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Research Paper

TOTAL OUTER- CONNECTED DOMINATING SETS AND TOTAL OUTER- CONNECTED DOMINATION POLYNOMIALS OF COMPLETE BIPARTITE GRAPH $K_{2,n}$

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ABSTRACT

Let $G = (V, E)$ be a simple graph. A set $D \subseteq V(G)$ is a total outer − connected dominating set of G if D is total dominating, and the induced subgraph $G[V(G) - D]$ is a connected graph. Let $K_{2,n}$ be the complete bipartite graph and $\tilde{D}_{tc}(K_{2,n}, i)$ denote the family of all total outer- connected dominating sets of $K_{2,n}$ with cardinality i. Let $\tilde{d}_{tc}(K_{2,n}, i) = |\tilde{D}_{tc}(K_{2,n}, i)|$. In this paper, we obtain the recursive formula for $\tilde{d}_{tc}(K_{2,n}, i)$. Using this recursive formula, we construct the polynomial $\tilde{D}_{tc}(K_{2,n}, x) = \sum_{i=2}^{2+n} \tilde{d}_{tc}(K_{2,n}, i)x^i$, which we call the total outer − connected domination polynomial of $K_{2,n}$ and obtain some properties of this polynomial.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For any vertex $v \in V(G)$, the open neighbourhood of v is the set $N_G(v) = u \in V(G)/uv \in E(G)$ and the closed neighbourhood of v is the set $N_G[v] = N(v) \cup v$. For a set $S \subseteq V$, the open

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neighbourhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$.

A dominating set of G is a set $D \subseteq V(G)$ such that $N_G[V] \cap D \neq \phi$, for all $v \in V(G)$. The domination number of G is the minimum cardinality of a dominating set of G and it is denoted by $\gamma(G)$. Similarly, a total dominating set of G is a set $D \subseteq V(G)$ such that for each $v \in V(G)$, $N_G(v) \cap D \neq \phi$. The total domination number $\gamma_{t(G)}$ of G is the minimum cardinality of a total dominating set of G.

Joanna Cyman introduced the concept total outer connected domination [7] and further explored in [8] together with Joanna Raczek.B.S. Panda and Arti Pandey contributed NP Completeness in total outer connected domination [19].Some bounds and generaliation of total outer connected domination was obtained by Nader Jafari Rad and Lut Volkmann [17]. In [9] Farshad Kazemnejad Behnaz Pahlavsay, Elisa Palezzato and Michele Torielli obtained some results in total outer connected domination number of middle graphs.

Definition 1.1. [12] A simple graph or multigraph is bipartite if its vertices can be partitioned into two sets in such a way that no edge joins two vertices in the same set. If m and n are the orders of the partite sets,then the graph is said to be an m-by-n bipartite graph.

Definition 1.2. [12] A complete bipartite graph is a simple bipartite graph in which each vertex in one partite set is adjacent to all vertices in the other partite set. If the two partite sets have cardinalities m and n, then this graph is denoted as $k_{m,n}$. Let $K_{2,n}$ be the complete bipartite graph with $n+2$ vertices. Figure 1 shows $K_{2,n}$.

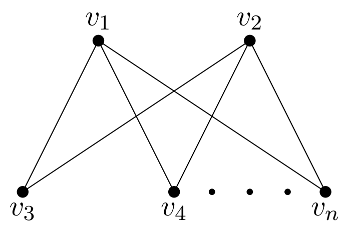

Figure 1:$K_{2,n}$

Definition 1.3. [7] Let G be a simple connected graph. A set $D \subseteq V(G)$ is a total outer connected dominating set of G if D is total dominating and the induced subgraph $G[V(G) - D]$ is a connected graph. The total outer connected domination number of G, denoted by $\tilde{\gamma}_{tc}(G)$, is the minimum cardinality of a total outer connected dominating set of G.

Definition 1.4. Let G be a simple connected graph. Let $\tilde{D}_{tc}(G,i)$ denote the family of all the total outer connected dominating sets of G with cardinality i and let $\tilde{d}_{tc}(G,i) = |\tilde{D}_{tc}(G,i)|$. Then the total outer connected domination polynomial $\tilde{D}_{tc}(G,x)$ of G is defined as $\tilde{D}_{tc}(G,i) = \sum_{i=\tilde{\gamma}_{tc}(G)}^{|V(G)|} \tilde{d}_{tc}(G,i)x^i$, where $\tilde{\gamma}_{tc}(G)$ is the total outer connected domination number of G.

Total outer connected domination plays a major role in facility location,wireless sensor networks communication networks etc., In facility location it was used to determine optimal location for hospitals, fire station etc., to serve all points in a network. Total outer connected domination might be relevant if the unserved areas need to remain connected for certain operations or communication. It plays a minimum number of sensors to cover an area and maintain connectivity among the unmonitored regions or between sensors. Total outer connected domination polynomial provide a systematic way to count the number of sets of various sizes which can be important for placing networks or locations.

This has inspired us to find some sets and recursive formula in complete bipartite graph. $K_{2,n}$. In [7],[8],[17],[19], we study total outer connected dominating sets of some graphs.

In this paper, we obtain total outer connected dominating sets and total outer connected domination polynomial of complete bipartite graph $K_{2,n}$. We denote the set $\{1, 2, ..., n+2\}$ by $[n+2]$ throughout this paper.

2. Total Outer- Connected Dominating Sets of $K_{2,n}$

Lemma 2.1. *For every* $n \in \mathbb{N}$, $\tilde{\gamma}_{tc}(K_{2,n}) = 2$.

Lemma 2.2. *For every* $n \in \mathbb{N}$,

(i) $\tilde{D}_{tc}(K_{2,n}, i) = \phi$ *if and only if* $i < \tilde{\gamma}_{tc}(K_{2,n})$ *or* $i > n+2$.
(ii) $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$ *if and only if* $\tilde{\gamma}_{tc}(K_{2,n}) \leq i \leq n+2$.

Proof. By the definition of total outer- connected domination number, there are at least two dominating sets in $\tilde{D}_{tc}(K_{2,n}, i),\ when\ i = \tilde{\gamma}_{tc}(K_{2,n})$. All super sets are again a total outer-connected dominating set, $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi\ if\ \tilde{\gamma}_{tc}(K_{2,n}) \leq i \leq n+2$. Clearly, by definition of the total outer-connected domination number, $\tilde{D}_{tc}(K_{2,n}, i) = \phi\ if\ i < \tilde{\gamma}_{tc}(K_{2,n})$. Also, $\tilde{D}_{tc}(K_{2,n}, i) = \phi\ if\ i > n+2$. □

Lemma 2.3. *For every* $n \in \mathbb{N}$,

(i) *If* $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi, \tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, *then* $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.
(ii) *If* $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi, \tilde{D}_{tc}(K_{2,n-1}, i) = \phi$, *then* $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.
(iii) *If* $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi, \tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, *then* $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.

Proof. (i) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, by lemma 2.2 (i) and (ii), $i < 3\ or\ i > n+2\ and\ 2 < i \leq n+1 < n+2$. Therefore, by Lemma 2.2(ii), $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.

(ii) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) = \phi$, by lemma 2.2 (i) and (ii), $3 \leq i \leq n+2\ and\ i < 2\ or\ i > n+1$. We get $2 \leq i \leq n+2$. Therefore, by lemma 2.2(ii), $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.

(iii) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, by Lemma 2.2$(ii)$, $3 \leq i \leq n+2\ and\ 2 \leq i \leq n+1$. We get $2 \leq i \leq n+2$. Therefore, by lemma 2.2(ii), $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$.

□

Lemma 2.4. *Suppose* $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$, *we have*

(i) $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi, \tilde{D}_{tc}(K_{2,n}, i) \neq \phi$ *and* $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$ *if and only if* $i = 2$.
(ii) $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$, $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$ *and* $\tilde{D}_{tc}(K_{2,n-1}, i) = \phi$ *if and only if* $i = n+2$.
(iii) $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi, \tilde{D}_{tc}(K_{2,n}, i) \neq \phi$ *and* $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$ *if and only if* $2 < i \leq n+1$

Proof. (i) $(\Rightarrow)$ Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi$, by lemma 2.2$(ii)$, $i < 3\ or\ i > n+2$

$$\Rightarrow i \leq 2 \tag{2.1}$$

Since $\tilde{D}_{tc}(K_{2,n}, i) \neq \phi$, by Lemma 2.2($ii$), $2 \leq i \leq n+2$

$$\Rightarrow i \geq 2 \tag{2.2}$$

From (2.1) and (2.2), $i = 2$

($\Leftarrow$) Suppose $i = 2$, then by lemma 2.2(i) and (ii), we have

$$\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi, \tilde{D}_{tc}(K_{2,n}, i) \neq \phi \ and \ \tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$$

(ii) ($\Rightarrow$) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$, by lemma 2.2($ii$), $3 \leq i \leq n+2$

$$\Rightarrow i \leq n+2 \tag{2.3}$$

Since $\tilde{D}_{tc}(K_{2,n-1}, i) = \phi$, by lemma 2.2($i$), $i < 2 \ or \ i > n+1$

$$\Rightarrow i \geq n+2 \tag{2.4}$$

From (2.3) and (2.4), $i = n+2$

($\Leftarrow$) Suppose $i = n+2$ then $i - 1 = n+1$, by Lemma 2.2(ii),we have the result.

(iii) ($\Rightarrow$) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$, by lemma 2.2($ii$),

$$3 \leq i < n+1 \tag{2.5}$$

Since $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, by lemma 2.2($ii$),

$$3 < i \leq n+1 \tag{2.6}$$

From (2.5) and (2.6),

$$3 \leq i \leq n+1$$

($\Leftarrow$) Suppose, $3 \leq i \leq n+1$, then by lemma 2.2(ii), we have the result.

□

Theorem 2.5. *For every $n \in \mathbb{N}$ and $i \geq 2$*

(i) *If $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, then*

$$\tilde{D}_{tc}(K_{2,n}, i) = \left\{ \begin{array}{c} \{1,3\}, \{1,4\}, ...\{1, n+2\} \\ \{2,3\}, \{2,4\}, ...\{2, n+2\} \end{array} \right\}$$

(ii) *If $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) = \phi$, then*

$$\tilde{D}_{tc}(K_{2,n}, i) = \{[n+2]\}$$

(iii) *If $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, then*

$$\tilde{D}_{tc}(K_{2,n}, i) = \left\{ \begin{array}{c} \{X_1 \cup \{n+2\} / X_1 \in \tilde{D}_{tc}(K_{2,n-1}, i-1)\} \cup \\ \{X_2 / X_2 \in \tilde{D}_{tc}(K_{2,n-1}, i) \end{array} \right\}$$

if $2 < i < n$ or $i = n+1$

(iv) *If $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$, then*

$$\tilde{D}_{tc}(K_{2,n}, i) = \left\{ \begin{array}{c} \{X_1 \cup \{n+2\} / X_1 \in \tilde{D}_{tc}(K_{2,n-1}, i-1)\} \cup \\ \{X_2 \cup \{4, 5, ..., n-1, n, n+1\} / X_2 \in \tilde{D}_{tc}(K_{2,1}, 2) \end{array} \right\}$$

if $i = n$

Proof. (i) Since $\tilde{D}_{tc}(K_{2,n-1}, i-1) = \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$. By lemma 2.4 ($i$), $i = 2$. Clearly, $\{1,3\}, \{1,4\}, ...\{1, n+2\}, \{2,3\}, \{2,4\}, ...\{2, n+2\}$ are the only sets with two elements which are totally outer- connected dominate all the elements of $K_{2,n}$. No other set with cardinality 2 totally outer- connected dominates $K_{2,n}$. So,

$$\tilde{D}_{tc}(K_{2,n}, i) = \left\{ \begin{array}{l} \{1,3\}, \{1,4\}, ...\{1, n+2\} \\ \{2,3\}, \{2,4\}, ...\{2, n+2\} \end{array} \right\}$$

(ii) We have $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) = \phi$. By lemma 2.4($ii$),we have $i = n+2$. So,

$$\tilde{D}_{tc}(K_{2,n}, i) = \{[n+2]\}$$

(iii) We have $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$. Let X_1 be a total outer- connected dominating set of $K_{2,n-1}$ with cardinality $i-1$. All the elements of $\tilde{D}_{tc}(K_{2,n-1}, i-1)$ ends with $n-1$ and n. Therefore $n-1 \in X_1$ and $n \in X_1$ adjoin $n+2$ with X_1. Hence every X_1 of $\tilde{D}_{tc}(K_{2,n-1}, i-1)$ belongs to $\tilde{D}_{tc}(K_{2,n}, i)$ by adjoining $n+2$ only. Let X_2 be a total outer- connected dominating set of $K_{2,n-1}$ with cardinality i. All the elements of $\tilde{D}_{tc}(K_{2,n-1}, i)$ ends with $n+2$. Therefore every X_2 of $\tilde{D}_{tc}(K_{2,n-1}, i)$ belongs to $\tilde{D}_{tc}(K_{2,n}, i)$.

Conversely, suppose $Z \in \tilde{D}_{tc}(K_{2,n}, i)$. Here all the elements of $\tilde{D}_{tc}(K_{2,n}, i)$ ends with $n+2$ or $n+1$ or n. Suppose $n+2 \in Z$, then $Z = X_1 \cup \{n+2\}$ where X_1 ends with n and $n-1$, $X_1 \in \tilde{D}_{tc}(K_{2,n-1}, i-1)$. Suppose $n+1 \in Z$,$n+2 \notin Z$, then $Z = X_2$ where X_2 ends with $n+2$, $X_2 \in \tilde{D}_{tc}(K_{2,n-1}, i)$. Suppose $n \in Z$,$n+1 \notin Z$,$n+2 \notin Z$, then $Z = X_2$ where X_2 ends with $n+2$, $X_2 \in \tilde{D}_{tc}(K_{2,n-1}, i)$.

(iv) We have $\tilde{D}_{tc}(K_{2,n-1}, i-1) \neq \phi$ and $\tilde{D}_{tc}(K_{2,n-1}, i) \neq \phi$. Let X_1 be a total outer- connected dominating set of $K_{2,n-1}$ with cardinality $i-1$. All the elements of $\tilde{D}_{tc}(K_{2,n-1}, i-1)$ ends with n and $n+1$. Therefore, $n+1 \in X_1$ and $n \in X_1$ adjoin $n+2$ with X_1. Hence every X_1 of $\tilde{D}_{tc}(K_{2,n-1}, i-1)$ belongs to $\tilde{D}_{tc}(K_{2,n}, i)$ by adjoining $n+2$ only. Let X_2 be a total outer- connected dominating set of $K_{2,1}$ with cardinality 2. All the elements of $\tilde{D}_{tc}(K_{2,1}, 2)$ ends with 3. Therefore, $3 \in X_2$ adjoin $\{4, 5, ..., n-1, n, n+1\}$ with X_2. Hence every X_2 of $\tilde{D}_{tc}(K_{2,1}, 2)$ belongs to $\tilde{D}_{tc}(K_{2,n}, i)$ by adjoining $\{4, 5, ..., n-1, n, n+1\}$.

Conversely, suppose $Z \in \tilde{D}_{tc}(K_{2,n}, i)$. Here all the elements of $\tilde{D}_{tc}(K_{2,n}, i)$ ends with $n+1$ or $n+2$. Suppose $n+1 \in Z$, then $Z = X_2 \cup \{4, 5, ..., n-1, n, n+1\}$ where X_2 ends with 3, $X_2 \in \tilde{D}_{tc}(K_{2,1}, 2)$. Suppose $n+2 \in Z$, then $Z = X_1 \cup \{n+2\}$ where X_1 ends with n and $n+1$, $X_1 \in \tilde{D}_{tc}(K_{2,n-1}, i-1)$.

□

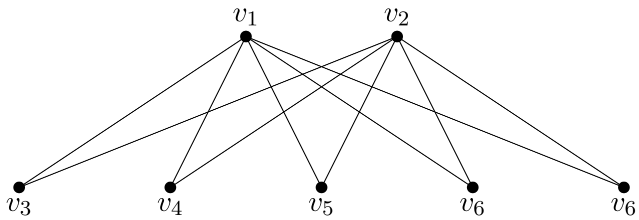

Figure 2:$K_{2,5}$

Consider this example, here $n = 5$.

By theorem 2.5 (i), we get, $\tilde{D}_{tc}(K_{2,5}, 2) = \left\{ \begin{array}{l} \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\} \\ \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\} \end{array} \right\}$

By theorem 2.5 (ii), we get, $\tilde{D}_{tc}(K_{2,5},7) = \{1,2,3,4,5,6,7\}$
By theorem 2.5 (iii), we get, $i = 3,4,6$.
For $i = 3$, $X_1 \in \tilde{D}_{tc}(K_{2,4},2) = \left\{ \begin{array}{c} \{1,3\},\{1,4\},\{1,5\},\{1,6\} \\ \{2,3\},\{2,4\},\{2,5\},\{2,6\} \end{array} \right\}$. Also, $X_2 \in \tilde{D}_{tc}(K_{2,4},3) = \left\{ \begin{array}{c} \{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\} \\ \{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\} \end{array} \right\}$. By combining this, we get, $\tilde{D}_{tc}(K_{2,5},3) = \left\{ \begin{array}{c} \{1,3,7\},\{1,4,7\},\{1,5,7\},\{1,6,7\},\{2,3,7\} \\ \{2,4,7\},\{2,5,7\},\{2,6,7\},\{1,3,4\},\{1,3,5\} \\ \{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\} \\ \{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\} \end{array} \right\}$.
For $i = 4$, $X_1 \in \tilde{D}_{tc}(K_{2,4},3) = \left\{ \begin{array}{c} \{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\} \\ \{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\} \end{array} \right\}$.
Also, $X_2 \in \tilde{D}_{tc}(K_{2,4},4) = \left\{ \begin{array}{c} \{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\} \\ \{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\} \end{array} \right\}$. By combining this, we get, $\tilde{D}_{tc}(K_{2,5},4) = \left\{ \begin{array}{c} \{1,3,4,5\},\{1,3,5,7\},\{1,3,6,7\},\{1,4,5,7\},\{1,4,6,7\} \\ \{1,5,6,7\},\{2,3,4,7\},\{2,3,5,7\},\{2,3,6,7\},\{2,4,5,7\} \\ \{2,4,6,7\},\{2,5,6,7\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\} \\ \{1,4,5,6\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\} \end{array} \right\}$.
For $i = 6$, $X_1 \in \tilde{D}_{tc}(K_{2,4},5) = \left\{ \begin{array}{c} \{1,2,3,4,5\},\{1,2,3,4,6\},\{1,2,3,5,6\} \\ \{1,2,4,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\} \end{array} \right\}$. Also, $X_2 \in \tilde{D}_{tc}(K_{2,4},6) = \{1,2,3,4,5,6\}$. By combining this, we get,
$\tilde{D}_{tc}(K_{2,5},6) = \left\{ \begin{array}{r} \{1,2,3,4,5,7\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},\{1,2,4,5,6,7\} \\ \{1,3,4,5,6,7\}\{2,3,4,5,6,7\},\{1,2,3,4,5,6\} \end{array} \right\}$.
By theorem 2.5 (iv), we get, $\tilde{D}_{tc}(K_{2,5},5)$.
Here, $X_1 \in \tilde{D}_{tc}(K_{2,4},4) = \left\{ \begin{array}{c} \{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\} \\ \{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\} \end{array} \right\}$. Also, $X_2 \in \tilde{D}_{tc}(K_{2,1},2) = \{1,3\},\{2,3\}$. By combining this, we get,
$\tilde{D}_{tc}(K_{2,5},5) = \left\{ \begin{array}{c} \{1,3,4,5,7\},\{1,3,4,6,7\},\{1,3,5,6,7\},\{1,4,5,6,7\},\{2,3,4,5,7\} \\ \{2,3,4,6,7\},\{2,3,5,6,7\},\{2,4,5,6,7\},\{1,3,4,5,6\},\{2,3,4,5,6\} \end{array} \right\}$.

3. Total Outer- Connected Domination Polynomial of $K_{2,n}$

Theorem 3.1. *Let $\tilde{D}_{tc}(K_{2,n},i)$ be the family of all total outer- connected dominating sets with cardinality i and let $\tilde{d}_{tc}(K_{2,n},i) = |\tilde{D}_{tc}(K_{2,n},i)|$.*

(i) *If $i = 2,n$. Then $\tilde{d}_{tc}(K_{2,n},i) = 2\tilde{d}_{tc}(K_{2,n-1},i) - \tilde{d}_{tc}(K_{2,n-2},i) \forall n \geq 2$*
(ii) *Otherwise $\tilde{d}_{tc}(K_{2,n},i) = \tilde{d}_{tc}(K_{2,n-1},i-1) + \tilde{d}_{tc}(K_{2,n-1},i)$.*
with intial values $\tilde{D}_{tc}(K_{2,1},x) = 2x^2 + x^3$, $\tilde{D}_{tc}(K_{2,2},x) = 4x^2 + 4x^3 + x^4$

Proof. (i) It follows from theorem 2.5.(i) and (iv)
(ii) It follows from theorem 2.5.(ii) and (iii)

□

Theorem 3.2. *The following properties hold for co-efficients of $\tilde{D}_{tc}(K_{2,n},x)$:*

(i) $\tilde{d}_{tc}(K_{2,n},n+2) = 1$,*for every $n \in N$*
(ii) $\tilde{d}_{tc}(K_{2,n},n+1) = n+2$,*for every $n \geq 2$*

n/i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	2	1												
2	0	4	4	1											
3	0	6	6	5	1										
4	0	8	12	8	6	1									
5	0	10	20	20	10	7	1								
6	0	12	30	40	30	12	8	1							
7	0	14	42	70	70	42	14	9	1						
8	0	16	56	112	140	112	56	16	10	1					
9	0	18	72	168	252	252	168	72	18	11	1				
10	0	20	90	240	420	504	420	240	90	20	12	1			
11	0	22	110	330	680	924	924	680	330	110	22	13	1		
12	0	24	132	440	1010	1604	1848	1604	1010	440	132	24	14	1	
13	0	26	156	572	1450	2614	3452	3452	2614	1450	572	156	26	15	1

TABLE 1. $\tilde{d}_{tc}(K_{2,n}, i)$, the number of total outer- connected dominating sets of $K_{2,n}$ with cardinality i

(*iii*) $\tilde{d}_{tc}(K_{2,n}, 2) = 2n$, *for every* $n \in N$

(*iv*) $\tilde{d}_{tc}(K_{2,n}, 3) = \sum_{i=1}^{n-1} \tilde{d}_{tc}(K_{2,i}, 2)$, *for every* $n \geq 3$

(*v*) $\tilde{d}_{tc}(K_{2,n}, n-1) = \tilde{d}_{tc}(K_{2,n-1}, n-2) + \tilde{d}_{tc}(K_{2,n-1}, n-1)$, *for every* $n \geq 4$

Proof. (*i*) Since $\tilde{D}_{tc}(K_{2,n}, n+2) = \{[n+2]\}$, we have $\tilde{d}_{tc}(K_{2,n}, n+2) = 1$.

(*ii*) Since $\tilde{D}_{tc}(K_{2,n}, n+1) = \{[n+2] - \{x\}/x \in [n+2]\}$, we have $\tilde{d}_{tc}(K_{2,n}, n+1) = n+2$.

(*iii*) By induction on n. The result is true for $n = 1$. L.H.S: $\tilde{d}_{tc}(K_{2,1}, 2) = 2$ $(from\ table)$. R.H.S: $2(1) = 2$. Therefore the result is true for $n = 1$. Now suppose the result is true for all natural numbers less than n. By theorem 3.1(*i*),

$$\begin{aligned} \tilde{d}_{tc}(K_{2,n}, 2) &= 2\tilde{d}_{tc}(K_{2,n-1}, 2) - \tilde{d}_{tc}(K_{2,n-2}, 2) \\ &= 2[2(n-1)] - 2(n-2) \\ &= 4n - 2n \\ &= 2n \end{aligned}$$

.

(*iv*) We prove this theorem by induction on n. Suppose $n = 3$, $\tilde{d}_{tc}(K_{2,3}, 3) = 6 = \sum_{i=1}^{2} \tilde{d}_{tc}(K_{2,i}, 2)$. Therefore the result is true for $n = 1$. Now suppose the result is true for all natural numbers less than n. By theorem 3.1(*ii*),

$$\begin{aligned} \tilde{d}_{tc}(K_{2,n}, 3) &= \tilde{d}_{tc}(K_{2,n-1}, 2) + \tilde{d}_{tc}(K_{2,n-1}, 3). \\ &= \tilde{d}_{tc}(K_{2,n-1}, 2) + \sum_{i=1}^{n-2} \tilde{d}_{tc}(K_{2,i}, 2). \\ &= \sum_{i=1}^{n-1} \tilde{d}_{tc}(K_{2,i}, 2). \end{aligned}$$

(*v*) By theorem 3.1(*ii*), we have, $\tilde{d}_{tc}(K_{2,n}, n-1) = \tilde{d}_{tc}(K_{2,n}, n-2) + \tilde{d}_{tc}(K_{2,n-1}, n) - \tilde{d}_{tc}(K_{2,n-1}, n) + \tilde{d}_{tc}(K_{2,n-1}, n-1)$. Therefore, $\tilde{d}_{tc}(K_{2,n}, n-1) = \tilde{d}_{tc}(K_{2,n-1}, n-2) + \tilde{d}_{tc}(K_{2,n-1}, n-1)$.Therefore, we have the result.

□

4. Conclusion

Total outer-connected domination ensures that any field dealing with complex networks or systems where you need to identify a robust, influential core while maintaining the integrity and connectivity. This enhances critical infrastructure management like power grids, water networks,transportation by identifying a minimal set of crucial substations, junction or hubs. All other parts of the grid or network are supplied. It is particularly important for resource allocation and deployment like emergency services, military, Humanitarain aid, security and surveillance systems.

Complete Bipartite graph are well known in recommendation systems like user- item interactions. One set of vertices represents users, and the other represents items. An edge indicates a user's interaction. While actual recommendation systems use more complete bipartite graph represents the theoretical ideal where every user could potentially interact with every item.

This kind of research has many applications because total outer- connected domination insights into the network design, monitoring, logistics, supply chain management, environmental monitoring. This is valuable in planning, developmnt of new networks. We can expand this study by examining various other generalized graphs like complete bipartite graphs available in the litrature. We explore total outer- connected dominating sets and polynomials in these configurations.

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