/3},$$

where c_1 and c_2 are constants of integration.

Theorem 3.4. *For the vector field $X = X^{\phi}(v, r, \theta, \phi)\, \partial_{\phi}$, the 2-Killing equations (3.44)–(3.50) admit the solution*

$$X^{\phi}(\phi) = \big(3c_1\phi + 3c_2\big)^{1/3},$$

where c_1 and c_2 are constants of integration.

4. **Conclusion**

In this paper, we investigated Killing and 2-Killing vector fields in the Vaidya spacetime. For Killing symmetries, we showed that when the mass function depends on the null coordinate ($m'(v) \neq 0$), the spacetime admits only the three rotational Killing vectors generating the $SO(3)$ algebra. In the static case ($m'(v) = 0$), an additional timelike Killing vector appears, corresponding to the Schwarzschild spacetime. For 2-Killing vector fields, we restricted to vector fields with a single non-vanishing component. Under this assumption, nontrivial solutions arise only in the v- and ϕ-directions, while the r- and θ-components admit only trivial solutions. Future work may focus on solving the full 2-Killing system without restricting to single-component vector fields. It would also be of interest to explore the conserved quantities associated with these generalized symmetries.

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