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Research Paper

ON THE GAUSS MAPPING OF HYPERSURFACES

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ABSTRACT

In this paper, we establish a one-to-one correspondence between some hypersurfaces and the images of them under their Gauss mappings. Then, we introduce a class of hypersurfaces, whose images under their Gauss mappings are the same. Moreover, we prove the evenness of the cardinal number of the cross section of a compact hypersurface $S \subseteq \mathbb{R}^m$ with a curve $\alpha : \mathbb{R} \to \mathbb{R}^m$ which is nowhere tangent to S, and tends to infinity in directions $\pm\infty$. Finally, we clarify how the results, relate to existing literature on Gauss mappings of minimal or finite-type surfaces. Some illustrative examples and an application are also provided.

1. Introduction and Preliminaries

The Gauss mapping and its properties have been of great interest in differential geometry, and the works of many mathematicians include several approaches on this subject. For instance, the initial results on the connection between total curvature and the Gauss mapping of complete minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are obtained in [41, 42].
The existence and uniqueness of an immersion from a surface or a hypersurface into $\mathbb{R}^n$ with a given metric (or conformal structure) and a given Gauss mapping has been studied by several authors, for example [27, 28]. The authors proved that, there exists a conformal non-minimal immersion from a simply-connected surface in $\mathbb{R}^n$ with a given Gauss mapping

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if and only if a set of differential equations depending on the conformal structure and the Gauss mapping is satisfied. The uniqueness problem of such non-minimal immersions has been studied in [1].

In [37], the authors combined the methods used by the author of [41, 42] with the results of [22] and [29] to put into final form the connection between total curvature and the Gauss mapping of complete minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$. In [49], the author showed that if the components of the Gauss mapping are eigenfunctions of the Jacobi-Steklov operator, then the surface must be rotationally symmetric.

In [28], the authors studied the properties of the generalized Gauss mapping, particularly those related to the geometry of $S \subseteq \mathbb{R}^n$ and its conformal structure.

The main object in [23] is to study the properties of the Gauss mapping of a surface immersed in $\mathbb{R}^3$, particularly those related to the geometry of the immersion and the conformal structure determined by its second fundamental form.

In [26] some relations between the Laplace Beltrami operator and the curvatures of helicoidal surfaces in 3-Euclidean space are established, and Bour's theorem on the Gauss mapping with some examples are given.

In [5], the author investigated the timelike Aminov surface with respect to its Gauss mapping in Minkowski space-time.

Some of the other kindes of Gauss mappings such as, finite type Gauss mapping, pointwise 1-type Gauss mapping, L_1-pointwise 1-type Gauss mapping, and their properties are considered in [2, 8, 11, 14, 18, 19, 20, 21, 32, 33, 34, 35, 45, 50].

One of the formidable problems in differential geometry is the classification of manifolds [25, 36]. Here, as a special case, we investigate the capability of the Gauss mapping to solve this problem for hypersurfaces. Then, as another application of this mapping, we prove a theorem on the cardinality of the cross section of a compact hypersurface $S \subset \mathbb{R}^m$ with a curve $\alpha : \mathbb{R} \to \mathbb{R}^m$ which is nowhere tangent to S and tends to infinity in each direction.

Before advancing any further, let us continue by some notions on the essential ingredients of this manuscript. Let $\mathbb{R}$ be the set of real numbers, and let U be an open subset of $\mathbb{R}^m$ for $m \geq 2$. By a "hypersurface in $\mathbb{R}^m$" (simply hypersurface), we mean a non-empty set $S = f^{-1}(c)$ for $c \in \mathbb{R}$, where $f : U \to \mathbb{R}$ is a smooth function such that $\nabla f(s) \neq 0$ for all $s \in S$ [43, 44]. If $Z \subseteq S$ and if Z is a hypersurface, then Z is called a sub-hypersurface of S. A curve in S is a smooth function of an open interval $I \subseteq \mathbb{R}$ into S. The tangent space of S at $s \in S$ which is denoted by T_sS, is the set of all of the velocity vectors of all of the curves $\alpha : I \to S$ passing through s [25].

If $A \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$, then a function $F : A \to \mathbb{R}^m$ is called smooth if F can be extended to a smooth function on some open set containing A [47].

A vector field on $A \subseteq \mathbb{R}^m$ is a function $\mathbf{X} : A \to A \times \mathbb{R}^m$ such that $\mathbf{X}(a) = (a, X(a))$ and $X : A \to \mathbb{R}^m$ is smooth. A tangent vector field on S is a vector field $\nu : S \to TS = \bigcup_{s\in S} T_sS$ where $\nu(s) \in T_sS$. An oriented hypersurface is a pair $(S, \mathbf{N}_S)$, where S is a hypersurface and $\mathbf{N}_S : S \to S \times \mathbb{R}^m$ given by $\mathbf{N}_S(s) = (s, N_S(s))$ is the unit normal vector field on S induced by $\pm\nabla f$. It is called the orientation of S. If $(Z, \mathbf{N}_Z)$ is an oriented hypersurface and if $Z \subseteq S$ and $\mathbf{N}_Z = (\mathbf{N}_S)_{|Z}$, then $(Z, \mathbf{N}_Z)$ is called an oriented sub-hypersurface of $(S, \mathbf{N}_S)$. For an oriented hypersurface $(S, \mathbf{N}_S)$, the Gauss mapping $N_S : S \to \mathbb{R}^m$ maps S into $S^{m-1} = \{x \in \mathbb{R}^m | \ \|x\| = 1\}$, and $N_S(S) = \{N_S(s) | s \in S\}$ is called the image of $(S, \mathbf{N}_S)$

under its Gauss mapping [6, 40].
If a hypersurface S is connected, then there exist on S exactly two orientations which induced by the vector fields $\pm\nabla f$.

Theorem 1.1. [47] *Let $U \subseteq \mathbb{R}^m$ be an open subset, $f : U \to \mathbb{R}$ be a smooth function, and let $S = f^{-1}(c)$ be a hypersurface. If $\frac{\partial f}{\partial x_m}(s) \neq 0$, then there exists an open subset $A \subseteq \mathbb{R}^{m-1}$ containing $(s_1, \dots, s_i, \dots, s_{m-1})$ and an open interval I_m containing s_m with the following property:*
For each $x \in A$, there exists a unique smooth function $g : A \to I_m$ such that $f(x, g(x)) = c$.

The following corollary is a consequence of Theorem 1.1.

Corollary 1.2. *Let S be a hypersurface and $s \in S$. Then there exists an open neighborhood $V \subseteq U$ of s such that*

(1) *$V \cap S$ is the graph of a smooth function,*
(2) *$V \cap S$ is connected.*

Let $U \subseteq \mathbb{R}^m$ be an open set and let $\mathbf{X} : U \to U \times \mathbb{R}^m$ be a vetcor field on U given by $\mathbf{X}(u) = (u, X_1(u), \dots, X_m(u))$, where $X_i : U \to \mathbb{R}$ is a smooth function for $i = 1, \dots, m$. Let I be an open interval and let $\alpha : I \to U$ given by $\alpha(t) = (x_1(t), \dots, x_m(t))$ be a smooth function, then α is said to be an integral curve of $\mathbf{X}$ if $\alpha'(t) = \mathbf{X}(\alpha(t))$, where $\alpha'(t) = (\alpha(t), \frac{dx_1}{dt}(t), \dots, \frac{dx_m}{dt}(t))$. This equation which is changed into an autonomus system of differentail equations

$$\frac{dx_i}{dt}(t) = X_i(x_1(t), \dots, x_m(t)) \quad (1 \leq i \leq m),$$

establishes a one-to-one correspondence between vector fields on U and autonomus systems of differentail equations [30].

Theorem 1.3. [46] *Let $U \subset \mathbb{R}^m$ be an open set and $\mathbf{X}$ be a vector field on U. Let $t_0 \in \mathbb{R}$ and $u_0 \in U$. Then,*

(1) *There exists an open interval I with $t_0 \in I$ and an integral curve $\alpha : I \to U$ with initial condition $\alpha(t_0) = u_0$,*
(2) *If I and J are two intervals with $t_0 \in I \cap J$ and if $\alpha : I \to U, \beta : J \to U$ are two integral curves of $\mathbf{X}$ with initial condition $\alpha(t_0) = \beta(t_0) = u_0$, then $\alpha(t) = \beta(t)$ for $t \in I \cap J$.*

The extended form of the preceding theorem to hypersurfaces can be provided as follows:

Theorem 1.4. [43] *Let S be a hypersurface and ν be a tangent vector field on S. Let $t_0 \in \mathbb{R}$ and $s_0 \in S$. Then,*

(1) *There exists an open interval I containing t_0 and an integral curve $\alpha : I \to S$ with initial condition $\alpha(t_0) = s_0$,*
(2) *If I and J are two intervals with $t_0 \in I \cap J$ and if $\alpha : I \to S, \beta : J \to S$ are some integral curves of ν with initial condition $\alpha(t_0) = \beta(t_0) = s_0$, then $\alpha(t) = \beta(t)$ for $t \in I \cap J$.*

Corollary 1.5. [43] *Let $U \subseteq \mathbb{R}^m$ be open, $f : U \to \mathbb{R}$ be a smooth function and $S = f^{-1}(c)$ be a hypersurface for $c \in \mathbb{R}$. Let $\mathbf{X}$ be a vector field on U whose restriction to S is tangent* *to S. If $\alpha : I \to U$ is an integral curve of $\mathbf{X}$ with initial condition $\alpha(\bar{t}) \in S$ for some $\bar{t} \in I$, then $\alpha(I) \subset S$.*

2. New hypersurfaces

Here we introduce some useful methods of manufacturing new hypersurfaces from known ones. At first, we put aside some sets which as consequences of Corollary 1.2, are not hypersurfaces.

Corollary 2.1. *A single point in $\mathbb{R}^m$ is not a hypersurface.*

Corollary 2.2. *The image of an oriented hypersurface under its Gauss mapping, in general, is not a hypersurface.*

In the following, we prove some theorems on the union of hypersurfaces.

Theorem 2.3. *Let $(S_1, \mathbf{N}_{S_1})$ and $(S_2, \mathbf{N}_{S_2})$ be compact oriented hypersurfaces such that $S_1 \cap S_2 = \emptyset$. Then $(S_1 \cup S_2, \mathbf{N}_{S_1} \cup \mathbf{N}_{S_2})$ is also a compact oriented hypersurface. Moreover, the image of $S_1 \cup S_2$ under its Gauss mapping, is the union of the images of $(S_1, \mathbf{N}_{S_1})$ and $(S_2, \mathbf{N}_{S_2})$ under their Gauss mappings.*

Proof. Suppose that $f : U \to \mathbb{R}, g : V \to \mathbb{R}$ are smooth functions and let $u, v \in \mathbb{R}$ be such that $S_1 = f^{-1}(u), S_2 = g^{-1}(v)$. Without loss of generality, it can be assumed that ∇f(res. ∇g) is a non-zero vector field on U (res. V). Since $\mathbb{R}^m$ is a normal topological space, it can be assumed that $U \cap V = \emptyset$. Put $W = U \cup V$ and define the function $h : W \to \mathbb{R}$ by

$$h(w) = \begin{cases} f(w) - u & \text{if } w \in U \\ g(w) - v & \text{if } w \in V. \end{cases}$$

Obviously, h is a smooth function on W such that $S_1 \cup S_2 = h^{-1}(0)$, and $\nabla h(w) \neq 0$ for all $w \in W$. $\square$

Corollary 2.4. *Let $(S_i, \mathbf{N}_{S_i})$ for $1 \leq i \leq k$ be a compact oriented hypersurface and let $S_i \cap S_j = \emptyset$ for all $1 \leq i < j \leq k$. Then $(\bigcup_{i=1}^{i=k} S_i, \bigcup_{i=1}^{i=k} \mathbf{N}_{S_i})$ is a compact oriented hypersurface, and $(S_i, \mathbf{N}_{S_i})$ for $(1 \leq i \leq k)$ is an oriented sub-hypersurface of $(\bigcup_{i=1}^{i=k} S_i, \bigcup_{i=1}^{i=k} \mathbf{N}_{S_i})$.*

Corollary 2.5. *The image of a compact oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping, is the union of the images of its connected components under their Gauss mappings as oriented sub-hypersurfaces of $(S, \mathbf{N}_S)$.*

The following example shows that the compactness is essential in the hypotheses of Theorem 2.3.

Example 2.6. Let $S_1 = \{(x, y)|y = 0,\ 2 < |x| < 4\}$ and $S_2 = \{(x, y)|x^2 + y^2 = 4\}$. Then S_1 is a non-compact hypersurface and $S_1 \cap S_2 = \emptyset$, but $S_1 \cup S_2$ is not a hypersurface (Figure 1).

Now, a useful method of manufacturing new hypersurfaces from the known one is provided.

Theorem 2.7. *Let $(S, \mathbf{N}_S)$ be an oriented hypersurface. Let T be an open subset of S and $\mathbf{N}_T = \mathbf{N}_S|_T$. Then $(T, \mathbf{N}_T)$ is an oriented sub-hypersurface of $(S, \mathbf{N}_S)$, and the image of T under its Gauss mapping is a subset of the image of S under its Gauss mapping.*

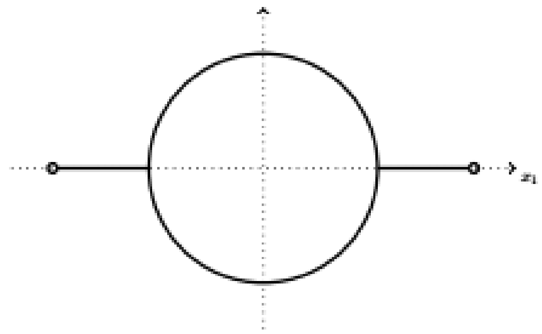

FIGURE 1. $S_1 \cup S_2$ is not a hypersurface.

Proof. There exists an open set $U \subseteq \mathbb{R}^m$ and a smooth function $f : U \to \mathbb{R}$ such that $S = f^{-1}(c)$ for some $c \in \mathbb{R}$, and $\nabla f(s) \neq 0$ for $s \in S$. There exists an open subset of $O \subseteq \mathbb{R}^m$ such that $T = O \cap S$. Put $W = U \cap O$ and define $g : W \to \mathbb{R}$ by $g = f_{|W}$. Then $T = g^{-1}(c)$ and $\nabla g(t) = \nabla f(t) \neq 0$ for $t \in T$. □

In the following, we introduce some subsets of S^{m-1} that are the images of some oriented hypersurfaces under their Gauss mappings.

Theorem 2.8. *For an open subset U of S^{m-1}, there exists an oriented hypersurface $(S, \mathbf{N}_S)$ whose image under its Gauss mapping is U.*

Proof. Obviously $S^{m-1} = f^{-1}(0)$ for $f : \mathbb{R}^m \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2 - 1$$

oriented by ∇f. Let O be an open subset of $\mathbb{R}^m$ such that $U = O \cap S^{m-1}$. Define $g : O \to \mathbb{R}$ by $g = f|_O$. Then $(S = g^{-1}(0), \mathbf{N}_S)$, where $\mathbf{N}_S(s) = (s, s)$ for $s \in S$, is a hypersurface with the required property. □

Theorem 2.9. *If all of the points of a non-empty set $A \subset S^{m-1}$ are isolated, then there exists an oriented hypersurface $(S, \mathbf{N}_S)$ whose image under its Gauss mapping is A.*

3. A ONE-TO-ONE CORRESPONDENCE

In this part, we establish a one-to-one correspondence between some hypersurfaces and the images of them under their Gauss mappings.

3.1. **Hyperplanes in $\mathbb{R}^m$.** Let $0 \neq a \in \mathbb{R}^m$, by a hyperplane in $\mathbb{R}^m$ we mean the set

$$P = \{x \in \mathbb{R}^m | \prec a, x \succ = c\},$$

where $c \in \mathbb{R}$ and $\prec a, x \succ$ is the scalar product of the vectors a and x. Evidently, the image of a hyperplane under its Gauss mapping is a single point. In the following, we provide a converse for this statement.

Theorem 3.1. *Let $(S, \mathbf{N}_S)$ be a hypersurface, and let $U \subseteq \mathbb{R}^m$ be an open set containing S. Let $\{v_0\} \subset \mathbb{R}^m$ be the image of S under its Gauss mapping. Let $s \in S$ and*

$$P_s = \{x \in \mathbb{R}^m | \prec x, v_0 \succ = \prec s, v_0 \succ\}.$$

Then there exists an open ball B_s with center s such that $B_s \cap P_s \subseteq S$.

Proof. Without loss of generality let $S = f^{-1}(c)$ for some smooth function $f : U \to \mathbb{R}$. Corollary 1.2 implies that for $s \in S$ there exists an open ball $B_s \subseteq U$ with center s such that $B_s \cap S$ is connected. Let $x_0 \in B_s \cap P_s$, $w = x_0 - s$, and consider the vector field $\mathbf{X}(u) = (u, w)$ for $u \in U$. The restriction of $\mathbf{X}$ to S is tangent to S and its integral curve through s is $\alpha : I \to B_s$, $\alpha(t) = (1-t)s + tx_0$ where I is an open interval containing $\{0, 1\}$. Since $\alpha(0) = s \in S$, then Corollary 1.5 asserts that $\alpha(t) \in S$ for all $t \in I$. Therefore $x_0 = \alpha(1) \in S$. □

Theorem 3.2. *Let the image of a path connected oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping be $\{v_0\} \subset \mathbb{R}^m$. Let $s \in S$ and*

$$P_s = \{x \in \mathbb{R}^m | \prec x, v_0 \succ = \prec s, v_0 \succ\}.$$

Then $S \subseteq P_s$.

Proof. Suppose that $S = f^{-1}(c)$ for some smooth function $f : U \to \mathbb{R}$. Let $x_0 \in S$ and $\alpha : [p, q] \to S$ be a curve such that $\alpha(p) = s, \alpha(q) = x_0$. For each $t \in [p, q]$ there exists a ball $B_{\alpha(t)} \subseteq U$ with the properties stated in Theorem 3.1. Since $\alpha[p, q]$ is a compact connected subspace of S, then there exist $t_1, \ldots, t_n \in [p, q]$ such that $\alpha[p, q] \subset B_{\alpha(t_1)} \cup \cdots \cup B_{\alpha(t_n)}$ and $B_{\alpha(t_i)} \cap B_{\alpha(t_{i+1})} \neq \emptyset$ for $i = 1, \ldots, n-1$. These relations together with the continuty of α imply that $B_{\alpha(t_i)} \cap B_{\alpha(t_{i+1})} \cap \alpha[p, q] \neq \emptyset$ for $i = 1, \ldots, n-1$. Because of Theorem 3.1, it can be assumed that the restriction of S to $B_{\alpha(t_i)}$ is $P_{\alpha(t_i)} \cap S$, so $P_{\alpha(t_i)} \cap P_{\alpha(t_{i+1})} \cap S \neq \emptyset$ for $i = 1, \ldots, n-1$. Suppose that $x_i = \alpha(r_i) \in P_{\alpha(t_i)} \cap P_{\alpha(t_{i+1})} \cap S$ for some $r_i \in [p, q]$ and $i = 1, \ldots, n-1$, then

$$\begin{aligned}
\prec s, v_0 \succ &= \prec \alpha(p), v_0 \succ \\
&= \prec \alpha(r_1), v_0 \succ = \prec \alpha(r_2), v_0 \succ \\
&= \cdots = \prec \alpha(r_{n-1}), v_0 \succ = \prec \alpha(q), v_0 \succ = \prec x_0, v_0 \succ .
\end{aligned}$$

Therefore, $\prec x_0, v_0 \succ = \prec s, v_0 \succ$, or $x_0 \subseteq P_s$. □

Corollary 3.3. *The image of a path connected oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping is a single point if and only if S is a portion of a hyperplane.*

3.2. **Cylinders in $\mathbb{R}^m$.** Let $U \subseteq \mathbb{R}^{m-1}$, and let $S = f^{-1}(c)$ for a smooth function $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\nabla f(s) \neq 0$ for all $s \in S$. Suppose that $\hat{C} = \hat{f}^{-1}(c)$ for the (smooth) function $\hat{f} : U \times \mathbb{R} \to \mathbb{R}$ given by

$$\hat{f}(x_1, \ldots, x_{m-1}, x_m) = f(x_1, \ldots, x_{m-1}).$$

Obviously if $s \in \hat{C}$ then $s + te_m \in \hat{C}$ for $e_m = (\overbrace{0, \ldots, 0}^{(m-1)\ times}, 1)$ and $t \in \mathbb{R}$. This property of $\hat{C}$ can be considered as a definition of a cylinder in $\mathbb{R}^m$. In fact, by a cylinder in $\mathbb{R}^m$ we mean a hypersurface S with the following property:
There exists a vector $a \in \mathbb{R}^m$ such that for any $s \in S$ and an open interval $I = I_s$ containing 0, $t \in I$ implies that $s + ta \in S$.

In the following, we give a necessary and sufficient condition on a hypersurface S to be a cylinder.

Theorem 3.4. *Let $0 \neq a \in \mathbb{R}^m$, $(S, \mathbf{N}_S)$ be a hypersurface and*

$$P = \{x \in \mathbb{R}^m | \prec a, x \succ = 0\}.$$

If S is a cylinder, then the image of S under its Gauss mapping is contained in P.

Proof. Let N_S be the Gauss mapping of $S = f^{-1}(c)$ for some smooth function $f : U \to \mathbb{R}$. Then $N_S(\alpha(t)) \perp \frac{d\alpha}{dt}(t)$ for any smooth function $\alpha : I \to S$ and $t \in I$, where I is an open interval containing 0. Pick $s \in S$ and consider the function $\alpha(t) = s + ta$ for $s \in S$ and $t \in I$. Then $\frac{d\alpha}{dt}(t) = a$ and $\prec a, N_S(\alpha(t)) \succ = 0$. Therefore $N_S(\alpha(t))$ is contained in P for $t \in I$, so $N_S(s) = N_S(\alpha(0)) \in P$. □

Theorem 3.5. *Let $0 \neq a \in \mathbb{R}^m$ and $(S, \mathbf{N}_S)$ be an oriented hypersurface. Let the image of S under its Gauss mapping is contained in the hyperplane*

$$P = \{x \in \mathbb{R}^m | \prec a, x \succ = 0\},$$

then S is a cylinder.

Proof. Let N_S be the Gauss mapping of $S = f^{-1}(c)$ for some smooth function $f : U \to \mathbb{R}$. Consider the constant vector field $\mathbf{X}(u) = (u, a)$ for $u \in U$. The integral curve of $\mathbf{X}$ through $s \in S$ is $\alpha(t) = s + ta$ while $t \in I$ for some open interval I around 0. Therefore by our hypotheses $\prec a, N_S \succ = 0$, so the restriction of $\mathbf{X}$ to S is tangent to S and $\alpha(0) = s \in S$. Thus, Corollary 1.5 implies that $\alpha(t) \in S$ for all $t \in I$. □

Corollary 3.6. *Let $0 \neq a \in \mathbb{R}^m$ and $(S, \mathbf{N}_S)$ be an oriented hypersurface. Then the image of S under its Gauss mapping is contained in the hyperplane*

$$P = \{x \in \mathbb{R}^m | \prec a, x \succ = 0\},$$

if and only if S is a cylinder.

3.3. **Hypersurfaces of revolution in $\mathbb{R}^m$.** In the following, we give a necessary and sufficient condition for the image of a hypersurface of revolution under its Gauss mapping to locate in an affine plane, or in a right circular cone [44].
Let ℓ be a half line (or a line segment) in $\mathbb{R}^2$ with equation $x_2 = \rho x_1 + \theta$ for some $0 \neq \rho, \theta \in \mathbb{R}$ such that $x_2 > 0$ for x_1 in an open interval I. Define the function $f : I \times \mathbb{R}^+ \times \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{(m-2)\ times} \to \mathbb{R}$ by

$$f(x_1, x_2, \ldots, x_m) = \sqrt{x_2^2 + \cdots + x_m^2} - \rho x_1 - \theta$$

for $m \geq 2$. Let $S_\ell = f^{-1}(0)$ and put

$$y_i = \frac{x_i}{\sqrt{\rho^2 + 1}\sqrt{x_2^2 + \cdots + x_m^2}}$$

for $2 \leq i \leq m$. It can be seen that S_ℓ is a hypersurface of revolution obtained by rotating ℓ around x_1 axis with unit normal

$$N_{S_\ell}(x_1, x_2, \ldots, x_m) = (\frac{-\rho}{\sqrt{\rho^2 + 1}}, y_2, \ldots, y_m),$$

so, the image of $(S_\ell, \mathbf{N}_{S_\ell})$ under its Gauss mapping is the circle

$$x_2^2 + \cdots + x_m^2 = \frac{1}{\rho^2 + 1}, \quad x_1 = \frac{-\rho}{\sqrt{\rho^2 + 1}}.$$

Therefore, it is in the affine plane $x_1 = \frac{-\rho}{\sqrt{\rho^2+1}}, 0 < \frac{|\rho|}{\sqrt{\rho^2+1}} < 1$. Conversely, let I be an open interval and $\mathcal{C}$ be the graph of a smooth function $g : I \to \mathbb{R}^+$, and let $f : I \times \mathbb{R}^+ \times \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{(m-2)\ times} \to \mathbb{R}$ be the function given by

$$f(x_1, x_2, \ldots, x_m) = \sqrt{x_2^2 + \cdots + x_m^2} - g(x_1).$$

Then $S_{\mathcal{C}} = f^{-1}(0)$ is the hypersurface of revolution obtained by rotating $\mathcal{C}$ around x_1 axis. Let the image of $S_{\mathcal{C}}$ under its Gauss mapping be in an affine plane $x_1 = b$ with $0 < |b| < 1$. Let $z_i = \frac{x_i}{\sqrt{x_2^2+\cdots+x_m^2}}$ for $2 \leq i \leq m$. Since the unit normal of $S_{\mathcal{C}}$ is

$$N_{\mathcal{S}_{\mathcal{C}}}(x_1, x_2, \ldots, x_m) = (\frac{-\frac{dg}{dx}(x_1)}{\sqrt{(\frac{dg}{dx}(x_1))^2 + 1}}$$
$$, \frac{z_2}{\sqrt{(\frac{dg}{dx}(x_1))^2 + 1}}, \ldots, \frac{z_m}{\sqrt{(\frac{dg}{dx}(x_1))^2 + 1}}),$$

it follows that $\frac{-\frac{dg}{dx}(x_1)}{\sqrt{(\frac{dg}{dx}(x_1))^2+1}} = b$ and hence $\frac{dg}{dx}(x_1) = \frac{\pm b}{\sqrt{1-b^2}}$. Thus $g(x_1) = \rho x_1 + \theta$ for some constants $0 \neq \rho, \theta \in \mathbb{R}$.

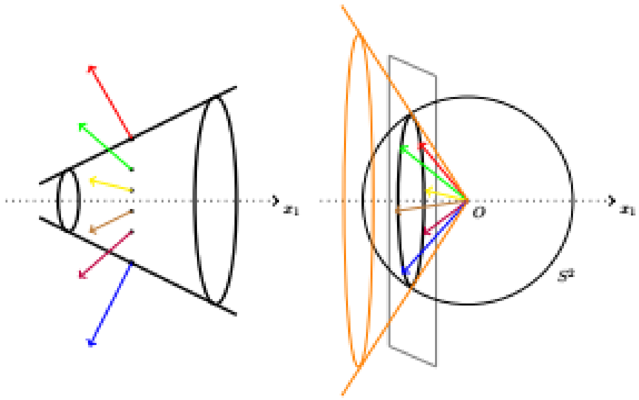

FIGURE 2. The Gauss mapping of a hypersurface of revolution obtaind by rotating a line segment (or a half line) $x_2 = \rho x_1 + \theta$, $0 \neq \rho, \theta \in \mathbb{R}$, $x_2 > 0$ around x_1 axis maps it in the intersection of an affine plane and a right circular cone.

It can also be seen that the image of $(S_\ell, \mathbf{N}_{S_\ell})$ under its Gauss mapping is in the $(m-1)$-dimensional right circular cone

$$x_1^2 = \rho^2(x_2^2 + \cdots + x_m^2), \rho \neq 0.$$

Conversely, if the image of $(S_{\mathcal{C}}, \mathbf{N}_{S_{\mathcal{C}}})$ under its Gauss mapping is in an $(m-1)$-dimensional right circular cone $x_1^2 = \rho^2(x_2^2 + \cdots + x_m^2)$ with some constant $0 \neq \rho \in \mathbb{R}$, then a calculation yields $\frac{dg}{dt}(x_1) = \pm\rho$ and $g(x_1) = \pm\rho x_1 + \theta$ for some $\theta \in \mathbb{R}$ (Figure 2). Some similar results hold for a line (or a line segment) in $\mathbb{R}^2$ with equation $x_2 = \theta$ for some $\theta \in \mathbb{R}^+$ defined on an open interval I (Figure 3).

Theorem 3.7. *Let I be an open interval and $g : I \to \mathbb{R}^+$ be a smooth function, and let*

$$f : I \times \mathbb{R}^+ \times \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{(m-2)\ times} \to \mathbb{R}$$

given by

$$f(x_1, x_2, \ldots, x_m) = \sqrt{x_2^2 + \cdots + x_m^2} - g(x_1).$$

Suppose that the image of the oriented hypersurface $S = f^{-1}(0)$ under its Gauss mapping, is denoted by $N_S(S)$. Then the following statements are equivalent:

(1) *$N_S(S)$ is in the affine plane $x_1 = b$ for some constant $0 < |b| < 1$.*
(2) *$N_S(S)$ is in the $(m-1)$-dimensional right circular cone*

$$x_1^2 = \rho^2(x_2^2 + \cdots + x_m^2)$$

for some constant $0 \neq \rho \in \mathbb{R}$.

(3) *$g(x) = \lambda x + \delta$ for some constants $0 \neq \lambda, \delta \in \mathbb{R}$.*

Moreover, for the functions g, f defined as above and $S = f^{-1}(0)$ the following statements are equivalent:

(1) *$N_S(S)$ is in the affine plane $x_1 = 0$.*
(2) *$N_S(S)$ is in the $(m-1)$-dimensional right circular cylinder $x_2^2 + \cdots + x_m^2 = 1$.*
(3) *$g(x) = \delta$ for some constant $\delta \in \mathbb{R}^+$.*

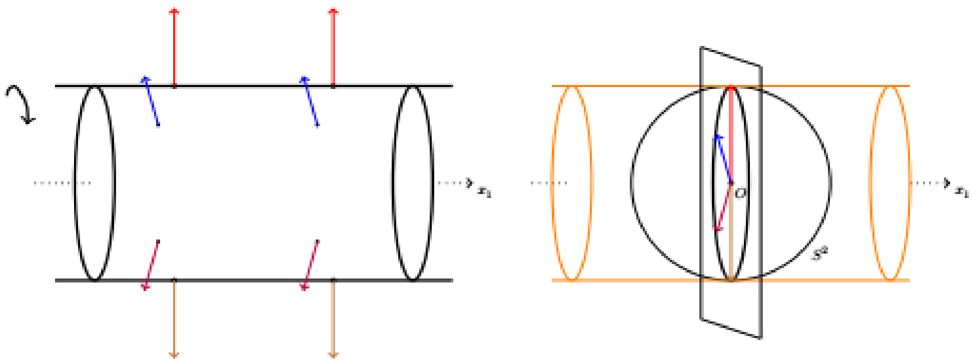

Figure 3. The Gauss mapping of a hypersurface of revolution obtaind by rotating a line (or a line segment) $x_2 = \theta > 0$ around x_1 axis maps it in the intersection of the plane $x_1 = 0$ and the right circular cylinder $x_2^2 + x_3^2 = 1$.

Theorem 3.8. *With any smooth function $g : I \to \mathbb{R}^+$ and*

$$f : I \times \mathbb{R}^+ \times \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{(m-2)\ times} \to \mathbb{R}$$

given by

$$f(x_1, x_2, \ldots, x_m) = \sqrt{x_2^2 + \cdots + x_m^2} - g(x_1),$$

the image of $(S = f^{-1}(0), \mathbf{N}_S)$ under its Gauss mapping is in the $(m-1)$-dimensional punctured sphere $S^{m-1} - \{\pm 1, 0, \ldots, 0\}$.

Theorem 3.9. *There exists a hypersurface of revolution $(S, \mathbf{N}_S)$ whose image under its Gauss mapping is exactly $S^{m-1} - \{\pm 1, 0, \ldots, 0\}$.*

4. Compact hypersurfaces in $\mathbb{R}^m$

In this section, we introduce a class of hypersurfaces with the same topological specifications whose images under their Gauss mappings are the same.

Theorem 4.1. *Let $S = f^{-1}(c)$ for some smooth function $f : U \to \mathbb{R}$ be a compact oriented hypersurface and $v \in S^{m-1}$. Then there exist points $s_1, s_2 \in S$ such that the vectors $N_S(s_1)$, $N_S(s_2)$ and v are parallel.*

Proof. The function $W : U \to \mathbb{R}$ given by $W(x) = e^{\prec x,v \succ}$ is smooth and takes its extremum points on S, say at s_{max} and s_{min}. By the Lagrange multiplier theorem

$$W(s)v = \nabla W(s) = \lambda \nabla f(s) = \lambda \|\nabla f(s)\| N_S(s)$$

for $s = s_{max}, s_{min}$, and so $N_S(s) \parallel v$. □

Theorem 4.2. *Let $(S, \mathbf{N}_S)$ be a compact oriented hypersurface and $v \in S^{m-1}$. Assume that S is a level set of some smooth function $f : U \to \mathbb{R}$ with non-zero gradient on S, where U either is the whole space $\mathbb{R}^m$ or an open ball of it. Then there exists a point $s \in S$ such that $N_S(s) = v$.*

Proof. Since $\| N_S(s) \| = \| v \| = 1$, then Theorem 4.1 implies that $N_S(s) = \pm v$ for $s = s_{max}, s_{min}$. Hence, it suffices to prove $N_S(s_{max}) \neq N_S(s_{min})$. There exists a disc $D(0; r) = \{x \in \mathbb{R}^m | \; \| x \| \leq r\} \subset U$ for some $r > 0$ such that $S \subset intD(0; r)$. Set $A(t) = s_{max} + tv, 0 \leq t \leq t_1$ for some $t_1 \in \mathbb{R}^+$ such that $A(t_1) \in \partial D(0; r)$, and set $C(t) = s_{min} - tv, 0 \leq t \leq t_2$ for some $t_2 \in \mathbb{R}^+$ such that $C(t_2) \in \partial D(0; r)$. Since $\partial D(0; r)$ is path connected for $m \geq 2$, so, there exists a continuous function $B : [l_1, l_2] \to \partial D(0; r)$ such that $B(l_1) = A(t_1)$ and $B(l_2) = C(t_2)$. Let the functions $\alpha : [t_1, t_1+l_2-l_1] \to \mathbb{R}$ and $\beta : [t_1+l_2-l_1, t_1+t_2+l_2-l_1] \to \mathbb{R}$ be defined by $\alpha(t) = t + l_1 - t_1$ and $\beta(t) = t_1 + t_2 + l_2 - l_1 - t$. The function ψ given by

$$\psi(t) = \begin{cases} A(t) & \text{if } t \in [0, t_1], \\ B \circ \alpha(t) & \text{if } t \in [t_1, t_1 + l_2 - l_1], \\ C \circ \beta(t) & \text{if } t \in [t_1 + l_2 - l_1, t_1 + t_2 + l_2 - l_1], \end{cases}$$

is continuous on $[0, t_1 + t_2 + l_2 - l_1]$, $\psi'_+(0) = (s_{max}, v)$, $\psi'_-(t_1 + t_2 + l_2 - l_1) = (s_{min}, v)$ and

$$\psi(t) \notin S, \forall t \in]0, t_1 + t_2 + l_2 - l_1[. \tag{4.1}$$

In fact $A(t) \notin S$ for $0 < t \leq t_1$ because

$$\frac{d}{dt}(W \circ A)(t) = \prec \nabla W(A(t)), \frac{dA}{dt}(t) \succ = W(A(t))\|v\|^2 > 0,$$

therefore W is increasing on ImA, and the maximum value of W on S is attained at $A(0) = s_{max}$. Moreover $C(t) \notin S$ for $0 < t \leq t_2$ because

$$\frac{d}{dt}(W \circ C)(t) = \prec \nabla W(C(t)), \frac{dC}{dt}(t) \succ = -W(A(t))\|v\|^2 < 0,$$

thus W is decreasing on ImC, and the minimum value of W on S is attained at $C(0) = s_{min}$. Finally $B \circ \alpha(t) \notin S$ for $t_1 \leq t \leq t_1 + l_2 - l_1$ since $B \circ \alpha(t) \in \partial D(0; r)$ and $\partial D(0; r) \cap S = \emptyset$. If $N_S(s_{max}) = N_S(s_{min})$ then

$$\frac{d}{dt}(f \circ \psi)(t)_{|t=0} = \prec \|\nabla f(s_{max})\| N_S(s_{max}), v \succ,$$

and

$$\frac{d}{dt}(f \circ \psi)(t)_{|t=t_1+t_2+l_2-l_1} = \prec \|\nabla f(s_{min})\| N_S(s_{min}), v \succ,$$

so the derivative $f \circ \psi$ has the same sign at both endpoints. Therefore $f \circ \psi$ would either be increasing at both end points or be decreasing at them. Since $(f \circ \psi)(0) = (f \circ \psi)(t_1 + t_2 +$

$l_2 - l_1) = c$ this would imply that there exists $a \in]0, t_1 + t_2 + l_2 - l_1[$ such that $(f \circ \psi)(a) = c$, contradicting (4.1). □

The following examples show that the compactness is essential in the hypotheses of Theorem 4.2.

Example 4.3. Let $f : \mathbb{R}^4 \to \mathbb{R}$ given by $f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$ and $S = f^{-1}(1)$. Then S is a non-compact hypersurface. A simple calculation shows that the image of the oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping is a proper subset of S^3.

Example 4.4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = x_1^2 + x_2^2$ and $S = f^{-1}(1)$. Then S is a non-compact hypersurface. A simple calculation shows that the image of the oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping is a proper subset of S^2 (Figure 4).

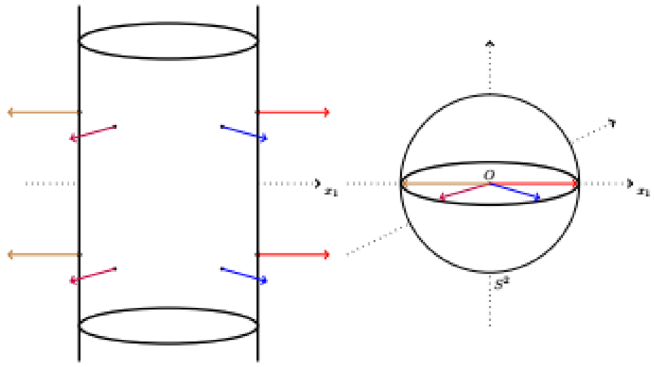

FIGURE 4. The image of the circular cylinder $x_1^2 + x_2^2 = 1$ under its Gauss mapping is a proper subset of S^2.

The following examples show that the converse of Theorem 4.2 is not true.

Example 4.5. Let $f : \mathbb{R}^m \to \mathbb{R}$ given by

$$f(x_1, \cdots, x_m) = (x_1^2 + \cdots + x_m^2 - 1)(x_1 - 2),$$

and $S = f^{-1}(0)$. A simple calculation shows that the image of the oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping is S^{m-1}, but S is neither compact nor connected.

Example 4.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = (x_1^2 - 1)^2 + x_2^2$ and $S = f^{-1}(c)$ for some $0 < c < 1$. A simple calculation shows that $N_S(S) = S^1$ and S is compact but not connected.

Example 4.7. Let $g_i : \mathbb{R}^m \to \mathbb{R}$ given by

$$g_i(x_1, \ldots, x_m) = x_1^2 + \cdots + x_m^2 - i$$

for $i = 2, 3$, and $S_i = g_i^{-1}(0)$. Then Theorem 2.3 shows that $S = S_1 \cup S_2$ is a hypersurface whose image under its Gauss mapping is S^{m-1}, but however it is not diffeomorphic to S^{m-1}.

In the following example, the image of a path connected oriented hypersurface $(S, \mathbf{N}_S)$ under its Gauss mapping is S^1, but S is not compact.

Example 4.8. Let $U = \mathbb{R}^2 - \{0\}$ and $f : U \to \mathbb{R}$ be defined by $f(r, \theta) = r - \theta$, where $r > 0$ and $\theta > 0$ are the polar coordinates of the non-zero point (x, y). A simple calculation shows

that
$$\frac{\partial f}{\partial x} = \frac{r\cos\theta + \sin\theta}{r}, \quad \frac{\partial f}{\partial y} = \frac{r\sin\theta - \cos\theta}{r}.$$
Therefore $\nabla f(r,\theta) \neq 0$ for all $(r,\theta) \in U$, so $S = \{(r,\theta) \in U | r = \theta\} = f^{-1}(0)$, the *Archimedean spiral*, is a hypersurface. Let ϕ be an angle such that $\cos\phi = \frac{r}{\sqrt{r^2+1}}$ and $\sin\phi = \frac{-1}{\sqrt{r^2+1}}$, then the Gauss mapping of S at $(r,\theta) \in S$ is
$$\begin{aligned} N_S(r,\theta) &= \left(\frac{r\cos\theta + \sin\theta}{\sqrt{r^2+1}}, \frac{r\sin\theta - \cos\theta}{\sqrt{r^2+1}}\right) \\ &= (\cos(\theta+\phi), \sin(\theta+\phi)). \end{aligned}$$
Since $\theta = r$ and $\phi = -\sin^{-1}\frac{1}{\sqrt{r^2+1}}$ are continuous functions of r, then $\theta + \phi$ changes continuously in $]-\frac{\pi}{2}, +\infty[$. Hence, $N_S : S \to S^1$ is a surjective function (Figure 5).

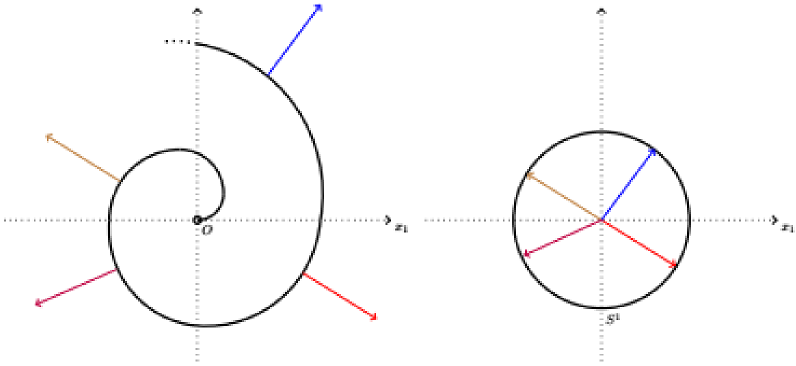

FIGURE 5. Archimedean spiral is a path connected oriented hypersurface. It is a non-compact space, but the image of it under its Gauss mapping is S^1.

The following example shows that the images of two diffeomorphic non-compact hypersurface under their Gauss mappings may be not same.

Example 4.9. Put $U_1 = \mathbb{R}^3 - \{(0,0,1)\}$ and define $f_1 : U_1 \to \mathbb{R}$ by $f_1(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2$. The image of $(S_1, \mathbf{N}_{S_1})$ for $S_1 = f_1^{-1}(1)$ (the *punctured Riemann sphere*) under its Gauss mapping with orientations induced by the vector fields $\pm\nabla f_1$ is $\pm S_1$. Now, let $U_2 = \mathbb{R}^3$ and define $f_2 : U_2 \to \mathbb{R}$ by $f_2(x_1,x_2,x_3) = x_3$. The image of $(S_2, \mathbf{N}_{S_2})$ for $S_2 = f_2^{-1}(0)$ under its Gauss mapping with orientations induced by the vector fields $\pm\nabla f_2$ is $(0,0,\pm 1)$. Although the *stereographic projection* makes S_1 and S_2 diffeomorphic, but clearly the images of them under their Gauss mappings are not diffeomorphic.

Any one-to-one continuous function f from a compact topological space X to a Hausdorff space has a continuous inverse on $f(X)$ [39]. Thus, the following statement is valid:

Corollary 4.10. *If $(S, \mathbf{N}_S)$ is a compact oriented hypersurface and N_S is one-to-one, then N_S^{-1} is continuous, and S is (path) connected.*

5. Gauss mapping, some cross sectional results

In the following, we show that the behavior of the Gauss mapping has a basic role in determination of the evenness or oddness of the cardinality of $Im\alpha \cap S$, for the curve $\alpha : \mathbb{R} \to \mathbb{R}^m$ which is nowhere tangent to the compact hypersurface S and tends to infinity in each direction.

Theorem 5.1. *Let $a, b \in \mathbb{R}$, $a < b$, $(S, \mathbf{N}_S)$ be an oriented hypersurface, and let $U \subseteq \mathbb{R}^m$ be an open set containing S.*

(1) *If $\alpha : \mathbb{R} \to U$ is a curve in U such that $\prec N(\alpha(t)), \frac{d\alpha}{dt}(t) \succ \neq 0$ for $\alpha(t) \in S$ and $\alpha(t) \cap S = \emptyset$ for $t \in]-\infty, a[\cup]b, +\infty[$, then the cardinal number of $Im(\alpha) \cap S$ is finite.*

(2) *If $\alpha(t_1), \alpha(t_2) \in S$ for $t_1 \neq t_2$ and $\alpha(t) \cap S = \emptyset$ for $t \in]t_1, t_2[$, then*

$$\prec N(\alpha(t_1)), \frac{d\alpha}{dt}(t_1) \succ \prec N(\alpha(t_2)), \frac{d\alpha}{dt}(t_2) \succ < 0.$$

Proof. (1) Let $S = f^{-1}(c)$ for the smooth function $f : U \to \mathbb{R}$. Define the smooth function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(\alpha(t))$, then $\frac{dg}{dt}(t) = \|\nabla f(\alpha(t))\| \prec N_S(\alpha(t)), \frac{d\alpha}{dt}(t) \succ \neq 0$ when $\alpha(t) \in S$. If $Im(\alpha) \cap S$ is not finite, then the Bolzano-Weierstrass theorem besides the compactness of $\alpha[a, b]$ would imply that $Im(\alpha) \cap S$ has an accumulation point $\bar{s} = \alpha(\bar{t}) \in S$ for some $\bar{t} \in [a, b]$. Now, without loss of generality it can be assumed that there exist sequences $\{\alpha(u_n)\}_{n\in\mathbb{N}}$ and $\{\alpha(v_n)\}_{n\in\mathbb{N}}$, each tends to $\alpha(\bar{t})$ such that $\prec N_S(\alpha(u_n)), \frac{d\alpha}{dt}(u_n) \succ > 0$, $\prec N_S(\alpha(v_n)), \frac{d\alpha}{dt}(v_n) \succ < 0$, hence $\prec N_S(\alpha(\bar{t})), \frac{d\alpha}{dt}(\bar{t}) \succ = 0$ [3]. This contradics the hypotheses of Theorem.

(2) Let for instance $\prec N_S(\alpha(t_1)), \frac{d\alpha}{dt}(t_1) \succ > 0$, $\prec N_S(\alpha(t_2)), \frac{d\alpha}{dt}(t_2) \succ > 0$. Then $\frac{dg}{dt}(t_1) > 0$ and $\frac{dg}{dt}(t_2) > 0$, so, the smoothness of g implies that there exists t_0, $t_1 < t_0 < t_2$ such that $g(t_0) = c$. This implies that $\alpha(t_0) \in S$ which is a contradiction.

□

The following example shows that the assumption "$\alpha(t) \cap S = \emptyset$ for $t \in]-\infty, a[\cup]b, +\infty[$" in Theorem 5.1(1) is not necessary.

Example 5.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1^2 + x_2^2$, $S = f^{-1}(1) \subset \mathbb{R}^2$ and let $\alpha(t) = (\sqrt{3} + \cos t, \sin t)$ for $t \in \mathbb{R}$. Then $(S, \mathbf{N}_S)$ is an oriented hypersurface, $\alpha(t) \cap S \neq \emptyset$ for $t = 2k\pi \pm \frac{5\pi}{6}, k \in \mathbb{Z}$, and $\prec N(\alpha(t)), \frac{d\alpha}{dt}(t) \succ \neq 0$ for $\alpha(t) \in S$, but the cardinal number of $Im(\alpha) \cap S$ is finite.

Theorem 5.3. *Let $a, b \in \mathbb{R}$, $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function, $(S, \mathbf{N}_S)$ be an oriented hypersurface such that $S = f^{-1}(c)$ for some $c \in \mathbb{R}$. Suppose that $\alpha : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ such that $\prec N(\alpha(t)), \frac{d\alpha}{dt}(t) \succ \neq 0$ when $\alpha(t) \in S$, and $\alpha(t) \notin S$ for all $t < a$ and $t > b$. If S is compact and $lim_{t\to-\infty}\|\alpha(t)\| = lim_{t\to+\infty}\|\alpha(t)\| = +\infty$, then the cardinal number of $Im(\alpha) \cap S$ is even.*

Proof. Theorem 5.1 implies that $\alpha(t_i) \cap S \neq \emptyset$ for a finite set of indices, for example, $i = 1, 2, \ldots, k$ such that $t_1 < t_2 < \cdots < t_{k-1} < t_k$. Suppose to get a contradiction that $\prec N(\alpha(t_1)), \frac{d\alpha}{dt}(t_1) \succ > 0$, $\prec N(\alpha(t_k)), \frac{d\alpha}{dt}(t_k) \succ > 0$. Therefore, for the smooth function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(\alpha(t))$ we have $\frac{dg}{dt}(t_1) > 0$, $\frac{dg}{dt}(t_k) > 0$, so $g(t) > c$ for $t > t_k$, and $g(t) < c$ for $t < t_1$. Since S is compact, there exists a disc $D(0; r) \subset \mathbb{R}^m$ such that $S \subset D(0; r)$ and $S \cap \partial D(0; r) = \emptyset$. The conditions

$$lim_{t\to-\infty}\|\alpha(t)\| = lim_{t\to+\infty}\|\alpha(t)\| = +\infty,$$

imply that there exist $w_k > t_k$ and $w_1 < t_1$ such that $\alpha(w_1), \alpha(w_k) \in \partial D(0; r)$ and $g(w_1) < c$, $g(w_k) > c$. Since $\partial D(0; r)$ is path connected, there exists a continuous function $\beta : [u, v] \to \partial D(0; r)$ for some $u, v \in \mathbb{R}$, such that $\beta(u) = \alpha(w_1)$ and $\beta(v) = \alpha(w_k)$. Let $h : [u, v] \to \mathbb{R}$ is defined by $h(t) = f(\beta(t))$, then $h(u) = f(\beta(u)) = f(\alpha(w_1)) = g(w_1) < c$ and $h(v) = f(\beta(v)) = f(\alpha(w_k)) = g(w_k) > c$. Therefore by Bolzano's intermediate value theorem, there exists $z \in]u, v[$ such that $h(z) = f(\beta(z)) = c$ or $\beta(z) \in S \cap \partial D(0; r)$, which is a contradiction. $\square$

Remark 5.4. The set $\mathbb{R}$ as the domain of α in Theorems 5.1 and 5.3 may be changed into an open interval, provided that the other hypotheses relating to this curve are modified accordingly.

The following example shows that if S is not compact, then for the curve α in Theorem 5.3, the cardinal number of $Im(\alpha) \cap S$ may be even or odd.

Example 5.5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = x_2^3 - 3x_1^2 x_2 - x_3$ and $S = f^{-1}(0)$. Then $(S, \mathbf{N}_S)$, the *monkey saddle* is a non-compact oriented hypersurface. The curve $\alpha(t) = (t, 3t, 18t)$ crosses S at three points $(-1, -3, -18), (0, 0, 0), (1, 3, 18) \in S$ for $t = -1, 0, 1$ respectively. Moreover,

$$\prec N(\alpha(t)), \frac{d\alpha}{dt}(t) \succ = \prec \frac{1}{\sqrt{900t^4 + 1}}(-18t^2, 24t^2, -1), (1, 3, 18) \succ = \frac{18(3t^2 - 1)}{\sqrt{900t^4 + 1}} \neq 0$$

for $t = 0, \pm 1$. Therefore, each of these points occures non tangentially. But $\beta(t) = (\sqrt{3}t, 3t, 1 - t^2)$ crosses S at two points $(-\sqrt{3}, -3, 0), (\sqrt{3}, 3, 0) \in S$ for $t = -1, 1$ respectively, and

$$\prec N(\beta(t)), \frac{d\beta}{dt}(t) \succ = \prec \frac{1}{\sqrt{1296t^4 + 1}}(-18\sqrt{3}t^2, 18t^2, -1), (\sqrt{3}, 3, -2t) \succ = \frac{2t}{\sqrt{1296t^4 + 1}} \neq 0$$

for ± 1, and β crosses S not tangentially at these two points (Figure 6).

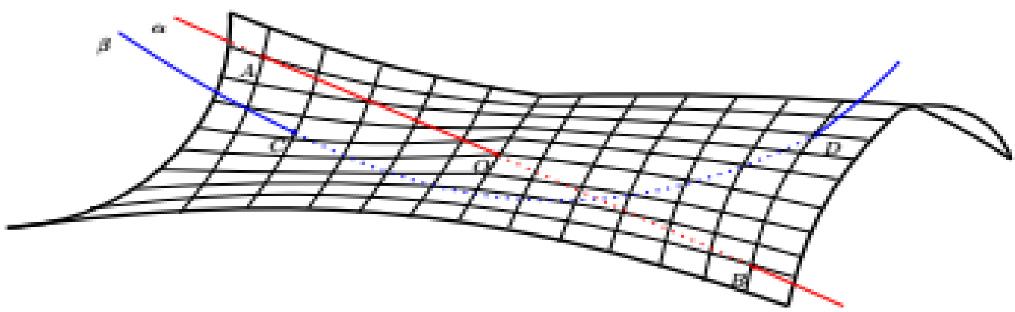

FIGURE 6. The monkey saddle is a non-compact oriented hypersurface. The curve α (res. β) is nowhere tangent to S and tends to infinity in directions $\pm\infty$. Moreover, the cardinal number of $Im(\alpha) \cap S$ (res. $Im(\beta) \cap S$) is odd (res. even).

5.1. **Application.** The complex zeros of a real polynomial occure in conjugate pairs. Hence, if $p_n(z) = \Sigma_{k=0}^{k=n} a_k z^k$ for real cofficients a_k, then the cardianl number of the set

$$Z(p_n) = \{u + iv |\ v \neq 0, p_n(u + iv) = 0\}$$

is even. Here, we explain this assertion as an application of the considerations in this section. Since the fundamental theorem of algebra implies that p_n for $n \geq 2$ can be decomposed into irreducible quadratic expressions on $\mathbb{R}$, then it suffices to paying attention to p_2 with $a_2 = 1$.

Theorem 5.6. [4] *Any real polynomial $p_2(z) = a_0 + a_1 z + z^2$ where $a_1^2 - 4a_0 < 0$ has two different complex zeros with equal Euclidean norms.*

In fact, suppose that $u+iv \in Z(p_2)$, then $u=-\frac{a_1}{2}$, $v^2=a_0-\frac{a_1^2}{4}$ and $u^2+v^2=a_0>0$. Define the curve $\alpha:\mathbb{R}\to\mathbb{R}^2$ and the function $f:\mathbb{R}^2\to\mathbb{R}$ by $\alpha(t)=(u,vt)$ and $f(x,y)=x^2+y^2$. Then $S=f^{-1}(a_0)$ oriented by gradient vector field, is a compact hypersurface and $lim_{t\to\pm\infty}\|\alpha(t)\|=+\infty$. Furthermore, $\alpha(t)\in S$ if and only if $t=\pm 1$ and

$$\prec N(\alpha(t)),\frac{d\alpha}{dt}(t)\succ=\prec(\frac{u}{\sqrt{a_0}},\frac{vt}{\sqrt{a_0}}),(0,v)\succ=\frac{v^2}{\sqrt{a_0}}t\neq 0.$$

Hence, as stated in Theorem 5.3, α crosses S at two different points with equal Euclidean norms, which both are the zeros of p_2.

6. Conclusions

Classification of smooth manifolds (required to be Hausdorff and second-countable) is one of the difficult problems in differential geometry. Although some new results has been made for higher dimensions, but the subject, in general, is still horrible. The strong Whitney embedding theorem states that any smooth real k-dimensional manifold M can be smoothly embedded in $\mathbb{R}^{2k}$ [46]. Therefore, the classification of submanifolds of $\mathbb{R}^m$ or some specific categories of them, for example "submanifolds of $\mathbb{R}^m$ of finite type" is valuable. These submanifolds are defined by introducing the properties of some special functions defined on them. For a submanifold M of $\mathbb{R}^m$, a smooth function ϕ on M is said to be of finite type if it can be expressed as a finite sum of eigenvectors of the Laplacian operator Δ of M, that is, $\phi=c+x_1+\cdots+x_k$ where c is a constant map and $x_1,\ldots,x_k$ are non-constant maps such that $\Delta\phi=\lambda_i x_i,\lambda_i\in\mathbb{R}, i=1,2,\ldots,k$. If all coordinate functions of $\mathbb{R}^m$ restricted to M are of finite type, then M is said to be of finite type. Otherwise, M is said to be of infinite type. If $\lambda_1,\ldots,\lambda_k$ are all different, then M is said to be of k-type. In particular, if one of $\{\lambda_1,\ldots,\lambda_k\}$ is zero, then M is said to be of null k-type [12]. Since the late 1970's the classification of finite type submanifolds, as a special case of submanifolds of Euclidean spaces, has been of interest to researchers and some open problems and conjectures related to them has been published, for example in [15]. A detailed survey of the results up to 1996 on this subject is given in [7]. In this reference, among other things, 20 open problems and 10 conjectures has been provided. Some of them are directly about the classification of finite type surfaces, hypersurfaces of finite type, compact finite type hypersurfaces, complete non-compact hypersurfaces of finite type, null 2-type surfaces, and submanifolds of an Euclidean space of 2-type.
The study of finite type submanifolds, have received a growing attention with many progresses since the beginning of this century [17, 31, 38]. But, a significant number of the results have been achieved are related to low dimensions, i.e., finite type surfaces in $\mathbb{R}^3$.

Here, to introduce the achivements, we begin by recalling some required definitions and notations.

The parametric equation of a hypersurface of revolution in $\mathbb{R}^3$ as stated in section 3 is $x(u,v)=(u,g(u)\cos v,g(u)\sin v)$ where $g(u)>0$ for u in an open interval I and $v\in\mathbb{R}$. The hypersurface x is said to be of the polynomial kind if $g(u)$ is a polynomial function of u, and it is said to be of the rational kind if g is a rational function of u, i.e., g is the quotient of two polynomial functions of u [12].

Let $\alpha:I\to\mathbb{R}^3$ be a unit speed curve with $\kappa>0$. Let $x(s,v)=\alpha(s)+\epsilon(\cos vN(s)+\sin vB(s))$, where $\epsilon>0$ is constant, $s\in I$, N is the normal vector and B is a binormal vector of α. The surface with parametrization $x = x(s, v)$, denoted by $T_\epsilon(\alpha)$, is called tube of radius ϵ around α [6, 16, 40]. A tube in $\mathbb{R}^3$ is called an anchor ring if the curve α is a plane circle or an open portion of it [16].

A ruled surface in $\mathbb{R}^3$ is a surface swept out by a straight line moving through $\mathbb{R}^3$. The parametric equation of this surface is $x(u, v) = \alpha(u) + v\beta(u)$ where $\alpha, \beta : I \to \mathbb{R}^3$ are smooth functions defined on an open interval I with β never 0, and $v \in J$ for an open interval J. The funtions α and β are called base curve and director curve respectively. Cylinders, cones and helicoids are some special ruled surfaces. If $a \in \mathbb{R}^3$ is a constant vector and $\beta(u) = a$, then $x(u, v) = \alpha(u) + va$ specifies the parametrization of a cylinder in $\mathbb{R}^3$. A cylinder is circular if the base curve is a circle. Clearly for a cylinder S with director curve $\beta(u) = a$, the following holds: if $s \in S$, then $s + ta \in S$ for sufficient small values of t.

A cone is a ruled surface with parametrization of the form $x(u, v) = p + v\beta(u)$ for fixed $p \in \mathbb{R}^3$.

A helicoid admits the parametrization $x(u, v) = \alpha(u)+v\beta(u)$ where $\alpha(u) = (0, 0, bu), b \neq 0$ and $\beta(u) = (\cos u, \sin u, 0)$ for $u, v \in \mathbb{R}$ [6, 40].

A quadric surface in $\mathbb{R}^3$ is a surface $S : f(x_1, x_2, x_3) = 0$ for which f involves at most quadratic terms in x_1, x_2, x_3, that is $f(x_1, x_2, x_3) = \Sigma_{1\leq i,j\leq 3}a_{ij}x_ix_j + \Sigma_{1\leq i\leq 3}b_ix_i + c$. Ellipsoids, one-sheeted elliptic hyperboloids, elliptic hyperboloids of two sheets, elliptic paraboloids and hyperbolic paraboloids are some types of quadric surfaces [40].

The mean curvature H of a surface S in $s \in S$, is the average of the principal curvatures of S at s. A surface with mean curvature identically zero is called a minimal surface. Hyperplanes and helicoids are some examples of minimal surfaces [6, 40]. Hypersurfaces of revolution with parametrizations $x(u, v) = (u, g(u) \cos v, g(u) \sin v)$, in general, are not minimal surfaces, but a simple calculation shows that if $g(u) = \cosh u$, then x is minimal. Some of the other minimal surfaces are *Enneper's surface* and *Scherk's surface* [6, 40].

In the following, some of the theorems about finite type surfaces, related to the concepts and results of the present paper are provided.

Theorem 6.1. [12] *The spheres and circular cylinders are the surfaces of finite type in* $\mathbb{R}^3$.

Theorem 6.2. [16] *Every circular cylinder in* $\mathbb{R}^3$ *is of* 2*-type.*

Theorem 6.3. [13] *The only null* 2*-type surfaces in* $\mathbb{R}^3$ *are open portions of circular cylinders.*

Theorem 6.4. [12] *Let S be a surface of revolution of the polynomial kind. Then S is a surface of finite type if and only if S is either an open portion of a plane, or an open portion of a circular cylinder.*

Theorem 6.5. [12] *Let S be a surface of revolution of the rational kind. Then S is a surface of finite type if and only if S is an open portion of a plane.*

Theorem 6.6. [16] *Circular cylinders are the only tubes of finite type in* $\mathbb{R}^3$.

Theorem 6.7. [13] *Null* 2*-type surfaces in* $\mathbb{R}^3$ *are open portions of circular cylinders.*

Theorem 6.8. [10] *A ruled surface in* $\mathbb{R}^3$ *is of finite type if and only if it is an open portion of a plane, of a circular cylinder, or of a helicoid.*

Theorem 6.9. [24] *Planes are the only finite type cones in* $\mathbb{R}^3$.

Theorem 6.10. [9] *The only quadrics of finite type in Euclidean* 3*-space are the circular cylinder and the sphere.*

Theorem 6.11. [12] *Minimal surfaces are of* 1*-type.*

Theorem 6.12. [48] *A surface in* $\mathbb{R}^3$ *is of* 1*-type if and only if either it is a minimal surface of* $\mathbb{R}^3$*, or it is an open portion of an ordinary sphere.*

Using the statements of this article and the above theorems, one can obtain some new results about hypersurfaces of finite type and images of them under their Gauss mappings. In the following, some of these results are provided.

Corollary 3.6, Theorem 3.7 and Theorem 6.6 imply that:

Theorem 6.13. *The image of a tube of finite type in* $\mathbb{R}^3$ *under its Gauss mapping is contained in a hyperplane.*

As a consequence of Corollary 3.6, Theorem 3.7 and Theorem 6.7 we have:

Theorem 6.14. *The image of a null* 2*-type hypersurface in* $\mathbb{R}^3$ *under its Gauss mapping is contained in a hyperplane.*

Corollary 3.3 and Theorem 6.9 assert that:

Theorem 6.15. *Let* S *be a hypersurface in* $\mathbb{R}^3$*. Then* S *is of finite type cone if and only if the image of* S *under its Gauss mapping is a single point.*

Corollary 3.3, Corollary 3.6, Theorem 3.7 and Theorem 6.4 show that:

Theorem 6.16. *Let* S *be a surface of revolution of the polynomial kind. If* S *is a surface of finite type, then the image of* S *under its Gauss mapping is a single point or is contained in a hyperplane.*

The following Theorem is a consequene of Corollary 3.3, Corollary 3.6 and Theorem 6.5.

Theorem 6.17. *Let* S *be a surface of revolution of the rational kind. Then* S *is a surface of finite type if and only if the image of* S *under its Gauss mapping is a single point.*

As a result of above considerations and Theorems 6.11 and 6.12 we have:

Theorem 6.18. *Enneper's surface, Scherk's surface and the hypersurface of revolution with parametrization*

$$x(u, v) = (u, \cosh u \cos v, \cosh u \sin v)$$

are of 1*-type.*

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