Journal of Finsler Geometry and its Applications
Vol. 7, No. 2 (2026), pp 68-81
https://doi.org/10.22098/jfga.2026.19659.1208

On pressure of dynamical systems induced by probability bi-sequences

Mehdi Rahimi[a*] and Asghar Ghodrati[b]

[a,b]Department of Mathematics, Faculty of Science,
University of Qom, Qom, Iran.

E-mail: m10.rahimi@gmail.com
E-mail: asghar.ghodrati@yahoo.com

Abstract. In this paper, we introduce a new family of metrics on topological dynamical systems, induced by probability bi-sequences, which generalizes both the classical Bowen metric and the mean metric. Using these metrics, we define measure-theoretic and topological pressure and show that these quantities coincide with the classical topological pressure and the sum of measure-theoretic entropy and integral of the potential function, respectively. The results hold under the following mild condition on the probability bi-sequence $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$:

$$\limsup_{n\to\infty} \frac{(\gamma_n^*)^{-1}}{n} < \infty,$$

where $\gamma_n^* := \min_{0\leq i\leq n} \gamma_{i,n}$. This condition is satisfied for a broad class of bi-sequences, including the uniform weights $\gamma_{m,n} = 1/(n+1)$ that recover the mean metric case. As an application, we extend the pressure versions of Katok's entropy formula to this more general setting. Our work unifies and generalizes several previous results on pressure in mean metrics.

Keywords: Katok's entropy formula, measure-theoretic pressure, probability bi-sequence.

---

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53B40, 53C30

1. **Introduction**

By a topological dynamical system, abbreviated by TDS, we mean a couple (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map. Note that X is naturally equipped by the Borel σ- algebra. The measure-theoretic and topological pressure [18] are generalizations of the measure-theoretic and topological entropy [1, 10, 15, 20] for dynamical systems. They are connected together via the well-known variational principle [22]. They play an important role in statistical mechanics and thermodynamic formalism in ergodic theory and dynamical systems [19]. Other mathematical approaches are also applied in modeling thermodynamic systems [16, 17]. The topological pressure also plays an important role in dimension theory in dynamical systems [12]. The Bowen metric d_n, given by

$$d_n(x,y) := \max\{d(T^i(x), T^i(y)) : \ i = 0, 1, \cdots, n-1\}$$

and the corresponding Bowen dynamical ball

$$B_n(x,\epsilon) = \{y \in X : \quad d_n(x,y) < \epsilon\}$$

are applied in the definition of topological and measure-theoretic pressure. They are also used by Katok to construct the Katok's entropy formula [9]. Other versions of Katok's formula are also established in the setting of random dynamical systems [24] and amenable group actions [23].

There are pressure versions of the Katok's entropy formula in different settings [3, 5]. One may find local approaches to the measure-theoretic pressure of dynamical systems in [2, 14].

The above mentioned studies were formulated in case of Bowen metric. Replacing the Bowen metric by the mean metric, defined by

$$\tilde{d}_n(x,y) := \frac{1}{n}\sum_{i=0}^{n-1} d(T^i(x), T^i(y)),$$

and the Bowen ball by the mean Bowen ball

$$B_{\tilde{d}_n}(x,\epsilon) := \{y \in X \ : \ \tilde{d}_n(x,y) < \epsilon\},$$

similar studies are carried out on topological entropy of the whole system [4], Katok's entropy formula [7], conditional entropy [6] and Brin-Katok formula for systems with mistake [11].

Following [8], we extend the concept of mean metric to a more general case. We introduce probability bi-sequences and corresponding induced metrics, which generalize the mean metric, and will introduce topological and measure-theoretic pressure induced by these general metrics. We prove that, independent of the choice of the probability bi-sequence, the new concepts are equivalent to the classical topological and measure-theoretic pressure. Our results generalize the results in [8].

2. **Preliminary Facts**

In this section, we review some facts, which are related to the paper. Let (X,T) be a TDS and $f \in C(X)$. The topological pressure of f with respect to T, denoted by $P_{\text{top}}(T,f)$, is defined by Ruelle [18]. The definition of topological pressure is given, using open covers, spanning sets and separated sets. The Bowen metric

$$d_n(x,y) := \max\{d(T^i(x),T^i(y)): \ i=0,1,\cdots,n-1\}$$

plays a vital role in the definition of topological pressure. In [8], the Bowen metric is replaced by the mean metric

$$\tilde{d}_n(x,y) := \frac{1}{n}\sum_{i=0}^{n-1} d(T^i(x),T^i(y))$$

to define measure-theoretic and topological pressure.

Write $(S_nf)(x) = \sum_{i=0}^{n-1} f(T^i(x))$. Given $n \in \mathbb{N}$, $\epsilon > 0$, $0 < \delta < 1$ and $\mu \in E(X,T)$, a set S is an $(n,\epsilon,\delta,\mu)'$-spanning set if $\mu(\bigcup_{x\in S} B_{\tilde{d}_n}(x,\epsilon)) > 1-\delta$. Huang and Wang [8] defined the following quantity:

$$\tilde{P}_\mu(T,f) := \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log \tilde{P}_\mu(T,f,n,\epsilon,\delta)}{n},$$

where

$$\tilde{P}_\mu(T,f,n,\epsilon,\delta) := \inf\Big\{\sum_{x\in S}\exp(\sup_{y\in B_{\tilde{d}_n}(x,\epsilon)}(S_nf)(y)) : S \text{ is a } (n,\epsilon,\delta,\mu)' \\ -\text{spanning set}\Big\}.$$

Also, a set $F \subseteq X$ is an $(n,\epsilon)'$-spanning set for X with respect to T, if for any $x \in X$, there exists $y \in F$ with $\tilde{d}_n(x,y) \le \epsilon$. Similarly, a set $E \subseteq X$ is an $(n,\epsilon)'$-separated set for X with respect to T, if for any $x,y \in X$, $x \neq y$ implies $\tilde{d}_n(x,y) > \epsilon$. The following quantities are also defined in [8]:

$$\tilde{P}_{top}(T,f) := \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\tilde{P}(T,f,n,\epsilon)$$

and

$$\tilde{P}^*_{top}(T,f) := \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\tilde{P}^*(T,f,n,\epsilon),$$

where

$$\tilde{P}(T,f,n,\epsilon) := \sup\{\sum_{x\in E} e^{(S_nf)(x)} \ : \ S \text{ is an } \ (n,\epsilon)' - \text{separated set for } X\}$$

and

$$\tilde{P}^*(T,f,n,\epsilon) := \inf\Big\{\sum_{x\in F}\exp(\sup_{y\in B_{\tilde{d}_n}(x,\epsilon)}(S_nf)(y)) : F \text{ is an } \ (n,\epsilon)' - \\ \text{spanning set for } X\Big\}.$$

The following theorems are stated and proved in [8].

Theorem 2.1. *Let (X,T) be a TDS, and $f \in C(X)$. For each $\mu \in E(X,T)$ and $0 < \delta < 1$, we have*

$$\tilde{P}_\mu(T,f) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{\log \tilde{P}_\mu(T,f,n,\epsilon,\delta)}{n} = h_\mu(T) + \int_X f d\mu.$$

Theorem 2.2. *Let (X,T) be a TDS and $f \in C(X)$. Then*

$$\tilde{P}_{top}(T,f) = \tilde{P}^*_{top}(T,f) = P_{top}(T,f).$$

In the next section, we will extend the previous results for more general metrics, covering mean metric.

3. **Probability bi-sequences and pressure.**

In this section, we first introduce the concept of probability bi-sequence and the corresponding metric. Then we use them to define a new version of topological and measure-theoretic pressure which extend the concepts in [8].

In the rest of the paper, (X,T) is a TDS, $C(X)$ is the Banach space of continuous functions on X, and $E(X,T)$ is the set of ergodic measures on X.

Definition 3.1. *A sequence $\Gamma = \{\gamma_{m,n}\}_{m,n \geq 0}$ is said to be a probability bi-sequence if $\sum_{i=0}^{n} \gamma_{i,n} = 1$ for all $n \geq 1$.*

Example 3.2. *$\Gamma_0 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} = \frac{1}{n+1}$ is a probability bi-sequence. Also, given every sequence of positive numbers $\{a_n\}_{n \geq 0}$, if $S_n := \sum_{j=0}^{n} a_j$, then $\Gamma_1 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} := \frac{a_m}{S_n}$ is a probability bi-sequence.*

Definition 3.3. *Let (X,T) be a TDS and $\Gamma = \{\gamma_{m,n}\}_{m,n \geqslant 0}$ be a probability bi-sequence. Given $n \in \mathbb{N}$ and $x, y \in X$, set:*

$$d_{n,\Gamma}(x,y) := \sum_{i=0}^{n-1} \gamma_{i,n-1} d(T^i(x), T^i(y)).$$

Obviously, $d_{n,\Gamma}$ is a metric on X. For $\epsilon > 0$, $x \in X$ and $n \in \mathbb{N}$, the (Γ,n)-ball centered at x with radius ϵ and length n is defined by

$$B_{n,\Gamma}(x,\epsilon) := \big\{ y \in X : d_{n,\Gamma}(y,x) < \epsilon \big\}.$$

Note that, if $\Gamma_0 = \{\gamma_{m,n}\}_{m,n\geq 0}$ with $\gamma_{m,n} = \frac{1}{n+1}$ then $d_{n,\Gamma_0} = \tilde{d}_n$ is the mean metric.

Definition 3.4. *Let (X,T) be a TDS, $\epsilon > 0$, $0 < \delta < 1$, $\mu \in E(X,T)$ and $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence. A set $S \subseteq X$ is an $(\Gamma, n, \epsilon, \delta, \mu)$-spanning set if $\mu(\bigcup_{x \in S} B_{n,\Gamma}(x,\epsilon)) > 1 - \delta$.*

Let also,

$$P_{\mu,\Gamma}(T,f,n,\epsilon,\delta) := \inf\Big\{ \sum_{x\in S} \exp(\sup_{y\in B_{n,\Gamma}(x,\epsilon)} (S_n f)(y)) : S \ \textit{is a} \ (\Gamma,n,\epsilon,\delta,\mu) \\ -\textit{spanning set for } X \Big\}.$$

and

$$P_{\mu,\Gamma}(T,f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log P_{\mu,\Gamma}(T,f,n,\epsilon,\delta)}{n}.$$

Remark 3.5. *We will show that the previous definition does not depend on* δ.

Definition 3.6. *A set* $F \subseteq X$ *is an* (Γ,n,ϵ)*-spanning set for* X *with respect to* T *if for any* $x\in X$*, there exists* $y\in F$ *such that* $d_{n,\Gamma}(x,y)\leq\epsilon$*. Now, set*

$$P^*_{\Gamma}(T,f,n,\epsilon) := \inf\Big\{ \sum_{x\in F} \exp(\sup_{y\in B_{n,\Gamma}(x,\epsilon)} (S_n f)(y)) : F \ \textit{is a} \,(\Gamma,n,\epsilon) \\ -\textit{spanning set for}\, X \Big\}.$$

Definition 3.7. *A set* $E\subseteq X$ *is an* (Γ,n,ϵ)*-separated set of* X *with respect to* T *if for* $x,y\in X$*,* $x\neq y$ *implies* $d_{n,\Gamma}(x,y)>\epsilon$*. Set*

$$P_{\Gamma}(T,f,n,\epsilon) := \sup\Big\{ \sum_{x\in E} e^{(S_n f)(x)} \ : \ E \ \textit{is an} \ (\Gamma,n,\epsilon) \\ -\textit{separated set for } X \Big\}.$$

Finally, define

$$P^*_{\Gamma}(T,f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P^*_{\Gamma}(T,f,n,\epsilon),$$

$$P_{\Gamma}(T,f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_{\Gamma}(T,f,n,\epsilon).$$

As in [8], we need to use mistake dynamical ball $B_{n,\Gamma}(g;x,\epsilon)$ where g is a mistake function. Recall that a function $g:\mathbb{N}\times(0,1]\to\mathbb{R}$ is a mistake function if $\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{g(n,\epsilon)}{n}=0$. We use the mistake function $g(n,\epsilon)=n\epsilon$, however, the definition of mistake function may be different in other texts [8, 13, 21].

Let $\Lambda_n := \{0,1,2,\cdots,n-1\}$. Given any subset $\Lambda\subseteq\Lambda_n$, set $d_{\Lambda}(x,y) := \max\{d(T^i(x),T^i(y)) : i\in\Lambda\}$ and $B_{\Lambda}(x,\epsilon) := \{y\in X : d_{\Lambda}(x,y)<\epsilon\}$.

Definition 3.8. *Let* $0<\epsilon\leq 1$*,* $n\in\mathbb{N}$ *and* $\Gamma=\{\gamma_{m,n}\}_{m,n\geq 0}$ *be a probability bi-sequence. The* Γ*- mistake ball centred at* x *with radious* ϵ *and length* n *is* *defined as follows:*

$$\begin{aligned} B_{n,\Gamma}(g;x,\epsilon) &:= \Big\{ y \in X : \sum_{i=0}^{n-1} \gamma_{i,n-1}\, \chi_{B(T^i(x),\epsilon)}(T^i(y)) > 1 - \frac{g(n,\epsilon)}{n} \Big\} \\ &= \Big\{ y \in X : \sum_{i=0}^{n-1} \gamma_{i,n-1}\, \chi_{B(T^i(x),\epsilon)}(T^i(y)) > 1 - \epsilon \Big\} \\ &= \bigcup_{\Lambda \in I_\Gamma(g;n,\epsilon)} B_\Lambda(x,\epsilon), \end{aligned}$$

where

$$I_\Gamma(g;n,\epsilon) := \Big\{ \Lambda \subseteq \Lambda_n : \sum_{i \in \Lambda} \gamma_{i,n-1} > 1 - \epsilon \Big\}.$$

The following lemma is an analogous version of Lemma 2.1 in [8].

Lemma 3.9. *For any $x \in X$, $n \in \mathbb{N}$ and $\epsilon > 0$, we have:*

$$B_n(x,\epsilon) \subseteq B_{n,\Gamma}(x,\epsilon) \subseteq B_{n,\Gamma}(g;x,\sqrt{\epsilon}).$$

Proof. The inclusion $B_n(x,\epsilon) \subseteq B_{n,\Gamma}(x,\epsilon)$ is obvious.
Now, set

$$I_1 := \{ i \in \Lambda_n \ : d(T^i(x), T^i(y)) < \sqrt{\epsilon} \}$$

and

$$I_2 := \{ i \in \Lambda_n \ : d(T^i(x), T^i(y)) \geqslant \sqrt{\epsilon} \},$$

where $\Lambda_n := \{0,1,2,\cdots n-1\}$. Let $y \in B_{n,\Gamma}(x,\epsilon)$. We have

$$\begin{aligned} \epsilon > \sum_{i=0}^{n-1} \gamma_{i,n-1} d(T^i(x),T^i(y)) &= \sum_{i \in I_1} \gamma_{i,n-1} d(T^i(x),T^i(y)) \\ &+ \sum_{i \in I_2} \gamma_{i,n-1} d(T^i(x),T^i(y)) \\ &\geqslant \sum_{i \in I_2} \gamma_{i,n-1} d(T^i(x),T^i(y)) \geqslant \sqrt{\epsilon} \sum_{i \in I_2} \gamma_{i,n-1}, \end{aligned}$$

so,

$$\sum_{i \in I_2} \gamma_{i,n-1} < \sqrt{\epsilon}. \tag{3.1}$$

Therefore,

$$\sum_{i \in I_1} \gamma_{i,n-1} > 1 - \sqrt{\epsilon}. \tag{3.2}$$

Note that $i \in I_1$ if and only if $\chi_{B(T^i(x),\sqrt{\epsilon})}(T^i(y)) = 1$ and $i \in I_2$ if and only if $\chi_{B(T^i(x),\sqrt{\epsilon})}(T^i(y)) = 0$. So, (3.2) implies that:

$$\begin{aligned}\sum_{i=0}^{n-1} \gamma_{i,n-1}\chi_{B(T^i(x),\sqrt{\epsilon})}(T^i(y)) &= \sum_{i\in I_1} \gamma_{i,n-1}\chi_{B(T^i(x),\sqrt{\epsilon})}(T^i(y)) \\ &+ \sum_{i\in I_2} \gamma_{i,n-1}\chi_{B(T^i(x),\sqrt{\epsilon})}(T^i(y)) \\ &= \sum_{i\in I_1} \gamma_{i,n-1} > 1-\sqrt{\epsilon}.\end{aligned}$$

This implies that $y \in B_{n,\Gamma}(g; x, \sqrt{\epsilon})$. Therefore, we have

$$B_{n,\Gamma}(x,\epsilon) \subseteq B_{n,\Gamma}(g; x, \sqrt{\epsilon}). \square$$

Definition 3.10. *Let (X, T) be a TDS, $f \in C(X)$, $n \in \mathbb{N}$, $\epsilon > 0$ and $Z \subseteq X$. A set $F \subseteq Z$ is an (Γ, g, n, ϵ)-spanning set for Z with respect to T if for any $x \in Z$, there exists $y \in F$ and $\Lambda \in I_\Gamma(g, n, \epsilon)$ such that $d_\Lambda(x, y) \leq \epsilon$.*

A set $E \subseteq Z$ is an (Γ, g, n, ϵ)-separated set for Z with respect to T if for any $x, y \in E$, $x \neq y$ and any $\Lambda \in I_\Lambda(g; n, \epsilon)$ implies that $d_\Lambda(x, y) > \epsilon$.

Definition 3.11. *Let (X, T) be a TDS, $n \in \mathbb{N}$, $\epsilon > 0$ and $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence. Given $0 < \delta < 1$ and $\mu \in E(X, T)$, a set S is a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set if $\mu(\bigcup_{x\in S} B_{n,\Gamma}(g; x, \epsilon)) > 1 - \delta$. Define,*

$$P_{\mu,\Gamma}(g; T, f, n, \epsilon, \delta) := \inf\Big\{ \textstyle\sum_{x\in S} \exp(\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)}(S_n f)(y)) : \\ S \text{ is a } (\Gamma, g, n, \epsilon, \delta, \mu) - \text{spanning set for } X\Big\},$$

and set

$$P_{\mu,\Gamma}(g; T, f) := \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{\log P_{\mu,\Gamma}(g; T, f, n, \epsilon, \delta)}{n}.$$

The following lemma is an analogous version of Lemma 3.1 in [8].

Lemma 3.12. *Let (X, T) be a TDS, $f \in C(X)$ and $\Gamma = \{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence. Then, for any $\theta > 0$ there exists $0 < \epsilon_0 < 1$ such that for any $0 < \epsilon < \epsilon_0$ and $n \in \mathbb{N}$,*

$$\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)} (S_n f)(y) \leq (S_n f)(x) + n\theta + \epsilon||f||(\gamma^*_{n-1})^{-1}$$

where

$$||f|| := \max_{x\in X} |f(x)| \;\; and \;\; \gamma^*_{n-1} := \min_{0\leq i\leq n-1} \gamma_{i,n-1}.$$

Proof. Let $\theta > 0$ be given. Choose $0 < \epsilon < \epsilon_0$ due to uniform continuity of f such that $d(x, y) < \epsilon$ implies $|f(x) - f(y)| < \theta$. Let $y \in B_{n,\Gamma}(g; x, \epsilon)$ be given. There exists $\Lambda \in I_\Gamma(g; n, \epsilon)$ such that $y \in B_\Lambda(x, \epsilon)$. Note that, since $\Lambda \in I_\Gamma(g; n, \epsilon)$, if we set $\gamma^*_{n-1} := \min_{0\le i\le n-1} \gamma_{i,n-1}$, then

$$\gamma^*_{n-1}|\Lambda^c| \le \sum_{i\in\Lambda^c} \gamma_{i,n-1} \le \epsilon$$

or equivalently, $|\Lambda^c| \le \epsilon(\gamma^*_{n-1})^{-1}$. Therefore,

$$\begin{aligned}(S_n f)(y) &= \sum_{i\in\Lambda} f(T^i(y)) + \sum_{i\in\Lambda^c} f(T^i(y))\\ &\le \sum_{i=0}^{n-1} f(T^i(x)) + n\theta + ||f|||\Lambda^c|\\ &\le (S_n f)(x) + n\theta + \epsilon||f||(\gamma^*_{n-1})^{-1}. \quad \square\end{aligned}$$

Before we proceed, we review the following lemmas.

Lemma 3.13. *(see Theorem 2.1 in [5]) Let (X, T) be a TDS and $f \in C(X)$. For each $\mu \in E(X, T)$ and $0 < \delta < 1$, set*

$$P_\mu(T, f, n, \epsilon, \delta) := \inf\{\sum_{x\in S} e^{(S_n f)(x)} : S \textit{ is an } (n, \epsilon, \delta, \mu) - \textit{spanning set}\}.$$

Then

$$\lim_{\epsilon\to 0} \liminf_{n\to+\infty} \frac{\log P_\mu(T, f, n, \epsilon, \delta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to+\infty} \frac{\log P_\mu(T, f, n, \epsilon, \delta)}{n} = h_\mu(T) + \int_X f d\mu.$$

Lemma 3.14. *([12], Appendix II, Lemma 1) For each $\eta > 0$, there exists $0 < \gamma \le \eta$, a finite partition $\xi = \{C_1, C_2, \cdots, C_m\}$ and a finite open cover $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ of X, where $k \ge m$, such that the following properties hold:*

(1) $\mathrm{diam}(U_i) \le \eta$ *and* $\mathrm{diam}(C_j) \le \eta$, $1 \le i \le k$, $1 \le j \le m$.
(2) $\overline{U_i} \subset C_i$, $1 \le i \le m$ *where $\overline{U_i}$ denotes the closure of the set U_i.*
(3) $\mu(C_i \setminus U_i) \le \gamma$, $1 \le i \le m$ *and* $\mu(\bigcup_{i=m+1}^k U_i) \le \gamma$.
(4) $2\gamma \log m \le \eta$.

Now, we are ready to state and prove our main theorem.

Theorem 3.15. *Let (X, T) be a TDS, $f \in C(X)$ and $\Gamma = \{\gamma_{m,n}\}_{m,n\ge 0}$ be a probability bi-sequence such that $K := \limsup_{n\to\infty} \frac{(\gamma^*_n)^{-1}}{n} < \infty$, where $\gamma^*_n := \min_{0\le i\le n} \gamma_{i,n}$. For every $\mu \in E(X, T)$ and $0 < \delta < 1$, we have:*

$$P_{\mu,\Gamma}(T, f) = h_\mu(T) + \int f\, d\mu.$$

Proof. Since $B_n(x, \epsilon) \subseteq B_{n,\Gamma}(x, \epsilon) \subseteq B_{n,\Gamma}(g; x, \sqrt{\epsilon})$, then any $(n, \epsilon, \delta, \mu)$-spanning set is an $(\Gamma, n, \epsilon, \delta, \mu)$-spanning set and any $(\Gamma, n, \epsilon, \delta, \mu)$-spanning set is a $(\Gamma, g, n, \sqrt{\epsilon}, \delta, \mu)$-spanning set. For any $\theta > 0$, choose $0 < \epsilon_0 < 1$ as in Lemma 3.12. Then, for any $0 < \epsilon < \epsilon_0^2$, we have

$$\begin{aligned} P_{\mu,\Gamma}(T, f, n, \epsilon, \delta) &\leq \inf\Big\{\sum_{x\in S}\exp(\sup_{y\in B_{n,\Gamma}(g;x,\sqrt{\epsilon})}(S_n f)(y)) : \\ &\qquad S \text{ is an } (n, \epsilon, \delta, \mu) - \text{spanning set}\Big\} \\ &\leq \inf\Big\{\sum_{x\in S}\exp((S_n f)(x) + n\theta + \sqrt{\epsilon}||f||(\gamma^*_{n-1})^{-1}) : \\ &\qquad S \text{ is an } (n, \epsilon, \delta, \mu) - \text{spanning set}\Big\} \\ &= \exp\{n\theta + \sqrt{\epsilon}||f||(\gamma^*_{n-1})^{-1}\}P_\mu(T, f, n, \epsilon, \delta). \end{aligned}$$

Therefore,

$$\frac{\log P_{\mu,\Gamma}(T, f, n, \epsilon, \delta)}{n} \leq \theta + \sqrt{\epsilon}||f||\frac{(\gamma^*_{n-1})^{-1}}{n} + \frac{\log P_\mu(T, f, n, \epsilon, \delta)}{n}.$$

In light of Lemma 3.13, since, by assumption, $K := \limsup_{n\to\infty} \frac{(\gamma^*_{n-1})^{-1}}{n} < \infty$, we conclude that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T, f, n, \epsilon, \delta)}{n} \leq h_\mu(T) + \int f\,d\mu + \theta.$$

Letting $\theta \to 0$, we obtain

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T, f, n, \epsilon, \delta)}{n} \leq h_\mu(T) + \int f\,d\mu. \tag{3.3}$$

Now, we prove the inequality

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T, f, n, \epsilon, \delta)}{n} \geq h_\mu(T) + \int f\,d\mu.$$

For any $\theta > 0$, choose $0 < \epsilon_0 < 1$ as in Lemma 3.12. Then, for any $0 < \epsilon < \epsilon_0$, we have:

$$\begin{aligned} P_{\mu,\Gamma}(g; T, f, n, \epsilon, \delta) &\leq \inf\Big\{\sum_{x\in S}\exp(\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)}(S_n f)(y)) : \\ &\qquad S \textit{ is an } (\Gamma, n, \epsilon^2, \delta, \mu) - \text{spanning set}\Big\} \\ &\leq \inf\Big\{\sum_{x\in S}\exp(S_n f)(x)) + n\theta + \epsilon||f||(\gamma^*_{n-1})^{-1}) : \\ &\qquad S \textit{ is an } (\Gamma, n, \epsilon^2, \delta, \mu) - \text{spanning set}\Big\} \\ &\leq \exp\Big\{n\theta + \epsilon||f||(\gamma^*_{n-1})^{-1}\Big\}P_{\mu,\Gamma}(T, f, n, \epsilon^2, \delta). \end{aligned}$$

So,

$$P_{\mu,\Gamma}(T, f, n, \epsilon^2, \delta) \geq P_{\mu,\Gamma}(g; T, f, n, \epsilon, \delta)\exp(-n\theta - \epsilon||f||(\gamma^*_{n-1})^{-1}). \tag{3.4}$$

Now, fix $0 < \eta < 1 - \delta$ and take $\gamma > 0$ and a partition ξ and an open cover $\mathcal{U}$ with Lebesgue number 2ϵ as in Lemma 3.14. Fix $Z \subseteq X$ with $\mu(Z) > 1-\delta$ and set $t_n(x) = |\{0 \le l \le n-1 : T^l(x) \in \bigcup_{i=m+1}^k U_i\}|$. As in the proof of Theorem 1.1 in [8], applying the Shannon-McMillan-Breiman and the Birkhöff's ergodic theorems, there exists $A \subseteq Z$ and $N \ge 1$ with $\mu(A) \ge \mu(Z) - \gamma$ such that for every $x \in A$ and $n \ge N$, we have:

(1) $t_n(x) \le 2\gamma n$.
(2) $\mu(\xi_0^{n-1}(x)) \le \exp\{-(h_\mu(T,\xi) - \gamma)n\}$.
(3) $\int f d\mu - \gamma \le \frac{1}{n}(S_n f)(x) \le \int f d\mu + \gamma$,

where $\xi_0^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\xi$. Using part 2, for any $n \ge N$, we have:

$$|\xi_n^*| \ge \mu(A) \exp\{(h_\mu(T,\xi) - \gamma)n\}, \tag{3.5}$$

where $\xi_n^* := \{C \in \xi_0^{n-1} : C \cap A \ne \phi\}$.

Let S be a (Γ, g, n, ϵ)-spanning set for Z. Since $\mu(Z) > 1 - \delta$, then S is a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set for X. Let $S' = \{x \in S : \overline{B}_{n,\Gamma}(g; x, \epsilon) \cap A \ne \emptyset\}$. Then $A \subset \bigcup_{x \in S'} \overline{B}_{n,\Gamma}(g; x, \epsilon)$. Fix $x \in S'$ and set

$$\mathcal{Q}_{n,\epsilon} := \{\Lambda_x \in I_\Gamma(g; n, \epsilon) : \overline{B}_{\Lambda_x}(x, \epsilon) \cap A \ne \emptyset\},$$

and $Q(n,\epsilon) := |\mathcal{Q}_{n,\epsilon}|$. Since $|\Lambda_x| \ge n - \epsilon(\gamma_{n-1}^*)^{-1}$, using Stirling's formula, we will have:

$$Q(n,\epsilon) \le \sum_{j=0}^{[\epsilon(\gamma_{n-1}^*)^{-1}]} \binom{n}{j} \le \sum_{j=0}^{[\epsilon(\gamma_{n-1}^*)^{-1}]} \binom{[(\gamma_{n-1}^*)^{-1}]}{j} \le \exp\{(\gamma_{n-1}^*)^{-1}(\epsilon + \Delta_n)\},$$

where

$$\Delta_\epsilon := \lim_{n \to +\infty} \Delta_n = -\epsilon \log K\epsilon - \frac{1}{K}(1 - K\epsilon)\log(1 - K\epsilon).$$

Fix $\Lambda_x \in \mathcal{Q}_{n,\epsilon}$ and set $\xi_{\Lambda_x} := \bigvee_{j \in \Lambda_x} T^{-j}\xi$. Let $N(x, \Lambda_x)$ be the number of atoms of ξ_{Λ_x} which intersect $A \cap \overline{B}_{\Lambda_x}(x, \epsilon)$ and $N(x, \xi_0^{n-1})$ be the number of atoms of ξ_0^{n-1} which intersect $A \cap \overline{B}_{\Lambda_x}(x, \epsilon)$. Since each set $\overline{B}(T^l x, \epsilon)$ is contained in some $U_{i_l} \in \mathcal{U}$, then $T^{-l}U_{i_l}$ intersects at most m sets of the form $T^{-l}C_{i_l}$. Therefore, in light of part 1 above, we have $N(x, \Lambda_x) \le m^{2\gamma n}$ and so,

$$N(x, \xi_0^{n-1}) \le N(x, \Lambda_x) m^{\epsilon(\gamma_{n-1}^*)^{-1}} \le m^{2\gamma n + \epsilon(\gamma_{n-1}^*)^{-1}}.$$

Consequently,

$$\begin{aligned}
|\xi_n^*| &\le \sum_{x \in S'} |\{C \in \xi_0^{n-1} : C \cap A \cap \overline{B}_{n,\Gamma}(g; x, \epsilon) \ne \phi\}| \\
&\le Q(n,\epsilon) \sum_{x \in S'} N(x, \xi_0^{n-1}) \\
&\le |S'| \exp\{(\gamma_{n-1}^*)^{-1}(\epsilon + \Delta_n)\} \exp\{(2\gamma n + \epsilon(\gamma_{n-1}^*)^{-1}) \log m\} \\
&= |S'| \exp\{(\gamma_{n-1}^*)^{-1}(\epsilon + \Delta_n) + (2\gamma n + \epsilon(\gamma_{n-1}^*)^{-1}) \log m\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{x\in S}\exp\left(\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)}(S_nf)(y)\right) &\geq \sum_{x\in S'}\exp\left(\sup_{y\in B_{n,\Gamma}(g;x,\epsilon)}(S_nf)(y)\right)\\ &\geq |S'|\exp\{n\int fd\mu-n\gamma\}\\ &\geq \mu(A)\exp\{(h_\mu(T,\xi)+\int fd\mu-2\gamma)n\\ &- (\gamma^*_{n-1})^{-1}(\epsilon+\Delta_n)-(2n\gamma+\epsilon(\gamma^*_{n-1})^{-1})\log m\}.\end{aligned}$$

Since S is a $(\Gamma, g, n, \epsilon, \delta, \mu)$-spanning set, then

$$\begin{aligned}\frac{\log P_{\mu,\Gamma}(g;T,f,n,\epsilon,\delta)}{n} &\geq \frac{\log\mu(A)}{n}+h_\mu(T,\xi)+\int fd\mu-2\gamma\\ &- \frac{(\gamma^*_{n-1})^{-1}}{n}(\epsilon+\Delta_n)-\frac{(2\gamma n+(\gamma^*_{n-1})^{-1})}{n}\log m.\end{aligned}$$

In light of (3.4), letting $n\to\infty$ and $\eta\to 0$, and considering the facts that $0<\gamma<\eta$, $2\gamma\log m\leq\eta$, $\lim_{\epsilon\to 0}\Delta_\epsilon=0$, $\mathrm{diam}(\xi)\leq\eta$ and that $K:=\limsup_{n\to\infty}\frac{(\gamma^*_{n-1})^{-1}}{n}<\infty$, we conclude that:

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T,f,n,\epsilon^2,\delta)}{n}\geq -\theta+\int fd\mu+h_\mu(T).$$

Letting $\theta\to 0$, we will have:

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T,f,n,\epsilon^2,\delta)}{n}\geq\int fd\mu+h_\mu(T).$$

Therefore,

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log P_{\mu,\Gamma}(T,f,n,\epsilon,\delta)}{n}\geq\int fd\mu+h_\mu(T).$$

This completes the proof. □

The following theorem generalizes Theorem 1.2 in [8], by replacing the mean metric $\tilde{d}_n$ by the metric $d_{n,\Gamma}$.

Theorem 3.16. *Let (X,T) be a TDS, $f\in C(X)$ and $\Gamma=\{\gamma_{m,n}\}_{m,n\geq 0}$ be a probability bi-sequence such that $\limsup_{n\to\infty}\frac{(\gamma^*_n)^{-1}}{n}<\infty$. then,*

$$P^*_\Gamma(T,f)=P_\Gamma(T,f)=P_{top}(T,f).$$

Proof. Firstly, we prove that $P^*_\Gamma(T,f)=P_\Gamma(T,f)$. Given any $(\Gamma,n,\frac{\epsilon}{2})$-spanning subset F of X and any (Γ,n,ϵ)-separated subset F of X, one may define an injective map $\phi:E\to F$ by choosing for each $x\in E$ some $\phi(x)\in F$ with $d_{n,\Gamma}(x,\phi(x))\leq\frac{\epsilon}{2}$. Therefore, $P^*_\Gamma(T,f,n,\frac{\epsilon}{2})\geq P_\Gamma(T,f,n,\epsilon)$. This immediately implies that $P^*_\Gamma(T,f)\geq P_\Gamma(T,f)$. Given $\epsilon>0$ and $n\in\mathbb{N}$, choose a subset $D=\{x_1,x_2,\cdots,x_k\}$ of X such that:

(i)$x_m \in X \setminus \bigcup_{i=1}^{m-1} B_{n,\Gamma}(x_i,\epsilon)$ for $2 \le m \le k$.
(ii) The following hold

$$(S_n f)(x_1) = \sup_{x\in X}(S_n f)(x), \quad (S_n f)(x_m) = \sup_{x\in X\setminus\bigcup_{i=1}^{m-1} B_{n,\Gamma}(x_i,\epsilon)}(S_n f)(x),$$

for $2 \le m \le k$.

Obviously, D is a maximal (Γ, n, ϵ)-separated subset of X and also an (Γ, n, ϵ)-spanning subset of X. So,

$$\begin{aligned} P^*_\Gamma(T, f, n, \epsilon) &\le \sum_{x\in D} \exp(\sup_{y\in B_{n,\Gamma}(x,\epsilon)}(S_n f)(y)) \\ &= \sum_{x\in D} e^{(S_n f)(x)} \\ &\le \sup\{\sum_{x\in E} e^{(S_n f)(x)} : E \text{ is an } (\Gamma, n, \epsilon) - \text{separated set of } X\} \\ &= P_\Gamma(T, f, n, \epsilon). \end{aligned}$$

This easily results in $P^*_\Gamma(T, f) \le P_\Gamma(T, f)$ which completes the proof of equality.

Secondly, we turn to prove that $P_\Gamma(T, f) = P_{\text{top}}(T, f)$. By Lemma 3.12, if E is an (Γ, n, ϵ)-separated set for X, then E is an (n, ϵ)-separated set for X. Then,

$$P_\Gamma(T, f, n, \epsilon) \le \sup\{\sum_{x\in E} e^{(S_n f)(x)} : E \text{ is an } (n, \epsilon) - \text{sepatared set for } X\}.$$

Therefore, $P_\Gamma(T, f) \le P_{\text{top}}(T, f)$.

For each $\mu \in E(X, T)$ and $0 < \delta < 1$, if F is an (Γ, n, ϵ)-spanning set for X, then F should be an $(\Gamma, n, \epsilon, \delta, \mu)$-spanning set for X. Therefore,

$$P^*_\Gamma(T, f, n, \epsilon) \ge P_{\mu,\Gamma}(T, f, n, \epsilon, \delta).$$

Applying Theorem 3.15, we have:

$$P^*_\Gamma(T, f) = P_\Gamma(T, f) \ge h_\mu(T) + \int_X f d\mu.$$

Taking supremum over all $\mu \in E(X, T)$ and applying the variational principle, we obtain $P_\Gamma(T, f) \ge P_{\text{top}}(T, f)$. This completes the proof. □

Remark 3.17. *One should note that, the condition* $\limsup_{n\to\infty} \frac{(\gamma^*_n)^{-1}}{n} < \infty$ *is satisfied for a vast class of probability bi-sequences containing the first part of Example 3.2. Also, there are many cases of sequences as in the second part of Example 3.2 which satisfy the previous condition. For instance, if* $\{a_n\}_{n\ge 0}$ *is an increasing bounded above sequence of positive numbers or is a decreasing sequance of positive numbers with positive lower bound, then the corresponding probability bi-sequence as in Example 3.2 will satisfy the condition* $\limsup_{n\to\infty} \frac{(\gamma^*_n)^{-1}}{n} < \infty$.

Remark 3.18. *Setting* $\Gamma = \Gamma_0 = \{\gamma_{m,n}\}_{m,n\geq 0}$ *with* $\gamma_{m,n} = \frac{1}{n+1}$ *in Theorem 3.15 and Theorem 3.16 one may conclude Theorems 1.1 and 1.2 in* [8] *respectively. Note that, the mean metric case corresponds to* $\gamma_{m,n} = \frac{1}{n+1}$*, for which* $\gamma_n^* = \frac{1}{n+1}$*. Then* $(\gamma_n^*)^{-1} = n+1$*, so* $\limsup \frac{(\gamma_n^*)^{-1}}{n} = 1 < \infty$*, satisfying their condition.*

Acknowledgment: The authors would like to thank the referees for their comprehensive and useful comments which helped the improvement of this work to the present form.

References

1. R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc., **114** (1965), 309-319.
2. M. Carvalho, S. A. Pérez, *A formula for the metric pressure*, arXiv.org. math. arXiv:1901.07198. (2019).
3. W. Cheng, Y. Zhao, Y. Cao, *Pressures for asymptotically sub-additive potentials under a mistake function*, Discrete Contin. Dyn. Syst. Ser. A 32(2) (2013), 487-497.
4. M. Gröger and T. Jäger, *Some remarks on modified power entropy*, Dynamics and Numbers, Contem. Math., **669**(2016), 105-122.
5. L. He, J. Lv, L. Zhou, *Definition of measure-theoretic pressure using spanning sets*, Acta Math. Sinica, Engl. Ser. **20** (4) (2004), 709-718.
6. P. Huang, E. Chen, C. Wang, *Entropy formulae of conditional entropy in mean metrics*, Discrete and Continuous Dynamical Systems, **38**, No. 10 (2018), 5129-5144.
7. W. Huang, Z. Wang, X. Ye, *Measure complexity and Möbius disjointness*, Advances in Mathematics, **347** (2019), 827-858.
8. P. Huang, C. Wang, *Measure-theoretic and topological pressure in mean metrics*, Dynamical Systems, **34** (2019), 259-273.
9. A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. Inst. Hautes Études Sci. **51** (1980), 137-173.
10. A. N. Kolmogorov, *New metric invariant of transitive dynamical systems and endomorphisms of Lebesgue spaces*, Doklady of Russian Academy of Sciences, **119**, N5 (1958), 861-864.
11. C. Luo, Y. Zhao, *The Brin-Katok formula for dynamical systems admitting mistakes*, Dynamical Systems **36** (2021), 560-571.
12. Y. Pesin, *Dimension theory in dynamical systems: Contemporary views and applications*, The University of Chicago Press, ChicagoIL, 1977.
13. C. Pfister, W. Sullivan, *On the topological entropy of saturated sets*, Ergod. Theory Dynam. Syst. **27** (2007), 929-956.
14. M. Rahimi, A. Assari, *On local metric pressure of dynamical systems*, Period Math Hung **82**, 223-230 (2021).
15. M. Rahimi and A. Assari, A note on entropy of finitely ergodic compact topological dynamical systems, J. Finsler Geom. Appl. **2**(1) (2021), 142-150.
16. M. Rahimi and A. Assari, A family of graphs generated by Hilbert space, J. Finsler Geom. Appl. **6**(1) (2025), 114-124.
17. M. Rahimi, M. R. Mozaffari and A. Tayebi, A Mathematical Modeling of n-State Systems, Physica A: Statistical Mechanics and its Applications, 674 (2025) 130727.
18. D. Ruelle, *Statistical mechanics on compact set with* $\mathbb{Z}^\nu$ *action satisfying expansiveness and specification*, Trans. Amer. Math. Soc., **187** (1973), 237-251.

19. D. Ruelle, *Thermodynamic formalism*, Cambridge Mathematical Library. Cambridge University Press, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.
20. Ya. G. Sinai, *On the notion of entropy of a dynamical system.* Doklady of Russian Academy of Sciences, **124**(1959), 768-771.
21. D. Thompson, *Irregular sets, the beta-transformation and the almost specification property,* Trans. Amer. Math. Soc. **364**(2012), 5395-5414.
22. P. Walters, *A variational principle for the pressure of continuous transformations*, Amer. J. Math., **97**(1975), 937-971.
23. D. Zheng, E. Chen, and J. Yang, *On large deviations for amenable group actions*, Discrete Contin. Dyn. Syst. Ser. A **36**(12) (2017), 7191-7206.
24. Y. J. Zhu, *Two Notes on Measure-Theoretic Entropy of Random Dynamical Systems*, Acta Math. Sin. (Engl. Ser.), **25**(6) (2009), 961-970.

Received: 05.01.2026
Accepted: 11.05.2026