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Adapted connections on foliated manifolds

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Abstract. In this study, we define the local components of the adapted connection relative to the adapted frame field. We also calculate the covariant derivative of a tensor with respect to this connection. Furthermore, we present a classification of totally geodesic foliations and bundle-like metrics, along with the introduction of the local components of the torsion tensor associated with this connection. To illustrate our findings, we provide a relevant example.

Keywords: Semi-Riemannian manifold, Adapted connection.

1. **Introduction**

The general form of adapted linear connections on foliated semi-Riemannian manifolds have been expressed in [2]. Schouten-Van Kampen and Vranceanu connections are two important examples of adapted linear connections studied in several works such as ([3], [6], [7], [8], [9]).

We organize our present work as follows: after introduction in section 1, first we conclude all the local coefficients of the adapted connections with respect to an adapted frame field. Then the transversal and structural covariant derivatives of an adapted tensor fields have been find. Also, we obtained new

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characterizations of some classes of foliations. In the last section, we construct an example of a foliated semi-Riemannian manifold endowed with an adapted linear connection and prove some results.

In the following, we present the definitions and theorems that are used in the main results of the paper.

Let M be an $(n+p)-$dimentional manifold. If a distribution $\mathcal{D}$ on M is given, The real vector space of all symmetric bilinear mappings $g_x : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow \mathbf{R}$, is defined by $L^2_s(\mathcal{D}_x, \mathbf{R})$. Then we consider the vector bundle

$$L^2_s(\mathcal{D}, \mathbf{R}) = \bigcup_{x\in M} L^2_s(\mathcal{D}_x, \mathbf{R}).$$

over M, and given the following definitions.then the semi-Riemannian metric of constant index on $\mathcal{D}$ is a smooth section $g : x \longrightarrow g_x$ of $L^2_s(\mathcal{D}, \mathbf{R})$ such that each g_x is non-degenerate of constant index on $\mathcal{D}_x$ for all $x \in M$. We say that $(\mathcal{D}, g)$ is a semi-Riemannian distribution of constant index. If in particular $\mathcal{D} = TM$, then g becomes a semi-Riemannian metric on M and (M, g) is called a semi-Riemannian manifold [4].

Let $\mathcal{F} = \{L_t\}$ be a foliation on M with tangent distribution $\mathcal{D}$, that is, for any $x \in M$, $\mathcal{D}_x = T_xL_t$, where L_t is the leaf of $\mathcal{F}$ passing through x. We assume that $(\mathcal{D}, g)$ is semi-Riemannian distribution, then $(M, g, \mathcal{F})$ is a foliated semi-Riemannian manifold.

Then a complementary distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ in TM can be obtained. Indeed, since M is para-compact and of differentiability class C^∞, there exists on M a Riemannian metric of class C^∞. Then we can take $\mathcal{D}^\perp$ as the complementary orthogonal distribution to $\mathcal{D}$ with respect to that metric, that is, we have

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp. \tag{1.1}$$

Denote by $\mathcal{P}$ and $\mathcal{Q}$ the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively. The tensor field $F = \mathcal{P} - \mathcal{Q}$ is an almost product structure on M, that is $F^2 = I$, where I is the identity morphism on TM. Therefore we call $(M, \mathcal{D}, \mathcal{D}^\perp)$ an almost product manifold.

We first construct a local frame field adapted to (1.1) as follows. Let $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\},\ i \in \{1, \dots n\},\ \alpha \in \{n+1, \dots, n+p\}$, be a foliated chart on $(M, \mathcal{F})$, that is, $\mathcal{D}$ is locally represented on $\mathcal{U}$ by the natural field of frames $\{\frac{\partial}{\partial x^1}, \dots \{\frac{\partial}{\partial x^n}\}$. If $\{e_{n+1}, ..., e_{n+p}\}$ locally represents $\mathcal{D}^\perp$ on $\mathcal{U}$, then $\{\frac{\partial}{\partial x^i}, e_\alpha\}$ is a frame field on $\mathcal{U}$ with respect to (1.1). Now we express each $\{\frac{\partial}{\partial x^\alpha}\}$ with respect to this frame field:

$$\frac{\partial}{\partial x^\alpha} = A^i_\alpha \frac{\partial}{\partial x^i} + A^\beta_\alpha e_\beta. \tag{1.2}$$

Since $A = \begin{bmatrix} \delta^i_j & A^i_\alpha \\ 0 & A^\beta_\alpha \end{bmatrix}$ is the transition matrix from the frame field $\{\frac{\partial}{\partial x^i}, e_\alpha\}$ to the natural frame field $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\}$, therefore $[A^\alpha_\beta]$ is a non-singular matrix of functions on $\mathcal{U}$. thus

$$\frac{\delta}{\delta x^\alpha} = A^\beta_\alpha e_\beta,\ \alpha \in \{n+1, ..., n+p\},$$

also represent locally $\mathcal{D}^\perp$ on $\mathcal{U}$. So (1.2) becomes

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A^i_\alpha \frac{\partial}{\partial x^i},\ \alpha \in \{n+1, \dots n+p\}, \tag{1.3}$$

We call $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$ the adapted frame field on $(M, \mathcal{F})$. Vector fields of the form (1.3) have been used in [5] and [9]. Next we consider the dual vector bundles $\mathcal{D}^*$ and ${\mathcal{D}^\perp}^*$ to $\mathcal{D}$ and $\mathcal{D}^\perp$respectively. Then an adapted tensor field of type $(q, s; r, t)$ on the foliated manifold $(M, \mathcal{F})$ is an $F(M) - (q+r+s+t)$ multi-linear mapping

$$T : \Gamma(\mathcal{D}^*)^q \times \Gamma({\mathcal{D}^\perp}^*)^r \times \Gamma(\mathcal{D})^s \times \Gamma({\mathcal{D}^\perp}^t) \longrightarrow F(M).$$

In orther to define the local components of T we consider the dual adapted frame field $\{\delta x^i, dx^\alpha\}$ on $(M, \mathcal{F})$, where

$$\delta x^i = dx^i + A^i_\alpha dx^\alpha.$$

Thus, locally T is given by smooth functions

$$\begin{aligned} T^{i_1 \dots i_q\, \alpha_1 \dots \alpha_r}_{j_1 \dots j_s\, \beta_1 \dots \beta_t} = T\big(&\delta x^{i_1}, \dots, \delta x^{i_q}, dx^{\alpha_1}, \dots, dx^{\alpha_r}, \\ &\tfrac{\partial}{\partial x^{j_1}}, \dots, \tfrac{\partial}{\partial x^{j_s}}, \tfrac{\delta}{\delta x^{\beta_1}}, \dots, \tfrac{\delta}{\delta x^{\beta_t}}\big). \end{aligned} \tag{1.4}$$

Lemma 1.1. [1] *Let* $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$ *be an adapted frame field on* $(M, \mathcal{F})$. *Then we have*

$$a)\ \big[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\big] = I^i_{\alpha\beta} \frac{\partial}{\partial x^i}, \quad b)\ \big[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\big] = A_i{}^j{}_\alpha \frac{\partial}{\partial x^j}, \tag{1.5}$$

where we set

$$a)\ I^i_{\alpha\beta} = \frac{\delta A^i_\alpha}{\delta x^\beta} - \frac{\delta A^i_\beta}{\delta x^\alpha}, \quad b)\ A^j_{i\,\alpha} = \frac{\partial A^j_\alpha}{\partial x^i}. \tag{1.6}$$

Definition 1.2. A linear connection ∇ on an almost product manifold M is said to be an adapted linear connection if the following conditions are satisfied:

$$a)\ \nabla_X \mathcal{P}Y \in \Gamma(\mathcal{D}), \quad b)\ \nabla_X \mathcal{Q}Y \in \Gamma(\mathcal{D}^\perp), \forall X, Y \in \Gamma(TM). \tag{1.7}$$

We recall the following Theorem ([2], p. 16)

Theorem 1.3. *Let* $(M, \mathcal{D}, \mathcal{D}^\perp)$ *be an almost product manifold and* $\tilde{\nabla}$ *be linear connection on* M. *Then all adapted linear connections on* M *are given by*

$$\nabla_X Y = \mathcal{P}\tilde{\nabla}_X \mathcal{P}Y + \mathcal{Q}\tilde{\nabla}_X \mathcal{Q}Y + \mathcal{P}S(X, \mathcal{P}Y) + \mathcal{Q}S(X, \mathcal{Q}Y), \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$, where S is an arbitrary tensor field of type (1,2) on M.

$\tilde{\nabla}$ expressed locally as follows [1]

$$\begin{aligned}
(a)\quad & \tilde{\nabla}_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F^\gamma_{\alpha\beta} \frac{\delta}{\delta x^\gamma} + G^i_{\alpha\beta} \frac{\partial}{\partial x^i}, \\
(b)\quad & \tilde{\nabla}_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = H^\gamma_{i\alpha} \frac{\delta}{\delta x^\gamma} + K^j_{i\alpha} \frac{\partial}{\partial x^j}, \\
(c)\quad & \tilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = L^\gamma_{\alpha i} \frac{\delta}{\delta x^\gamma} + M^j_{\alpha i} \frac{\partial}{\partial x^j}, \\
(d)\quad & \tilde{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = N^\gamma_{ij} \frac{\delta}{\delta x^\gamma} + P^k_{ij} \frac{\partial}{\partial x^k}.
\end{aligned} \tag{1.9}$$

where

$$(F^\gamma_{\alpha\beta},\ G^i_{\alpha\beta},\ H^\gamma_{i\alpha},\ K^j_{i\alpha},\ L^\gamma_{\alpha i},\ M^j_{\alpha i},\ N^\gamma_{ij},\ P^k_{ij})$$

are the local coefficients with respect to the adapted frame field

$$\left\{ \frac{\partial}{\partial x^i},\ \frac{\delta}{\delta x^\alpha} \right\},$$

and satisfy the identities:

$$\begin{aligned}
&(a)\ \ F_{\alpha\beta}{}^{\gamma} = F_{\beta\alpha}{}^{\gamma}, && (b)\ G_{\alpha\beta}{}^{i} - G_{\beta\alpha}{}^{i} = I_{\alpha\beta}{}^{i}, \\
&(c)\ \ H_{i\alpha}{}^{\gamma} = L_{\alpha i}{}^{\gamma}, && (d)\ K_{i\alpha}{}^{j} - M_{\alpha i}{}^{j} = A_{i\alpha}{}^{j}, \\
&(e)\ \ H_{i\alpha}{}^{\gamma} = -\, g^{\gamma\beta}\, G_{\alpha\beta}{}^{j}\, g_{ij}, && (f)\ N_{ij}{}^{\gamma} = -\, g^{\gamma\alpha}\, M_{\alpha i}{}^{h}\, g_{hj}, \\
&(g)\ \ P_{ij}{}^{k} = P_{ji}{}^{k}.
\end{aligned} \tag{1.10}$$

2. **Adapted Connections**

It is the purpose of this section to obtain all the local coefficients of the adapted connection ∇ with respect to an adapted frame field $\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\}$. First, we put

$$
\begin{aligned}
&(a)\ S\left(\frac{\delta}{\delta x^\alpha},\frac{\delta}{\delta x^\beta}\right)=S^\gamma_{\alpha\beta}\frac{\delta}{\delta x^\gamma}+S^i_{\alpha\beta}\frac{\partial}{\partial x^i},\\
&(b)\ S\left(\frac{\delta}{\delta x^\alpha},\frac{\partial}{\partial x^i}\right)=S^\gamma_{\alpha i}\frac{\delta}{\delta x^\gamma}+S^j_{\alpha i}\frac{\partial}{\partial x^j},\\
&(c)\ S\left(\frac{\partial}{\partial x^i},\frac{\delta}{\delta x^\alpha}\right)=S^\gamma_{i\alpha}\frac{\delta}{\delta x^\gamma}+S^j_{i\alpha}\frac{\partial}{\partial x^j},\\
&(d)\ S\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)=S^\gamma_{ij}\frac{\delta}{\delta x^\gamma}+S^k_{ij}\frac{\partial}{\partial x^k}.
\end{aligned}
\tag{2.1}
$$

Next by using relations (1.9) and (2.1) we can state the following.

Lemma 2.1. *The adapted connections are locally given by the following formula:*

$$
\begin{aligned}
&(a)\ \nabla_{\frac{\delta}{\delta x^\beta}}\frac{\delta}{\delta x^\alpha}=\left(F^\gamma_{\alpha\beta}+S^\gamma_{\beta\alpha}\right)\frac{\delta}{\delta x^\gamma},\\
&(b)\ \nabla_{\frac{\delta}{\delta x^\alpha}}\frac{\partial}{\partial x^i}=\left(K^j_{i\alpha}+S^j_{\alpha i}\right)\frac{\partial}{\partial x^j},\\
&(c)\ \nabla_{\frac{\partial}{\partial x^i}}\frac{\delta}{\delta x^\alpha}=\left(L^\gamma_{\alpha i}+S^\gamma_{i\alpha}\right)\frac{\delta}{\delta x^\gamma},\\
&(d)\ \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}=\left(P^k_{ij}+S^k_{ji}\right)\frac{\partial}{\partial x^k}.
\end{aligned}
$$

Let $(T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t})$ be an adapted tensor field of type $(q,s;r,t)$. Then the transversal covariant derivative of T with respect to ∇ is given by

$$
\begin{aligned}
(T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t|\gamma}) &= \frac{\delta T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}}{\delta x^\gamma}+\Sigma^q_{x=1}T^{i_1\dots h\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}(K^{i_x}_{h\gamma}+S^{i_x}_{\gamma h})\\
&+ \Sigma^r_{y=1}(T^{i_1\dots i_q\alpha_1\dots\varepsilon\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t|\gamma}(F^{\alpha y}_{\varepsilon\gamma}+S^{\alpha_y}_{\gamma\varepsilon})\\
&- \Sigma^s_{z=1}T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots h\dots j_s\beta_1\dots\beta_t|\gamma}(K^h_{j_z\gamma}+S^h_{\gamma j_z})\\
&- \Sigma^t_{u=1}T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\varepsilon\dots\beta_t|\gamma}(F^\varepsilon_{\beta_u\gamma}+S^\varepsilon_{\gamma\beta_u}).
\end{aligned}
$$

Similarly, the structural covariant derivative of T with respect to ∇ is given by

$$\begin{aligned}(T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t||k}) &= \frac{\partial T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t}}{\partial x^k} + \Sigma_{x=1}^{q} T^{i_1\ldots h\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t}(P^{i_x}_{hk} + S^{i_x}_{kh}) \\ &+ \Sigma_{y=1}^{r}(T^{i_1\ldots i_q\alpha_1\ldots\varepsilon\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\beta_t|\gamma}(L^{\alpha y}_{\varepsilon k} + S^{\alpha_y}_{k\varepsilon}) \\ &- \Sigma_{z=1}^{s} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots h\ldots j_s\beta_1\ldots\beta_t|\gamma}(P^{h}_{j_z k} + S^{h}_{kj_z}) \\ &- \Sigma_{u=1}^{t} T^{i_1\ldots i_q\alpha_1\ldots\alpha_r}_{j_1\ldots j_s\beta_1\ldots\varepsilon\ldots\beta_t|\gamma}(L^{\varepsilon}_{\beta_u k} + S^{\varepsilon}_{k\beta_u}).\end{aligned}$$

Theorem 2.2. *The adapted transversal and structural covariant derivative of g_{ij} and $g_{\alpha\beta}$ are given by*

(a)

$$g_{ij|\gamma} = \frac{\delta g_{ij}}{\delta x^\gamma} - g_{hj}\big(K_{i\gamma}{}^{h} + S_{\gamma i}{}^{h}\big) - g_{ih}\big(K_{j\gamma}{}^{h} + S_{\gamma j}{}^{h}\big).$$

(b)

$$g_{\alpha\beta|\gamma} = \frac{\delta g_{\alpha\beta}}{\delta x^\gamma} - g_{\varepsilon\beta}\big(F_{\alpha\gamma}{}^{\varepsilon} + S_{\gamma\alpha}{}^{\varepsilon}\big) - g_{\alpha\varepsilon}\big(F_{\beta\gamma}{}^{\varepsilon} + S_{\gamma\beta}{}^{\varepsilon}\big).$$

(c)

$$g_{ij\|k} = \frac{\partial g_{ij}}{\partial x^k} - g_{hj}\big(P_{ik}{}^{h} + S_{ki}{}^{h}\big) - g_{ih}\big(P_{jk}{}^{h} + S_{kj}{}^{h}\big).$$

(d)

$$g_{\alpha\beta\|k} = \frac{\partial g_{\alpha\beta}}{\partial x^k} - g_{\varepsilon\beta}\big(L_{\alpha k}{}^{\varepsilon} + S_{k\alpha}{}^{\varepsilon}\big) + g_{\alpha\varepsilon}\big(L_{\beta k}{}^{\varepsilon} + S_{k\beta}{}^{\varepsilon}\big).$$

where

$$g_{\alpha\beta} = g\big(\tfrac{\delta}{\delta x^\alpha}, \tfrac{\delta}{\delta x^\beta}\big), \quad g_{ij} = g\big(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big).$$

We say that the Semi-Riemannian metric g on M is bundle-like for the foliation $\mathcal{F}$ if each geodesic in M which is tangent to the transversal distribution $\mathcal{D}^\perp$ at one point remains tangent for its entire length. g is bundle-like for $\mathcal{F}$ if and only if

$$h_s^\perp(QX, QY) = 0, \ \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where $h^\perp(\mathcal{Q}X, \mathcal{Q}Y) = \mathcal{P}\tilde{\nabla}_{\mathcal{Q}X}\mathcal{Q}Y$ is the second fundamental form of $\mathcal{D}^\perp$

Theorem 2.3. *Let $(M, g, \mathcal{F})$ be a foliated Semi-Riemannian manifold. Then the following conditions are equivalent:*

a) *g is a bundle-like metric for $\mathcal{F}$.*
b) $G^i_{\alpha\beta} = \frac{1}{2}\Gamma^i_{\alpha\beta}, \ \forall\alpha, \beta \in \{n+1, ..., n+p\}, \ i \in \{1, ..., n\}$.
c) $G^i_{\alpha\beta} + G^i_{\beta\alpha} = 0, \ \forall\alpha, \beta \in \{n+1, ..., n+p\}, \ i \in \{1, ..., n\}$.

Proof. Let g be bundle-like, so by (2.2) we have

$$\mathcal{P}\tilde{\nabla}_{\mathcal{Q}X}\mathcal{Q}Y+\mathcal{P}\tilde{\nabla}_{\mathcal{Q}Y}\mathcal{Q}X=0.$$

Since $\tilde{\nabla}$ is torsion-free, we have

$$\mathcal{P}\tilde{\nabla}_{\mathcal{Q}X}\mathcal{Q}Y=\tfrac{1}{2}\mathcal{P}\left[\mathcal{Q}X,\mathcal{Q}Y\right].$$

We replace $\mathcal{Q}X$ and $\mathcal{Q}Y$ by $\frac{\delta}{\delta x^{\beta}}$ and $\frac{\delta}{\delta x^{\alpha}}$. Therefore, by Lemma 1.1(a), we obtain $G_{\alpha\beta}^{\ i}=\frac{1}{2}I_{\alpha\beta}^{\ i}$, and since $I_{\alpha\beta}^{\ i}+I_{\beta\alpha}^{\ i}=0$, we have $G_{\alpha\beta}^{\ i}+G_{\beta\alpha}^{\ i}=0$. □

First, if each leaf of $\mathcal{F}$ is a totally geodesic submanifold of (M,g) we say that $\mathcal{F}$ is totally geodesic foliation. We present the next Theorem which have been proved by Bejancu and Farran [2]

Theorem 2.4. *$\mathcal{F}$ is totally geodesic if and only if one of the following conditions is satisfied:*

a) *The second fundamental form of $\mathcal{F}$ vanishes identically on M, i.e., we have*

$$h(\mathcal{P}X,\mathcal{P}Y)=\mathcal{Q}\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}Y=0,\ \forall X,Y\in\Gamma(TM). \tag{2.3}$$

b) *The shape operator of the structural distribution $\mathcal{D}$ vanishes identically on M, i.e., we have*

$$A_{\mathcal{Q}X}\mathcal{P}Y=-\mathcal{P}\tilde{\nabla}_{\mathcal{P}Y}\mathcal{Q}X=0,\ \forall X,Y\in\Gamma(TM). \tag{2.4}$$

From Theorem 2.4 and definition of ∇ we deduce the following.

Theorem 2.5. *Let (M,g) be a Semi-Riemannian manifold and $\mathcal{F}$ be a foliation on M. Then the following assertions are equivalent:*

a) *$\mathcal{F}$ is a totally geodesic foliation.*

b) $\nabla_{\mathcal{P}X}\mathcal{P}Y=\tilde{\nabla}_{\mathcal{P}X}\mathcal{P}Y+\mathcal{P}S(\mathcal{P}X,\mathcal{P}Y)$.

c) $\nabla_{\mathcal{P}Y}\mathcal{Q}X=\tilde{\nabla}_{\mathcal{P}Y}\mathcal{Q}X+\mathcal{Q}S(\mathcal{P}Y,\mathcal{Q}X)$.

The adapted connection ∇ has a torsion tensor field T given by

$$T(X,Y)=\nabla_XY-\nabla_YX-[X,Y],\ \forall X,Y\in\Gamma(TM). \tag{2.5}$$

In particular, on $\Gamma(\mathcal{D})$ is

$$T(\mathcal{P}X,\mathcal{P}Y)=\mathcal{P}S(\mathcal{P}X,\mathcal{P}Y)-\mathcal{P}S(\mathcal{P}Y,\mathcal{P}X).$$

Then with respect to an adapted frame field $\{\frac{\partial}{\partial x^i},\frac{\delta}{\delta x^{\alpha}}\}$ we put:

$$(a)\ \ T\left(\frac{\delta}{\delta x^{\alpha}},\frac{\delta}{\delta x^{\beta}}\right)=T_{\alpha\beta}^{\ \ \gamma}\frac{\delta}{\delta x^{\gamma}}+T_{\alpha\beta}^{\ \ i}\frac{\partial}{\partial x^{i}},$$

$$(b)\ \ T\left(\frac{\delta}{\delta x^{\alpha}},\frac{\partial}{\partial x^{i}}\right)=-T\left(\frac{\partial}{\partial x^{i}},\frac{\delta}{\delta x^{\alpha}}\right)=T_{i\alpha}^{\ \ \gamma}\frac{\delta}{\delta x^{\gamma}}+T_{i\alpha}^{\ \ j}\frac{\partial}{\partial x^{j}},$$

$$(c)\ \ T\left(\frac{\partial}{\partial x^{j}},\frac{\partial}{\partial x^{i}}\right)=T_{ij}^{\ \ \gamma}\frac{\delta}{\delta x^{\gamma}}+T_{ij}^{\ \ k}\frac{\partial}{\partial x^{k}}. \tag{2.6}$$

Then by direct calculations we obtain all components of T as in the next Lemma.

Lemma 2.6. *The local components of torsion tensor field of the adapted connection are given by*

$$\begin{aligned}
&(a)\ \ T_{\alpha\beta}{}^{\gamma} = S_{\beta\alpha}{}^{\gamma} - S_{\alpha\beta}{}^{\gamma}, && (b)\ \ T_{\alpha\beta}{}^{i} = -I_{\alpha\beta}{}^{i}, \\
&(c)\ \ T_{i\alpha}{}^{\gamma} = -(L_{\alpha i}{}^{\gamma} + S_{i\alpha}{}^{\gamma}), && (d)\ \ T_{i\alpha}{}^{j} = K_{i\alpha}{}^{j} + S_{\alpha i}{}^{j} - A_{i\alpha}{}^{j}, \\
&(e)\ \ T_{ij}{}^{\gamma} = 0, && (f)\ \ T_{ij}{}^{k} = S_{ji}{}^{k} - S_{ij}{}^{k}.
\end{aligned} \tag{2.7}$$

Proof. By direct calculations using Lemma 2.1 and (2.5). □

3. **Example**

We suppose that

$$\left\{ e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = e^{x+y}\frac{\partial}{\partial z} \right\}$$

is a basis for the tangent space $T(\mathbb{R}^3)$, where (x, y, z) denotes the standard coordinate of a point in $\mathbb{R}^3$.

We define a semi-Riemannian metric g on $\mathbb{R}^3$ as

$$g(X, Y) = -X^1Y^1 + X^2Y^2 + X^3Y^3,$$

where $X = X^1e_1 + X^2e_2 + X^3e_3$ and $Y = Y^1e_1 + Y^2e_2 + Y^3e_3$ are vector fields on $\mathbb{R}^3$.

We consider the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ spanned by $\{e_1, e_2\}$ and $\{e_3\}$ respectively. From (1.2) we obtain

$$\frac{\partial}{\partial z} = A_3^1\frac{\partial}{\partial x} + A_3^2\frac{\partial}{\partial y} + A_3^3 e_3.$$

Therefore,

$$A_3^1 = 0, \quad A_3^2 = 0, \quad A_3^3 = \frac{1}{e^{x+y}},$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-(x+y)} \end{bmatrix}.$$

By using (1.3), we infer that

$$\frac{\delta}{\delta z} = \frac{\partial}{\partial z} - A_3^1\frac{\partial}{\partial x} - A_3^2\frac{\partial}{\partial y} = \frac{\partial}{\partial z}.$$

$$\begin{aligned}
&\tilde{\nabla}_{\partial_1}\partial_1 = 0, && \tilde{\nabla}_{\partial_1}\partial_2 = 0, && \tilde{\nabla}_{\partial_1}\delta_3 = -f^{-2}\delta_3, \\
&\tilde{\nabla}_{\partial_2}\partial_1 = 0, && \tilde{\nabla}_{\partial_2}\partial_2 = 0, && \tilde{\nabla}_{\partial_2}\delta_3 = -f^{-2}\delta_3, \\
&\tilde{\nabla}_{\delta_3}\partial_1 = -f^{-2}\delta_3, && \tilde{\nabla}_{\delta_3}\partial_2 = -f^{-2}\delta_3, && \tilde{\nabla}_{\delta_3}\delta_3 = -f^{-2}\partial_1 + f^2\partial_2.
\end{aligned}$$

We define the tensor field S in (1.8) by

$$S(X,Y) = (-Y^1 + Y^3)X.$$

In consequence of the above discussions and equation (1.8), we obtain

$$\nabla_{\partial_1}\partial_1 = -\partial_1, \quad \nabla_{\partial_1}\partial_2 = 0, \quad \nabla_{\partial_1}\delta_3 = -f^{-2}\delta_3,$$

$$\nabla_{\partial_2}\partial_1 = -\partial_2, \quad \nabla_{\partial_2}\partial_2 = 0, \quad \nabla_{\partial_2}\delta_3 = f\partial_2 - f^{-2}\delta_3,$$

$$\nabla_{\delta_3}\partial_1 = -\delta_3, \quad \nabla_{\delta_3}\partial_2 = 0, \quad \nabla_{\delta_3}\delta_3 = f\delta_3.$$

where

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \delta_3 = \frac{\delta}{\delta z}.$$

By direct calculation, we obtain

$$h(\mathcal{P}X, \mathcal{P}Y) = 0, \quad \text{and} \quad h^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) = X^3Y^3(-f^{-2}\partial_1 + f^2\partial_2) \neq 0.$$

It follows that the foliation $\mathcal{F}$ determined by the involutive distribution $\mathcal{D}$ is totally geodesic and g is not bundle-like.

References

1. A. Bejancu, *Schouten-Van Kampen and Vrănceanu connnections on foliated manifolds*, Anale. Stin. Univ. AL.I. CUZU. IASI, Tomul LII, Math. (2006), 37-60.
2. A. Bejancu and H R and Farran, *Foliation and Geometric structures*, Mathematics and Its Applications, springer, Dordrecht, 2006.
3. A. Fridmann and J. A. Schouten, *Über die Geometrie der halbsymmeMatrischen Übertragungen*, Math. Z. **21** (1924), 211-223.
4. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, London, 1983.
5. B. L. Reinhart, *Foliated manifolds with bundle-like metric*, Ann. Math. **69**(1959), 119-132.
6. J. A. Schouten, *Ricci-Calculus, An introduction to tensor analysis and geometrical applications*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954.
7. J. A. Schouten and E.R. Van Kampen, *Zur Einbettungs-und Krümmungstheorie nichtholonomer Gebilde*, Math. Ann. **103**(1930), 752-783.
8. G. Vrănceanu, *Sur quelques points de Ia theorie des espaces non holonomes*, Bul. Fac. St. Cernăuti. **5**(1931), 177-205.
9. I. Vaisman, *Variétés riemanniennes feuilletées*, Czech. Math. J. **21**(1971), 46-75.

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