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Projective change between special cubic (α, β)-metric and Randers metric

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Abstract. In 1994, S. Basco and M. Matsumoto studied the concept of projective change between two Finsler spaces with (α, β)-metrics. Projective change between two Finsler metrics arises from Information Geometry. In the present paper, we find conditions to characterize the projective change between two (α, β)-metrics, such as special cubic (α, β)-metric and Randers metric on a manifold with dim $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms.

Keywords: Finsler space, Randers metric, special cubic (α, β)-metric, Projective change, Douglas metric, locally Minkowskian space.

1. Introduction

The study of projectively related Finsler metrics was initiated by Berwald and his studies mainly concern the 2-dimensional Finsler spaces [2]. Further Rapcsák [11], Szabó [18] and Báscó-Matsumoto [3, 9] have made substantial

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contributions on this topic. As Shen pointed out in [17], the problem of projectively related Finsler metrics is strongly connected to projectively related sprays.

In [15], Shen-Yu studied projectively related Randers metrics. They show that two Randers metrics are pointwise projectively related if and only if they have the same Douglas tensors and the corresponding Riemannian metrics are projectively related. In [10], Park and Lee studied the projective change between a Finsler space with (α, β)-metric and the associated Riemannian metric. In [5], Cui-Shen find necessary and sufficient conditions under which a Berwald metric and a Randers metric are projectively related. Then Zohrehvand-Rezaii found necessary and sufficient conditions under which a Matsumoto metric and a Randers metric are projectively related [22]. The projective change between two Finsler spaces have been studied by many geometers [4, 9, 14, 16, 19, 23, 24].

In Finsler geometry, a change of $L \to \bar{L}$ of a Finsler metric on a same underlying manifold M is called projective change if any geodesic in (M, L) remains to be a geodesic in $(M, \bar{L})$ and vice versa. Two Finsler metrics L and $\bar{L}$ on a manifold M are said to be projectively related if their geodesics as point sets are the same. Two Riemannian metrics α and $\bar{\alpha}$ are projectively related in Riemannian geometry if and only if their spray coefficients have the relation

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{1.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold, and (x^i, y^j) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics L and $\bar{L}$ are projectively related if and only if their spray coefficients have the relation [5]

$$G^i = \bar{G}^i + P(y) y^i, \tag{1.2}$$

where $P(y)$ is a scalar function on $TM \backslash \{0\}$ and homogeneous of degree one in y. The change from a Finsler metric L to another Finsler metric $\bar{L} = L + \bar{\beta}$ is called a Randers change, $\bar{\beta}$ is a nonzero one form on the based manifold satisfying $||\bar{\beta}|| < 1$. In [7], it has been proved that L is projectively related to its Randers change $\bar{L}$ if and only if $\bar{\beta}$ is closed.

An (α, β)-metric is a Finsler metric on a manifold M defined by $L = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M. Randers metrics $L = \alpha + \beta$ are the simplest (α, β)-metrics which were first introduced by physicist Randers from the standpoint of general relativity [12].

In 2023 Tripathi et al. [19, 20, 21] defined a special cubic (α, β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ on n-dimensional manifold M by taking $p = 3$ in a class p-power (α, β)-metrics $L = \alpha \left(1 + \frac{\beta}{\alpha}\right)^p$ [6]. The purpose of this paper is to find the relation between two Finsler spaces with special cubic (α, β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$

and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ respectively under projective change, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms.

2. Preliminaries

The terminology and notations are referred to [1, 8, 13]. Let $F^n = (M, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function $L(x, y)$. We use the following notations:

$$\begin{cases} (a) \quad g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2, \quad \dot{\partial} = \frac{\partial}{\partial y^i}, \\ (b) \quad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \\ (c) \quad h_{ij} = g_{ij} - l_i l_j, \\ (d) \quad \gamma^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}), \\ (e) \quad G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \;\; G^i_j = \dot{\partial}_j G^i, \;\; G^i_{jk} = \dot{\partial}_k G^i_j, \;\; G^i_{jkl} = \dot{\partial}_l G^i_{jk}. \end{cases}$$

For a given Finsler metric $L = L(x, y)$, the geodesics of L satisfy the following ODEs:

$$\frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx}{dt} \right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{g^{il}}{4} \left\{ [L^2]_{x^k y^l} y^k - [L^2]_{x^k} \right\}.$$

Let $\phi = \phi(s), |s| < b_0$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \le b < b_0. \tag{2.1}$$

If $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is 1-form satisfying $||\beta_x||_\alpha < b_0 \; \forall x \in M$, then $L = \phi(s), s = \beta/\alpha$, is called an (regular) (α, β)-metric. In this case, the fundamental form of the metric tensor induced by L is positive definite.

Let $\nabla \beta = b_{i|j} dx^i \otimes dx^j$ be covariant derivative of β with respect to α.

Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

β is closed if and only if $s_{ij} = 0$ [14]. Let $s_j = b^i s_{ij}$, $s^i_j = a^{il} s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and $r_{00} = r_{ij} y^i y^j$.

The geodesic coefficients G^i of L and geodesic coefficients G^i_α of α are related as follows

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\psi b^i + \Theta \alpha^{-1} y^i\}, \tag{2.2}$$

where

$$\begin{cases} \Theta = \dfrac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi' + (b^2 - s^2)\phi'')}, \\ Q = \dfrac{\phi'}{\phi - s\phi'}, \\ \psi = \dfrac{1}{2}\dfrac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{cases}$$

Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{2.3}$$

where G^i are the spray coefficients of L. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

It is simple to verify that the Douglas tensor is projective invariant by using (1.2). Observing that the spray coefficients of a Riemannian metric are quadratic forms, what one sees that the Douglas tensor vanishes (2.3). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{L}$. In the beginning, we compute the Douglas tensor of a general (α, β)-metric.

Let

$$\hat{G}^i = G^i_\alpha + \alpha Q s^i_0 + \psi\{-2Q\alpha s_0 + r_{00}\} b^i.$$

Then (2.2) becomes

$$G^i = \hat{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1} y^i.$$

Clearly, G^i and $\hat{G}^i$ are projective equivalent according to (1.2), they have the same Douglas tensor.

Let

$$T^i = \alpha Q s^i_0 + \psi\{-2Q\alpha s_0 + r_{00}\} b^i. \tag{2.4}$$

Then $\hat{G}^i = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} =& \hat{D}^i_{jkl} \\ =& \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m} y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right) \\ =& \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right). \end{aligned} \tag{2.5}$$

To compute (2.5) explicitly, we need the following identities

$$\begin{aligned} \alpha_{y^k} &= \alpha^{-1} y_k, \\ s_{y^k} &= \alpha^{-2}(b_k \alpha - s y_k), \end{aligned}$$

where $y_i = a_{il}y^l$. Here and from now on, α_{y^k} means $\frac{\partial \alpha}{\partial y^k}$, etc. Then

$$\begin{aligned}[\alpha Q s_0^m]_{y^m} =&\alpha^{-1} y_m Q s_0^m + \alpha^{-2} Q^{'}[b_m \alpha^2 - \beta y_m] s_0^m \\ =&Q^{'} s_0\end{aligned}$$

and

$$\begin{aligned}[\psi(-2Q\alpha s_0 + r_{00})b^m]y_m = \psi^{'}\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ +2\psi[r_0 - Q^{'}(b^2 - s^2)s_0 - Qss_0],\end{aligned}$$

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.4) we have

$$\begin{aligned}T^m_{y^m} = Q^{'} s_0 + \psi^{'}\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ +2\psi[r_0 - Q^{'}(b^2 - s^2)s_0 - Qss_0].\end{aligned} \tag{2.6}$$

Let L and $\bar{L}$ be two (α, β)-metrics, we assume that they have the same Douglas tensor, i.e. $D^i_{jkl} - \bar{D}^i_{jkl}$. From (2.3) and (2.5) we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$ such that

$$T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i = H^i_{00}, \tag{2.7}$$

where $H^i_{00} = H^i_{jk}y^j y^k, T^i$ and $T^m_{y^m}$ are given by (2.4) and (2.6) respectively.

3. Projective change between special cubic (α, β)-metric and Randers metric

In this section, we find the projective change between two (α, β)-metrics, i.e., special cubic (α, β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a same underlying manifold M of dimension $n \geq 3$. For (α, β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$, (2.1) shows that L is a regular Finsler metric if and only if 1-form β meets the requirement $||\beta_x||_\alpha < \frac{1}{2}$ for any $x \in M$. The geodesic coefficients are given by (2.2) with

$$\begin{cases} \Theta = \dfrac{3 - 12s}{2\{(1+s))^2 - 3s(1+s) + 6(b^2 - s^2)\}}, \\ Q = \dfrac{3}{1 - 2s}, \\ \psi = \dfrac{3}{2\{(1+s)^2 - 3s(1+s) + (b^2 - s^2)6\}}. \end{cases} \tag{3.1}$$

Substituting (3.1) into (2.2), we get

$$G^i = G^i_\alpha + \frac{3\alpha^2}{\alpha - 2\beta}s^i_0 + \left\{\frac{-6\alpha^2}{\alpha - 2\beta}s_0 + r_{00}\right\}\left\{\frac{6\alpha^2 b^i + (3\alpha - 12\beta)y^i}{2\{(\alpha+\beta)^2 - 3\beta(\alpha+\beta) + 6(\alpha^2 b^2 - \beta^2)\}}\right\}. \tag{3.2}$$

For Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$, (2.1) also shows that $\bar{L}$ is a regular Finsler metric if and only if $||\beta_x||_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.2) with

$$\bar{\Theta} = \frac{1}{2(1+s)}, \quad \bar{Q} = 1, \quad \bar{\psi} = 0. \tag{3.3}$$

First we prove the following lemma:

Lemma 3.1. *Let* $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ *and* $\bar{L} = \bar{\alpha} + \bar{\beta}$ *be two* (α,β)*-metrics on a manifold* M *with dimension* $n \geq 3$*. Then they have the same Douglas tensor if and only if both the metrics* L *and* $\bar{L}$ *are Douglas metrics.*

Proof. We first establish the sufficient condition. Let D^i_{jkl} and $\bar{D}^i_{jkl}$ be the appropriate Douglas tensors and let L and $\bar{L}$ be the Douglas metrics. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$ i.e., Douglas tensor of L and $\bar{L}$ is the same. We then establish the necessary condition. (2.7) holds if L and $\bar{L}$ have the same Douglas tensor. Upon replacing (3.1) and (3.3) in (2.7), we obtain

$$H^i_{00} = \frac{A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2 + H^i\alpha + I^i}{J\alpha^6 + K\alpha^5 + L\alpha^4 + M\alpha^3 + N\alpha^2 + O\alpha + P} - \bar{\alpha}\bar{s}^i_0, \tag{3.4}$$

where

$$\begin{cases} A^i = 3s^i_0(1+6b^2)^2 - 18s_0(1+6b^2)b^i, \\ B^i = \beta[s^i_0(-216b^4 - 108b^2 - 12) + s_0 b^i(216b^2 + 54)] - s_0\lambda y^i(18b^2 + 6) \\ + r_{00}(18b^2 + 3)b^i - r_0\lambda y^i(36\lambda b^2 + 6), \\ C^i = s^i_0(-216\beta^2 b^2 - 33\beta^2) + s_0(108\beta^2 b^i + 396\beta b^2\lambda y^i + 30\beta\lambda y^i) \\ + r_{00}(-72\beta b^2 b^i - 15\beta b^i - 3b^2\lambda y^i) + r_0(144\beta b^2\lambda y^i + 24\beta\lambda y^i) \\ D^i = s^i_0(576\beta^3 b^2 + 138\beta^3) + s_0(-288\beta^3 b^i - 720\beta^2 b^2\lambda y^i - 18\beta^2\lambda y^i) \\ + r_{00}(72\beta^2 b^2 b^i - 36\beta\lambda b^2 y^i) + r_0(-144\beta^2 b^2\lambda y^i - 24\beta^2\lambda y^i + 48\beta^2\lambda y^i) \\ E^i = 96\beta^4 s^i_0 - 420\beta^3 s_0\lambda y^i + r_{00}(84\beta^3 b^i - 180\beta^2 b^2\lambda y^i + 3\beta^2\lambda y^i) \\ + r_0(24\beta^3\lambda y^i - 192\beta^3\lambda y^i) \\ F^i = -384\beta^5 s^i_0 + 768\beta^4 s_0\lambda y^i - r_{00}(96\beta^4 b^i + 192\beta^3 b^2\lambda y^i - 36\beta^3\lambda y^i) \\ + 192\beta^4 r_0\lambda y^i \\ H^i = -180\beta^4 r_{00}\lambda y^i, \\ I^i = 192\beta^5 r_{00}\lambda y^i. \end{cases} \tag{3.5}$$

and

$$\begin{cases} J = (1+6b^2)^2, \\ K = -\beta(144b^4+60b^2+6), \\ L = \beta^2(144b^2-3), \\ M = \beta^3(336b^2+68), \\ N = -\beta^4(384b^2+60), \\ O = -192\beta^5, \\ P = 256\beta^6. \end{cases} \tag{3.6}$$

Then (3.4) is equivalent to

$$\begin{aligned} &A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2 + H^i\alpha + I^i \\ &= (J\alpha^6 + K\alpha^5 + L\alpha^4 + M\alpha^3 + N\alpha^2 + O\alpha + P)(H^i_{00} + \bar{\alpha}\bar{s}^i_0). \end{aligned} \tag{3.7}$$

Replacing y^i in (3.7) by $-y^i$ yields

$$\begin{aligned} &- A^i\alpha^7 + B^i\alpha^6 - C^i\alpha^5 + D^i\alpha^4 - E^i\alpha^3 + F^i\alpha^2 - H^i\alpha + I^i \\ &= (J\alpha^6 - K\alpha^5 + L\alpha^4 - M\alpha^3 + N\alpha^2 - O\alpha + P)(H^i_{00} - \bar{\alpha}\bar{s}^i_0). \end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8) yields

$$\begin{aligned} &B^i\alpha^6 + D^i\alpha^4 + F^i\alpha^2 + I^i \\ &= (J\alpha^6 + L\alpha^4 + N\alpha^2 + P)H^i_{00} + (K\alpha^4 + M\alpha^2 + O)\alpha\bar{\alpha}\bar{s}^i_0. \end{aligned} \tag{3.9}$$

Substracting (3.7) from (3.8) yields

$$\begin{aligned} &A^i\alpha^7 + C^i\alpha^5 + E^i\alpha^3 + H^i\alpha \\ &= (K\alpha^5 + M\alpha^3 + O\alpha)H^i_{00} + (J\alpha^6 + L\alpha^4 + N\alpha^2 + P)\bar{\alpha}\bar{s}^i_0. \end{aligned} \tag{3.10}$$

We split the proof into two cases

Case (i): If $\bar{\alpha} \neq \mu(x)\alpha$, then from (3.9), we see that $\alpha\bar{\alpha}\bar{s}^i_0$ is a homogeneous polynomial with respect to y. Therefore $\bar{s}^i_0 = 0$, which says that $\bar{\beta}$ is closed.

Case (ii): If $\bar{\alpha} = \mu(x)\alpha$, then (3.9) and (3.10) reduce to

$$\begin{aligned} &B^i\alpha^6 + D^i\alpha^4 + F^i\alpha^2 + I^i \\ &= (J\alpha^6 + L\alpha^4 + N\alpha^2 + P)H^i_{00} + \alpha^2\mu(x)\bar{s}^i_0(K\alpha^4 + M\alpha^2 + O) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + H^i \\ &= H^i_{00}(K\alpha^4 + M\alpha^2 + O) + \mu(x)\bar{s}^i_0(J\alpha^6 + L\alpha^4 + N\alpha^2 + P), \end{aligned} \tag{3.12}$$

(3.11) and (3.12) is equivalent to

$$\begin{aligned} I^i - H^i_{00}P =& [-B^i\alpha^4 - D^i\alpha^2 - F^i + H^i_{00}(J\alpha^4 + L\alpha^2 + N) \\ &+ \mu(x)\bar{s}^i_0(K\alpha^4 + M\alpha^2 + O)]\alpha^2 \end{aligned} \tag{3.13}$$

$$\begin{aligned} H^i - H^i_{00}O - P\mu(x)\bar{s}^i_0 =& [-A^i\alpha^4 - C^i\alpha^2 - E^i + H^i_{00}(K\alpha^2 + M) \\ &+ \mu(x)\bar{s}^i_0(J\alpha^4 + L\alpha^2 + N]\alpha^2. \end{aligned} \tag{3.14}$$

respectively. From (3.13) and (3.14), $I^i - H^i_{00}P$ and $H^i - H^i_{00}O - P\mu(x)\bar{s}^i_0$ has the factor α^2. From (3.5) and (3.6), we can see that $I^i = -16/15\beta H^i$ and $P = -4/3\beta O$. Thus H^i, $H^i_{00}O$ and $P\mu(x)\bar{s}^i_0$ has the factor α^2. Since $H^i_{00}O = -192\beta^5 H^i_{00}$, we conclude that for each i there exists a scalar function $\sigma^i(x)$ on M such that $H^i_{00} = \sigma^i(x)\alpha^2$. From (3.13) and (3.5), we can see that

$$r_{00} = \eta(x)\alpha^2. \tag{3.15}$$

On the other hand from (3.5), (3.6), (3.13), (3.14) and (3.15), we have that

$$[(1+6b^2)^2\mu(x)\bar{s}^i_0 - 3(1+6b^2)^2\bar{s}^i_0 + 18(1+6b^2)s_0b^i + 3b^2\lambda y^i\eta(x)]\alpha^6$$

has the factor β, thus

$$(1+6b^2)^2(\mu(x)\bar{s}^i_0 - 3s^i_0) + 18(1+6b^2)s_0b^i + 3b^2\lambda y^i\eta(x)$$

has the factor β, i.e. for each i there exists a scalar function $\xi^i(x)$ such that

$$(1+6b^2)^2(\mu(x)\bar{s}^i_0 - 3s^i_0) + 18(1+6b^2)s_0b^i + 3b^2\lambda y^i\eta(x) = \beta\xi^i(x),$$

By multiplying in a_{ik} and then differentiating with respect to y^j, we obtain

$$(1+6b^2)^2(\mu^{-1}\bar{s}_{kj} - s_{kj}) + 18(1+6b^2)b_ks_j + 3\lambda a_{jk}b^2\eta(x) = b_j\xi_k(x),$$

where $\xi_k(x) = \xi^i(x)a_{ik}$. From above we have

$$b_j\xi_k(x) + b_k\xi_j(x) = 6\lambda a_{jk}b^2\eta(x) + 18(1+6b^2)(b_ks_j + b_js_k).$$

By multiplying in y^jy^k, we have

$$2\beta\xi_0 = 6\lambda\alpha^2b^2\eta(x) + 18(1+6b^2)\beta s_0.$$

Thus $\eta(x) = 0$ and therefore $r_{00} = 0$.
Therefore

$$\begin{cases} A^i = 3(1+6b^2)^2s^i_0 - 18(1+6b^2)s_0b^i, \\ B^i = -216b^4s^i_0 - 108\beta b^2s^i_0 + 216\beta b^2s_0b^i - 12\beta s^i_0 + 54\beta s_0b^i - 18\lambda b^2s_0y^i \\ \qquad - 6s_0\lambda y^i, \\ C^i = -216\beta^2b^2s^i_0 - 33\beta^2s^i_0 + 108\beta^2s_0b^i + 396\beta b^2s_0\lambda y^i + 30\beta s_0\lambda y^i, \\ D^i = 576\beta^3b^2s^i_0 + 138\beta^3s^i_0 - 288\beta^3s_0b^i - 720\beta^2b^2s_0\lambda y^i, \\ E^i = 96\beta^4s^i_0 - 420\beta^3s_0\lambda y^i, \\ F^i = -384\beta^5s^i_0 + 768\beta^4s_0\lambda y^i, \\ H^i = 0, \\ I^i = 0. \end{cases} \tag{3.16}$$

From (3.13) and (3.16), we have

$$F^i - \mu(x)\bar{s}^i_0O - H^i_{00}P = \beta^4[-384\beta s^i_0 + 768s_0\lambda y^i + 192\beta\mu(x)\bar{s}^i_0 - 256\beta^2\sigma^i(x)]$$

has the factor α^2, thus for each i there exists a scalar function $\rho^i(x)$ on M such that

$$192\beta\mu(x)\bar{s}^i_0 - 384\beta s^i_0 + 768 s_0\lambda y^i - 256\beta^2\sigma^i(x) = \alpha^2\rho^i(x).$$

By multiplying in y_i, we have

$$768\alpha^2 s_0\lambda - 256\sigma^i(x)y_i\beta^2 = \alpha^2\rho^i(x)y_i,$$

therefore

$$\alpha^2[\rho^i(x)y_i - 768\lambda s_0] = -256\sigma^i(x)y_i\beta^2,$$

Thus $\sigma^i(x) = 0$ and $H^i_{00} = 0$. On the other hand from (3.14) and $H^i_{00} = 0$, we see that $P\mu(x)\bar{s}^i_0 = 256\beta^6\mu(x)\bar{s}^i_0$ has tha factor α. Therefore for each i there exists scalar function $\tau^i(x)$ on M such that

$$\mu(x)\bar{s}^i_0 = \alpha\tau^i(x),$$

by multiplying in a_{ij} and differentiating with respect to y^k we have

$$256\mu(x)^{-1}\bar{s}_{jk} = \alpha^{-1}y_k\tau_j(x),$$

by multiplying in $y^j y^k$, we obtain

$$\alpha\tau_0(x) = 0,$$

hence $\tau_j(x) = 0$ and $\bar{s}_{jk} = 0$, therefore $\bar{\beta}$ is closed. Anyway, $\bar{\beta}$ is closed. It is well known that Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Then $\bar{L}$ is a Douglas metric. Since L is projectively related to $\bar{L}$, then both L and $\bar{L}$ are Douglas metrics. We complete the proof. □

Now, we prove the following main theorem by using Lemma 3.1:

Theorem 3.2. *The Finsler metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if the following conditions are satisfied.*

$$\begin{cases} G^i_\alpha = G^i_{\bar{\alpha}} + Py^i, \\ b_{i|j} = 0, \\ d\bar{\beta} = 0, \end{cases} \tag{3.17}$$

where $b = ||\beta||_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, P is a scalar function.

Proof. We first establish the necessary condition. The Douglas tensor of two Finsler metrics is the same if L is projectively related to $\bar{L}$, since the Douglas tensor is invariant under projective changes between two Finsler metrics. Then lemma 3.1 leads us to the conclusion that L and $\bar{L}$ are Douglas metrics.

We know that Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed [5], i.e.,

$$d\bar{\beta} = 0. \tag{3.18}$$

and $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ is a Douglas metric if and only if

$$b_{i|j} = 0. \tag{3.19}$$

where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta = b_i y^i$ with respect to α [19]. In this case, β is closed. As β is closed, $s_{ij} = 0$ implies that $b_{i|j} = b_{j|i}$. Hence we have $s^i_0 = 0, s_0 = 0$.

By using (3.19), we have $r_{00} = r_{ij} y^i y^j = 0$. Upon replacing all these in (3.2), we obtain

$$G^i = G^i_\alpha. \tag{3.20}$$

The equation (3.20) shows Randers change between L and $\bar{\alpha}$, since L is projective to $\bar{L} = \bar{\alpha} + \bar{\beta}$. Here $\bar{\beta}$ is closed, thus L and $\bar{\alpha}$ are projectively related. Consequently, there exists a scalar function $P = P(y)$ on $TM\backslash\{0\}$ such that

$$G^i = G^i_{\bar{\alpha}} + P y^i. \tag{3.21}$$

From (3.20) and (3.21), we have

$$G^i_\alpha = G^i_{\bar{\alpha}} + P y^i. \tag{3.22}$$

(3.18) and (3.19) together with (3.22) complete the proof of the necessity.

Given that $\bar{\beta}$ is closed, proving that L is projectively related to $\bar{\alpha}$ suffices to establish the sufficiency. (3.20) is obtained by substituting (3.19) into (3.2). From (3.20) and (3.22), we have

$$G^i = G^i_{\bar{\alpha}} + P y^i. \tag{3.23}$$

i.e., L is projectively related to $\bar{\alpha}$. □

From the above theorem, immediately we get the following corollaries.

Corollary 3.3. *The Finsler metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha = G^i_{\bar{\alpha}} + P y^i,$$

where P is a scalar function.

Further, we assume that the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}_i y^i$ is a one form with $\bar{b}_i$=constants. Then (3.17) can be written as

$$\begin{aligned} G^i_\alpha &= P y^i, \\ b_{i|j} &= 0. \end{aligned} \tag{3.24}$$

Hence, we state

Corollary 3.4. *The Finlser metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ is projectively related to $\bar{L}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if (3.24) holds.*

4. **Conclusion**

In the 1920s, Elie Cartan and Tracey Thomas initiated projective differential geometry. Overdetermined systems of partial differential equations naturally arise in the simplest situation, which is provided by projective differential geometry. The Rapcsák theorem [11] is a notable theorem in projective differential geometry that plays an important role in projective geometry of Finsler spaces. This theorem establishes both the necessary and sufficient conditions for a Finsler space to be projective to another Finsler space. The most fundamental application of what has come to be known as the Bernstein-Gefland machinery is provided by projective differential geometry. As such, it is identical to conformal differential geometry. However, there are direct applications in Riemannian differential geometry.

In this paper, we have obtained some important results concerning projective change between special cubic (α, β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ and Randers metric in Lemma 3.1, Theorem 3.2, and Corollaries (3.3, 3.4).

References

1. P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of sprays and Finsler spaces with applications in Physics and Biology*, Kluwer academic publishers, London, 1985.
2. L. Berwald, *On Finsler and Cartan geometries, III. Two-dimensional Finsler spaces with rectilinear extremals*, Ann. Math., **42**(20) (1941), 84-112.
3. S. Bácśo and M. Matsumoto, *Projective change between Finsler spaces with (α, β)- metrics*, Tensor N.S., **55** (1994), 252-257.
4. G. Chen and X. Cheng, *A class of Finsler metrics projectively related to a Randers metric*, Publ. Math. Debrecen., **81**(3-4) (2012), 351-363.
5. N. Cui and Y. B. Shen, *Projective change between two classes of (α, β)-metrics*, Diff. Geom. Appl., **27** (2009), 566-573.
6. G. Yang, *On a class of Einstein-reversible Finsler metrics*, Differ. Geom. Appl., **60** (2018), 80-10.
7. M. Hashiguchi and Y. Ichijyō, *Randers spaces with rectilinear geodesics*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.), **13** (1980), 33-40.
8. M. Matsumoto, *Foundations of Finsler Geometry and special Finsler spaces*, Kaiseisha press Otsu, Saikawa, 1986.
9. M. Matsumoto, *Projective changes of Finsler metrics and projectively flat Finsler spaces*, Tensor N.S., **34** (1980), 303-315.
10. H. S. Park and Y. Lee, *Projective changes between a Finsler space with (α, β)-metric and associated Riemannian metric*, Canad. J. Math., **60** (2008), 443-456.
11. A. Rapcsák, *Über die bahntreuen Abbildungen metrisher Räume*, Publ. Math. Debrecen, **8** (1961), 285-290.
12. G. Randers, *On an asymmetric metric in the four-space of general relativity*, Phys. Rev., **59** (1941), 195-199.
13. H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin, 1959.
14. S. K. Narsimhamurthy and D. M. Vasantha, *Projective change between two Finsler spaces with (α, β)-metric*, Kyungpook Math. J., **52** (2012), 81-89.

15. Y. B. Shen and Y. Yu, *On projectively related Randers metric*, Inter. J. Math., **19**(5) (2008), 503-520.
16. Z. Shen, *On projectively related Einstein metrics in Riemann-Finsler geometry*, Math. Ann., **320** (2001), 625-647.
17. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.
18. Z. Szabó, *Positive definite Berwald spaces*, Tensor N.S., **35** (1981), 25-39.
19. B. K. Tripathi, S. Khan and V. K. Chaubey, *On Projectively Flat Finsler Space with a special cubic* (α, β)*- metric*, Filomat, **37**(26) (2023), 8975-8982.
20. B. K. Tripathi and S. Khan, *On weakly Berwald space with a special cubic* (α, β)*-metric*, Surv. Math. Appl., **18** (2023), 1-11.
21. B. K. Tripathi, S. Khan and M. K. Singh, *Conformally flat locally Minkowski Finsler space with a special cubic* (α, β)*-metric*, J. Adv. Math. Stud., **16**(3) (2023), 275-281.
22. M. Zohrehvand and M. M. Rezaii, *On projectively related of two special classes of* (α, β)*-metrics*, Diff. Geom. Appl., **29**(5) (2011), 660-669.
23. B. K. Tripathi and S. Prajapati, *On Berwald-Matsumoto type Finsler metrics*, J. Finsler Geom. Appl., **6**(2) (2025), 71-81.
24. P. K. Dwivedi, Sachin Kumar and C. K. Mishra, *On projectively flat Finsler space with* n*-power* (α, β)*-metric*, J. Finsler Geom. Appl., **5**(2) (2024), 14-24.

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