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# Shape and topological optimization for a fractional elliptic boundary problem

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**Abstract.** In this paper, we consider a shape optimization problem associated with the fractional Laplacian. We focus on $J(\Omega) = j(\Omega, u_\Omega)$ where $u_\Omega$ is the solution of (1.3). We give an existence of optimal shape using different methods. These results are based compactness and $\epsilon-$ cône property. We also establish the shape derivative and topological derivative of the functional using the minmax method.

**Keywords:** Shape optimization, shape derivative, optimal conditions, fractional Laplacian.

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## 1. **Introduction**

Fractional calculus is a theory of integrals and derivatives of arbitrary order, that unifies and generalizes the notions of integer-order differentiation and $n$-fold repeated integration. Warma [37] clarified the Neumann and Robin boundary conditions associated with the fractional Laplacian operator in open subsets of $\mathbb{R}^N$. It shows the existence and regularity of weak solutions. For questions regarding regularity, refer to the authors listed below. These types of problems were first studied in more detail by Cafarelli and Sylvestre [8]. In addition, Ros-Oton and Serra [28, 29, 30] provided another way of representing the regularity up to the boundary of the domain $\Omega$ for the solution $u$ of system (1.3). Fall [21] extended the results in [30], where $u$ is the non-local Schrodinger solution. Dalibart and Gerard-Varet proved the shape derivative in the case where $s = \frac{1}{2}$. Fall et al. [19] generalize Dalibart and Gerard-Varet's work [11] to the case of the fractional Laplacian, using shape optimization techniques. In [19] the authors studied the optimality of the latter. For questions regarding the existence of an optimal form, we refer to the works of Allaire, Henrot and Pierre [1], Allaire, Henrot and Bucur [4], who studied the existence of an optimal form in the case of the classical Laplacian. We also generalize the work to the p-Laplacian case. For this, an optimal form existence study was established using s-quasi opens. The cited works permit us to study the problem of the existence of an optimal form in a much more general way, using two different methods, but also proposes a new technique for calculating the shape derivative the minimax approach and establishes the topological derivative using the same theoretical framework.

Let $\mathcal{U}_{ad} \subset \mathbb{R}^N, N \geq 2$, denote the admissible set. Let $\Omega$ be a domain of $\mathcal{U}_{ad}$. The solution of the optimization problem

$$\min \{J(\Omega); \Omega \in \mathcal{U}_{ad}\} \tag{1.1}$$

with $J : \mathcal{U}_{ad} \to \mathbb{R}$ a shape functional defined by

$$J(\Omega) = C(N,2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\Omega(x) - u_\Omega(y)|^2}{|x-y|^{N+2s}} dxdy. \tag{1.2}$$

State $u_\Omega$ is a solution to the following problem (1.3).

The aim of this study is to prove existence, and optimal conditions. First, we will look at

$$\left\{ \begin{array}{lll} (-\Delta)^s u_\Omega = f & \text{in} & \Omega, \\ u_\Omega \ = 0 & \text{on} & \mathbb{R}^N \backslash \Omega. \end{array} \right. \tag{1.3}$$

with $0 < s < 1$. We look for the existence of solutions to the fractional elliptic boundary problem (1.3) and the shape optimization problem (1.1) under the hypothesis of $(s-\gamma)-$ convergence.

Then, the first-order optimality conditions of (1.1) is given by calculating the shape and topological derivatives of the functional (1.2) using the minimax method. This problem has already been studied by Fall et al. [19], in which the existence of a solution is obtained under the hypothesis of a uniform cone, and the form resolution is obtained using the vector field method.

In this study, motivated by [11] and [19], we aim to prove the existence of an optimal shape solution to the minimization problem (1.1). More precisely, our main results are as follows:

**Theorem 1.1.** *Let $F : \mathcal{A}(D) \longrightarrow (-\infty, +\infty]$ be a shape functional that is lower $(s-\gamma)$ semi-continuous weak. Subsequently, the following problem*

$$\min\{F(\Omega) : \Omega \in \mathcal{A}(D)\} \tag{1.4}$$

*has a solution.*

**Theorem 1.2.** *Let $\mathcal{O} = \{\Omega \subset \mathbb{R}^N,\ vol(\Omega) = c,\ \partial\Omega \in \mathcal{C}^2\}$, and $J$ be defined by (1.2). Then there exists a domain $\Omega \in \mathcal{O}$ such that*

$$J(\Omega) = \inf_{\Omega \in \mathcal{O}} J(\Omega)$$

*under the constraints*

$$\begin{cases} (-\Delta)^s u_\Omega = f & in \quad \Omega \\ u_\Omega = 0 & on \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.5}$$

**Theorem 1.3.** *Let $\Omega$ be the solution to the optimization problem $\min\{J(\Omega),\ \Omega \in \mathcal{O}\}$. If the function $\mathcal{R}(\epsilon)$ has a finite limit $\mathcal{R}(u,p)$, then the shape derivative of $J(\Omega)$ is given by*

$$\begin{aligned} DJ(\Omega, V) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |p(x)-p(y)|^2 K_0'(x,y)\,dxdy \\ &+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(p(x)-p(y))K_0'(x,y)\,dxdy \\ &- \int_\Omega (\nabla f\cdot V(0))p\,dx - \int_\Omega fp\,divV(0)\,dx + \mathcal{R}(u,p), \end{aligned}$$

*where $p$ is solution to the following adjoint equation*

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(\varphi'(x)-\varphi'(y))K_0(x,y)\,dxdy \\ &\qquad = -\frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\varphi'(x)-\varphi'(y))(p(x)-p(y))K_0(x,y)\,dxdy. \end{aligned}$$

**Theorem 1.4.** *Let $0 \le d < N$, $E$ verify **Hypothesis H1** and $t = \alpha_{N-d} r^{N-d}$. The topolo*
*gical derivative exists if the function $\mathcal{R}(t)$ has a finite limit denoted $R(u_{\Omega_0}, p_{\Omega_0})$. Therefore, the topological derivative of the function is given by:*

$$dJ = \lim_{t\longrightarrow 0}\frac{J(\Omega_t)-J(\Omega)}{\alpha_{N-d}r^{N-d}}$$

$$dJ = R(u_{\Omega_0}, p_{\Omega_0}) - f(x_0)\ p_{\Omega_0}(x_0).$$

*where $p_{\Omega_0}, u_{\Omega_0}$ are solutions of systems*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi'(x)-\phi'(y))\left[2\,C(N,s)\,(u_{\Omega_0}(x)-u_{\Omega_0}(y))+(p_{\Omega_0}(x)-p_{\Omega_0}(y))\right]}{\mid x-y\mid^{N+2s}}dxdy=0.$$

The paper is organized as follows: Section 2 is devoted to the preliminaries of fractional Laplacian operators, as well as the minmax theory. In Section 3, we study the existence of a solution to the shape optimization problem under the $(s-\gamma)-$ convergence hypothesis. At this level, we will use two hypotheses of solution existence, with $(s-\gamma)-$ convergence: the study of solution existence with $(s-\gamma)-$ convergence under the assumption of compactness of the admissible set and the study of solution existence with the $\epsilon-$ cône property. Section 4 is devoted to prove existence of the shape and topological derivatives of the functional under consideration. To accomplish this, we use the minimax theory.

## 2. Preliminaries

### 2.1. On the Fractional Problem.

The following results can be found in [8] and [37].

**Theorem 2.1.** *Let $s\in(0,1)$ and $p\in[1,+\infty)$, $q\in[1,p]$, $\Omega\subset\mathbb{R}^N$ be a bounded extension domain for $W^{s,p}$ and $T$ be a bounded subset of $L^p$. Suppose that*

$$\sup_{f\in T}\left(\int_\Omega\int_\Omega\frac{\mid f(x)-f(y)\mid^p}{\mid x-y\mid^{N+ps}}dxdy\right)<+\infty.$$

*Then $T$ is pre-compact in $L^q$.*

*Proof.* See [19]. □

**Corollary 2.2.** *Let $s\in(0,1)$, $p\in[1,+\infty)$ such that $sp<N$. If $q\in[1,p^*)$, $\Omega\subseteq\mathbb{R}^N$ is a bounded extension domain for $W^{s,p}$ and $T$ are bounded subsets of $L^p$. Suppose that*

$$\sup_{f\in T}\left(\int_\Omega\int_\Omega\frac{\mid f(x)-f(y)\mid^p}{\mid x-y\mid^{N+ps}}dxdy\right)<+\infty.$$

*Then $T$ is pre-compact in $L^q$.*

*Proof.* See [19]. □

**Definition 2.3.** *Let $\Omega\subset\mathbb{R}^N$ be an open set $u\in\mathcal{L}^1_s$. The distribution $(-\Delta)^s u$ is defined by :*

$$<(-\Delta)^s u,\varphi>=\int_{\mathbb{R}^N}u(-\Delta)^s\varphi dx,\ \forall\ \varphi\in C_c^\infty(\Omega). \tag{2.1}$$

*Saying that $(-\Delta)^s u=f$ in $D'(\Omega)$, is equivalent to the very weak formulation*

$$\int_{\mathbb{R}^N}u(-\Delta)^s\varphi dx=\int_\Omega f\varphi dx,\ \forall\ \varphi\in C_c^\infty(\Omega). \tag{2.2}$$

**Definition 2.4.** *We define* $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^s}}$*, as the completion of* $C_c^\infty(\Omega)$*, which is an Hilbert space with respect to the norm :*

$$\| \varphi \|_{D^{s,2}(\Omega)} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[\varphi(x) - \varphi(y)]^2}{| x - y |^{N+2s}} dxdy \right)^{\frac{1}{2}}. \tag{2.3}$$

*If* $u \in D^{s,2}(\Omega) \subset \mathcal{L}_s^1$ *satisfies :*
$(-\Delta)^s u = f$ *in* $D'(\Omega)$*, we have the weak formulation :*

$$< (-\Delta)^s u, \varphi >_{D^{s,2}(\Omega)} = \int_\Omega f \varphi dx, \; \forall \; \varphi \in D(\Omega), \tag{2.4}$$

*where*

$$< u, \varphi >_{D^{s,2}(\Omega)} = C(N,s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{| x - y |^{N+2s}} dxdy.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary, and $0 < s < 1$. Note here that for the space of smooth functions with compact support, we take the notation $C_c^\infty$ instead of $C_0^\infty$.
Consider the following bilinear form :

$$<,>_{D^{s,2}} : C_c^\infty(\Omega) \times C_c^\infty(\Omega) \longrightarrow \mathbb{R}$$
$$(u,v) \mapsto \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{| x - y |^{N+2s}} dxdy$$

which is the scalar product of $C_c^\infty(\Omega)$. We recall the Hilbert space $D^{s,2}(\Omega)$ as the completion of $C_c^\infty(\Omega)$.

**Lemma 2.5.** *If* $\Omega$ *is a bounded Lipschitz open set*

$$D^{s,2}(\Omega) = \left\{ u \in H^s(\mathbb{R}^N), \text{ such that } u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.$$

**Proposition 2.6.** *Let* $0 < s < 1$ *and* $\Omega$ *be a bounded open set subset of* $\mathbb{R}^N$*. Let* $f : \Omega \longrightarrow \mathbb{R}$ *be a measurable function with compact support. Then, there exists a positive constant* $C = C(N,s,\Omega)$ *depending on* $N$*,* $s$ *and* $\Omega$ *such that*

$$\| f \|_{L^2(\Omega)} \leq C \| f \|_{D^{s,2}(\Omega)}.$$

For more information on this subsection, the reader may also refer to [19]. We also have the following more general results:

**Definition 2.7.** *Let* $0 < s < 1$ *and* $p \in [2, +\infty)$*,* $N \geq sp$ *and* $\Omega$ *be a bounded open set of* $\mathbb{R}^N$*, with the Lipchitz boundary,*

$$[f]_{s,p} = \left( \int_\Omega \int_\Omega \frac{| f(x) - f(y) |^p}{| x - y |^{N+ps}} dxdy \right)^{\frac{1}{p}}$$

*be the Gagliardo semi norm of measurable function* $u$*.*
*1.* $W^{s,p}(\Omega)$ *are defined as follows*

$$W^{s,p}(\Omega) = \{ f \in L^p(\Omega) \text{ such that } [f]_{s,p} < +\infty \}$$

*endowed with the usual norm*

$$\| f \|_{W^{s,p}(\Omega)}= \left(\int_\Omega |f|^p_{L^p} + [f]_{s,p}\right)^{\frac{1}{p}}.$$

*2. Consider the closed linear subspace* $W_0^{s,p}(\Omega)$ *by*

$$W_0^{s,p}(\Omega) = \left\{ f \in W^{s,p}(\mathbb{R}^N) : \ f = 0; \, a.e \, in \, \mathbb{R}^N \setminus \Omega \right\}.$$

*equivalently renormed by setting* $\|f\|_{s,p} = [f]_{s,p}$.

**Definition 2.8.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. Given* $A \subset \Omega$, *for any* $0 < s < 1$ *and* $p \geq 1$, *the Gagliardo* $s-$*capacity of* $A$ *relatively to* $\Omega$ *as*

$$cap_s(A, \Omega) = \inf \left\{ [u]^p_s : \ u \in \mathcal{C}(\Omega), \ u \geq 0, \ A \subset \{u \geq 1\} \right\},$$

*where*

$$[u]^p_s = \int_\Omega \int_\Omega \frac{| \, u(x) - u(y) \, |^p}{| \, x - y \, |^{N+ps}} dxdy.$$

**Definition 2.9.** *A subset* $A$ *of* $\Omega$ *is a s-quasi open set if there exists a decreasing sequence* $\{w_k\}_{k\in\mathbb{N}}$ *of open subsets of* $\Omega$ *such that* $cap_s(w_k, \Omega) \longrightarrow 0$, *as* $k \longrightarrow +\infty$, *and* $A \cup w_k$ *is an open set for all* $k \in \mathbb{N}$.

We now provide a definition of $(s - \gamma)-$ convergence. This definition was inspired by $\gamma-$ convergence, [6, 7, 22].

**Definition 2.10.** *Let* $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{A}_s(\Omega)$ *and* $A \in \mathcal{A}_s(\Omega)$. *We say that* $A_k \xrightarrow{\gamma_s} A$ *if* $u^s_{A_k} \longrightarrow u^s_A$ *strongly in* $L^2(\Omega)$.

**Definition 2.11.** *Let* $0 < s < 1$ *be fixed and let* $F_s : \mathcal{A}_s(\Omega) \longrightarrow \mathbb{R}$ *be such that:* $F_s$ *is lower semi continuous with respect to the* $(s - \gamma)-$ *convergence; that is*

$$A_k \xrightarrow{\gamma_s} A \ \ implies \ \ F_s(A) \leq \lim_{k \longrightarrow +\infty} \inf F_s(A_k).$$

$F_s$ *is decreasing with respect to set inclusion; that is* $F_s(A) \geq F_s(B)$ *whenever* $A \subset B$.

For more information on this theory, please consult [20].

**Theorem 2.12.** *: Let* $\Omega_n$ *be an open sequence in the class* $\mathcal{O}_\epsilon$. *Then there exists an open* $\Omega \in \mathcal{O}_\epsilon$ *and an sub-sequence* $\Omega_{n_k}$ *which converges towards* $\Omega$ *both in the sense of Hausdorff, in the sense of the characteristic functions and in the sense of compact. Additionally,* $\overline{\Omega}_{n_k}$ *and* $\partial\Omega_{n_k}$ *converge in the Hausdorff sense to* $\overline{\Omega}$ *and* $\partial\Omega$.

*Proof.* See [19, 25]. $\square$

**Lemma 2.13.** *: Let $K$ be a compact and $B$ be a bounded open of $\mathbb{R}^N$. Let $\Omega_n$ be a sequence of open with $\overline{\Omega}_n \subset K \subset B$, verifying the ownership of the $\epsilon-$ cône.*
*Then there is an open $\Omega$ satisfying the ownership of the $\epsilon-$ cône and an extracted sequence $\Omega_{n_k}$ such as*

$$\Omega_{n_k} \xrightarrow{\ H\ } \Omega,\ \chi_{\omega_{n_k}} \xrightarrow{\ L^1 p.p\ } \chi_\Omega,$$

$$\overline{\Omega}_{n_k} \xrightarrow{\ H\ } \overline{\Omega},\ \partial\Omega_{n_k} \xrightarrow{\ H\ } \partial\Omega.$$

*This is allowed us to characterize the existence of a solution.*

*Proof.* See [19, 25]. □

2.2. **Some initial results for the minmax method.** In this subsection, we describe how the calculation of the topological derivative using the minmax approach, [13, 14, 26]. To First, we consider the following definitions and notations.

**Definition 2.14.** *A Lagrangian function is a function of the form*

$$(t, x, y) \mapsto L(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R} \quad \tau > 0$$

*where $X$ is a vector space, $Y$ is a non empty subset of the vector space and the function $y \mapsto L(t, x, y)$ is affine.*

Associate with the parameter $t$ the parametrized minimax

$$t \mapsto \mathrm{g}(t) = \inf_{x\in X} \sup_{y\in Y} L(t, x, y) : [0, \tau] \to \mathbb{R} \quad \text{and} \quad d\mathrm{g}(0) = \lim_{t\to 0^+} \frac{\mathrm{g}(t) - \mathrm{g}(0)}{t}.$$

When the limits exist, we will use the following notations

$$\begin{aligned} d_t L(0, x, y) &= \lim_{t\to 0^+} \frac{L(t, x, y) - L(0, x, y)}{t} \\ \varphi \in X,\ \ d_x L(t, x, y; \varphi) &= \lim_{\theta\to 0^+} \frac{L(t, x + \theta\varphi, y) - L(t, x, y)}{\theta} \\ \phi \in Y\ \ d_y L(t, x, y; \phi) &= \lim_{\theta\to 0^+} \frac{L(t, x, y + \theta\phi) - L(t, x, y)}{\theta}. \end{aligned}$$

Since $L(t, x, y)$ is affine in $y$, for all $(t, x) \in [0, \tau] \times X$,

$$\forall\, y, \psi \in Y\ \ d_y L(t, x, y; \psi) = L(t, x, \psi) - L(t, x, 0) = d_y L(t, x, 0, \psi). \tag{2.5}$$

The state equation at $t \geq 0$

$$\text{Find } x^t \in X \text{ such that for all } \psi \in Y,\ \ d_y L(t, x^t, 0; \psi) = 0. \tag{2.6}$$

The set of states $x^t$ at $t \geq 0$ is denoted

$$E(t) = \left\{ x^t \in X,\ \ \forall\, \psi \in Y,\ \ d_y L(t, x^t, 0; \psi) = 0 \right\}. \tag{2.7}$$

The adjoint equation at $t \geq 0$ is

$$\text{Find } p^t \in Y \ \text{ such that for all } \ \varphi \in X, \ \ d_x L(t, x^t, p^t; \varphi) = 0. \tag{2.8}$$

The set of solutions $p^t$ at $t \geq 0$ is denoted

$$Y(t, x^t) = \left\{p^t, \in Y, \ \ \forall\, \varphi \in X, \ \ d_x L(t, x^t, p^t; \varphi) = 0\right\}. \tag{2.9}$$

Finally the set of minimisers for the minimax is given by

$$X(t) = \left\{x^t \in X, \ \ \mathrm{g}(t) = \inf_{x \in X} \sup_{y \in Y} L(t, x, y) = \sup_{y \in Y} L(t, x^t, y)\right\}. \tag{2.10}$$

**Lemma 2.15. (Constrained infimum and minimax)**
*We have the following assertions*

(i)
$$\inf_{x \in X} \sup_{y \in Y} L(t, x, y) = \inf_{x \in E(t)} L(t, x, 0).$$

(ii) *The minimax $g(t) = +\infty$ if and only if $E(t) = \emptyset$. In this case $X(t) = X$.*

(iii) *If $E(t) \neq \emptyset$, then*
$$X(t) = \left\{x^t \in E(t) : \ \ L(t, x^t, 0) = \inf_{x \in E(t)} L(t, x, 0)\right\} \subset E(t)$$
*and $g(t) < +\infty$.*

*Proof.* See [13]. □

Here, we provide definitions and theorems of d-dimensional Minkowski content and d-rectifiability.

**Definition 2.16.** *Let $E$ be a subset of a metric space $X$. $E \subset X$ is d-rectifiable if it is the image of a compact subset $K$ of $\mathbb{R}^d$ by a continuous lipschitzian function $f : \mathbb{R}^d \to X$.*

Let $E$ be a closed compact set of $\mathbb{R}^N$ and $r \geq 0$, the distance function $d_E$ and the $r$-dilatation $E_r$ of $E$ are defined as follows:

$$d_E(x) = \inf_{x_0 \in E} |x - x_0|, \quad E_r = \{x \in \mathbb{R}^N : \ d_E(x) \leq r\}.$$

**Definition 2.17.** *Given $d$, $0 \leq d \leq N$ the upper and lower $d$-dimensional Minkowski contents of a set $E$ are defined by an $r$-dilatation of this set as follows*

$$M^{*d}(E) = \limsup_{r \to 0^+} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}; \quad M_*^d(E) = \liminf_{r \to 0^+} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}$$

*where $m_N$ is the Lebesgue measure in $\mathbb{R}^N$ and $\alpha_{N-d}$ is the volume of a ball of radius 1 in $\mathbb{R}^{N-d}$.*

We need the following assumption for everything that follows:

**Hypothesis (H0)**

Let $X$ be the vector space.

(i) : For all $t \in [0,\tau]$, $x^0 \in X(0)$, $x^t \in X(t)$ and $y \in Y$, the function $\theta \mapsto L(t, x^0 + \theta(x^t - x^0), y) : [0,1] \to \mathbb{R}$ is continuous. This implies that for almost all $\theta$ the derivative exists and is equal to $d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0)$ and it is the integral of its derivative. In particular

$$L(t, x^s, y) = L(t, x^0, y) + \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0)\, d\theta.$$

ii) : For all $t \in [0,\tau]$, $x^0 \in X(0)$, $x^t \in X(s)$ and $y \in Y$, $\phi \in X$ and for almost all $\theta \in [0,1]$, $d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$ exists and the functions $\theta \mapsto d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$ belong to $L^1[0,1]$

**Definition 2.18.** *Given $x^0 \in X(0)$ and $x^t \in X(t)$, the averaged adjoint equation is:*

$$\textit{Find } y^t \in Y \ \ \forall \phi \in X, \ \ \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)\, d\theta = 0.$$

*The set of solutions is denoted $Y(t, x^0, x^t)$.*
*$Y(0, x^0, x^0)$ clearly reduce to the set of standard adjoint states $Y(0, x^0)$ at $t = 0$.*

**Theorem 2.19.** *Consider the Lagrangian functional*

$$(t, x, y) \mapsto L(t, x, y) : [0,\tau] \times X \times Y \to \mathbb{R}, \ \ \tau > 0$$

*where $X$ and $Y$ are vector spaces and the function $y \mapsto L(t, x, y)$ is an affine. Assuming that **(H0)** and the following hypotheses are satisfied*
***(H1)** for all $t \in [0,\tau]$, $g(t)$ is finite, $X(t) = \{x^t\}$ and $Y(0, x^0) = \{p^0\}$ are singletons,*
***(H2)** $d_t L(0, x^0, y^0)$ exists,*
*and **(H3)** The following limit exists*

$$R(x^0, y^0) = \lim_{t \to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$$

*Then, $dg(0)$ exists and $dg(0) = d_t L(0, x^0, p^0) + R(x^0, p^0)$.*

*Proof.* See [13]. □

**Corollary 2.20.** *Consider the Lagrangian functional*

$$(t, x, y) \mapsto L(t, x, y) : [0,\tau] \times X \times Y \to \mathbb{R}, \ \ \tau > 0$$

*where $X$ and $Y$ are vector spaces and the function $y \mapsto L(t, x, y)$ is an affine. Assume that **(H0)** and the following assumptions are satisfied:*
***(H1a)** for all $t \in [0,\tau]$, $X(s) \neq \emptyset$, $g(t)$ is finite, and for each $x \in X(0)$, $Y(0, x) \neq \emptyset$,*
***(H2a)** for all $x \in X(0)$ and $p \in Y(0, x)$ $d_t L(0, x, p)$ exists,*

***(H3a)*** *there exist* $x^0 \in X(0)$ *and* $p^0 \in Y(0, x^0)$ *such that the following limit exists*

$$R(x^0, p^0) = \lim_{t \to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$$

*Then,* $dg(0)$ *exists and there exist* $x^0 \in X(0)$ *and* $p^0 \in Y(0, x^0)$ *such that* $dg(0) = d_t L(0, x^0, p^0) + R(x^0, p^0)$.

## 3. **Existence of Optimal Shape**

In this section, we are interested in the existence of an optimal form $\Omega$, solution of the problem (1.1). For the questions of existence shape we can refer to the work of Henrot and Pierre [25] and Allaire and Henrot [1]. In these works, the aforementioned authors use shape functionals dependent on non-fractional PDE solutions. We aim to replicate the same work in the fractional case by using a functional $J(\Omega)$, $u_\Omega$ is a solution of a fractional-type equation. Consequently, we obtain the following result indicating the existence of an optimal form using two distinct methods:

### 3.1. **Existence by compactness.**

Here, we weaken the assumptions, nevertheless the functional $J$ remains lower
$(s-\gamma)-$ semi-continuous for the topology of $(s-\gamma)-$convergence and we study the compactness of $\mathcal{A}(D)$ for this convergence. The idea is to penalize functional $J$. This gives us $F(\Omega) = J(\Omega) + \alpha_1[|\Omega| - c]^+$ where $\alpha \in \mathbb{R}^+$ is a penalization factor and

$$J(\Omega) = C(N, s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\Omega(x) - u_\Omega(y)|^2}{|x - y|^{N+2s}} dxdy \tag{3.1}$$

We will study a problem of the type

$$\min\{F(\Omega) : \Omega \in \mathcal{A}, |\Omega| \le c\}. \tag{3.2}$$

with $F : \mathcal{A}(D) \longrightarrow \mathbb{R}^-$ a constrained shape functional of a parabolic boundary problem with $u_\Omega$ as its solution and is defined by

$$\begin{cases} (-\Delta)^s u_\Omega = f \text{ in } \Omega \\ u = 0 \text{ on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{3.3}$$

**Proof of Theorem 1.1:** Under the constraint of problem (3.3), we begin by showing the lower semi-continuity of the functional. To do this, we set $m = \inf\{G(\Omega), \Omega \in \mathcal{A}(D)\}$ and $u_{\Omega_n} = u_n$. Since $u_n$ is a solution of (3.3), the $m > -\infty$; and there exists a minimizing sequence $\Omega_{n_k}$ contained in $\mathcal{A}(D)$ such that $G(\Omega) \longrightarrow m$.
Consider $u_n$ as a solution of the following problem

$$\begin{cases} (-\Delta)^s u_n = f \text{ in } \Omega_n \\ u_n = 0 \text{ on } \mathbb{R}^N \backslash \Omega_n. \end{cases} \tag{3.4}$$

We define the function $\tilde{u_n}$ by

$$\tilde{u_n} = \begin{cases} u_n \text{ if } x \in \Omega_n \\ 0 \text{ if } x \in D \backslash \Omega_n. \end{cases}$$

Now, from the variational formulas, we get

$$\int_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{\mid x - y \mid^{N+2s}} dxdy = \int_D f(x) v_{\Omega_{n_k}}(x) dx \quad \forall \ v \in D^{s,2}(\Omega). \tag{3.5}$$

We show that in [19] $(u_n)$ is bounded in $D^{s,2}(\Omega)$. There exists a sub-sequence $(u_{kl})_{l \geq 1}$ of $(u_k)_{k \geq 1}$ such that

$$u_{kl} \rightharpoonup u^*_\Omega \in D^{s,2}(\Omega),$$

$$u_{kl} \longrightarrow u^*_\Omega \in L^2(\Omega),$$

and

$$u_{kl} \rightharpoonup u^*_\Omega \in L^2(\Omega), \ \ if \ \ l \longrightarrow \infty.$$

By passing to the limit when $k \longrightarrow \infty$ and weak convergence, we obtain the following formulation

$$\int_{\mathbb{R}^{2N}} \frac{(u^*_\Omega(x) - u^*_\Omega(y))(\varphi(x) - \varphi(y))}{\mid x - y \mid^{N+2s}} dxdy = \int_D f(x)\varphi(x) dx, \ \ \forall \varphi \in D^{s,2}(\Omega) \tag{3.6}$$

which is the weak formulation of the following problem

$$\begin{cases} (-\Delta)^s u^*_\Omega = f \text{ in } \Omega \\ \\ u^*_\Omega = 0 \text{ on } \mathbb{R}^N \setminus D. \end{cases}$$

$\tilde{u_n}$ is bounded in $D^{s,2}(\Omega)$ there exist $M > 0$ thus that $\|\tilde{u}_{n_k}\| \leq M$ and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u^*_\Omega(x) - u^*_\Omega(y)|^2}{|x - y|^{N+2s}} dxdy \leq \lim_{k \longrightarrow \infty} \inf \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\Omega_{nk}}(x) - u_{\Omega_{nk}}(y)|^2}{|x - y|^{N+2s}} dxdy.$$

On the other hand, the lower semi-continuity of the Lebesgue measure leads

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u^*_\Omega(x) - u^*_\Omega(y)|^2}{|x - y|^{N+2s}} dxdy + \alpha \left( m_L(\Omega) - c \right)$$

$$\leq \lim_{k \longrightarrow \infty} \inf \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_{\Omega_{nk}}(x) - \tilde{u}_{\Omega_{nk}}(y)|^2}{|x - y|^{N+2s}} dxdy + \lim_{k \longrightarrow \infty} \inf \left( (m_L(\Omega_{nk}) - c \right).$$

Then we have :

$$F(\Omega) \leq \lim_{k \longrightarrow \infty} \inf F(\Omega_{nk}).$$

This completes the proof. □

3.2. **Existence under the $\epsilon-$ cône property.**

*Proof.* **of Theorem 1.2** . Because $u_\Omega$ is a solution of (1.5), then $u_\omega$ is bounded, therefore considering our function, we can use Holder's inequality. $u_\Omega$ is a solution of (1.5), then it is in $D^{s,2}(\Omega)$, for $0 < s < 1$.
Let us show that $J$ is bounded as:

$$J(\Omega) = C(N,s)\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u(x)-u(y))}{\mid x-y\mid^{N+2s}}dxdy$$

$$\mid J(\Omega)\mid = C(N,s)\left|\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(u(x)-u(y))}{\mid x-y\mid^{N+2s}}dxdy\right| > -\infty.$$

And on the other hand, using the Holder inequality we have

$$\mid J(\Omega)\mid \leq C(N,s)\left(\int_{\mathbb{R}^{2N}} \frac{(\mid u(x)-u(y)\mid)^2}{\mid x-y\mid^{N+2s}}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^{2N}} \frac{\mid (u(x)-u(y)\mid)^2}{\mid x-y\mid^{N+2s}}dxdy\right)^{\frac{1}{2}}$$

$$\mid J(\Omega)\mid \leq C(N,s)\parallel u\parallel^2_{D^{s,2}(\Omega)}.$$

Hence $J$ is bounded.
Let $m = \inf_{\Omega\in\mathcal{O}} J(\Omega)$, we have $\inf_{\Omega\in\mathcal{O}} J(\Omega) > -\infty$, so there exists a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}$ such that

$$J(\Omega_n) \longrightarrow m = \inf J(\Omega).$$

Since $\Omega_n \subset O$, there exists a compact set $K$ such that $\bar{\Omega}_n \subset K$. Then according to the compactness Theorem 2.12, there is an open set $\Omega$, with $|\Omega_n| = c$ and an extracted sequence $\Omega_{n_k}$ such that $\Omega_{n_k} \xrightarrow{H} \Omega$ and $\chi_{\Omega_{n_k}} \xrightarrow{a.e.} \chi_\Omega$
It remains to show that:

$$\lim J(\Omega_{n_k}) = J(\Omega) = \inf_{\Omega\in\mathcal{O}_\epsilon\ or\ \mathcal{O}_{ad}} J(\Omega).$$

Let us show that the sequence $u_{\Omega_{n_k}}$ is bounded in $D^{s,2}(\Omega_{n_k})$. Replacing $\Omega$ with $\Omega_{n_k}$ in the weak formulation, we obtain

$$\int_{\mathbb{R}^{2N}} \frac{(u_{\Omega_{n_k}}(x)-u_{\Omega_{n_k}}(y))(v_\Omega(x)-v_\Omega(y))}{\mid x-y\mid^{N+2s}}dxdy = \int_{\Omega_{n_k}} f(x)v_{\Omega_{n_k}}(x)dx \quad \forall\ v\in D^{s,2}(\Omega). \tag{3.7}$$

And from Proposition 2.6 we have

$$\left(\frac{2-C^2}{2}\right)\parallel u_{\Omega_{n_k}}\parallel^2_{D^{s,2}(\Omega_{n_k})} \leq m + \frac{1}{2}\parallel f\parallel^2_{L^2(\Omega_{n_k})}.$$

Therefore the sequence $u_{\Omega_{n_k}}$ is bounded in $D^{s,2}(\Omega_{n_k})$.
As $(u_{\Omega_{n_k}})$ is bounded in $D^{s,2}(\Omega_{n_k})$, there exists $u^*_\Omega \in D^{s,2}(\Omega)$ and an extracted subsequence $(u_{\Omega_{n_k}})_{k\geq 1}$ of $(u_{\Omega_{n_k}})$ still denoted by$(u_{\Omega_{n_k}})_{k\geq 1}$ such that:

$$(u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup u^*_\Omega \in D^{s,2}(\Omega),$$

$$(u_{\Omega_{n_k}})_{k\geq 1} \longrightarrow u^*_\Omega \in L^2(\Omega),$$

and

$$(u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup u^*_\Omega \in L^2(\Omega), \ \ if \ \ k \longrightarrow \infty.$$

By passing to the limit when $k \longrightarrow \infty$ and weak convergence, we obtain the following formulation

$$\int_{\mathbb{R}^{2N}} \frac{(u^*_\Omega(x) - u^*_\Omega(y))(\varphi(x) - \varphi(y))}{\mid x-y \mid^{N+2s}} dxdy = \int_\Omega f(x)\varphi(x)dx, \ \ \forall \varphi \in D^{s,2}(\Omega), \tag{3.8}$$

which is the weak formulation of the following problem $\begin{cases} (-\Delta)^s u^*_\Omega = f \text{ in } \Omega \\ u^*_\Omega = 0 \text{ on } \mathbb{R}^N \setminus \Omega. \end{cases}$

Finally by taking $\varphi = u_{\Omega_{n_k}}$ in (3.7), we obtain

$$\lim \left( \int_{\mathbb{R}^{2N}} \frac{\mid (u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) \mid^2}{\mid x-y \mid^{N+2s}} dxdy \right) = \lim \int_{\Omega_{n_k}} u_{\Omega_{n_k}} f(x)$$

$$= \int_\Omega f(x) u^*_\Omega = \int_{\mathbb{R}^{2N}} \frac{[u^*_\Omega(x) - u^*_\Omega(y)]^2}{\mid x-y \mid^{N+2s}} dxdy.$$

In the other hand, we have

$$\int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))]^2}{\mid x-y \mid^{N+2s}} = \int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y))]^2}{\mid x-y \mid^{N+2s}}$$

$$-2 \int_{\mathbb{R}^{2N}} \frac{(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y))(u_\Omega(x) - u_\Omega(y))}{\mid x-y \mid^{N+2s}} + \int_{\mathbb{R}^{2N}} \frac{[(u_\Omega(x) - u_\Omega(y))]^2}{\mid x-y \mid^{N+2s}}.$$

Then taking the limits in the right hand side after equality, as $k \longrightarrow \infty$

$$\lim_{k\longrightarrow\infty} \Big( \int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y))]^2}{\mid x-y \mid^{N+2s}} - 2 \int_{\mathbb{R}^{2N}} \frac{(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y))(u_\Omega(x) - u_\Omega(y))}{\mid x-y \mid^{N+2s}}$$

$$+ \int_{\mathbb{R}^{2N}} \frac{[(u_\Omega(x) - u_\Omega(y))]^2}{\mid x-y \mid^{N+2s}} \Big) = 0.$$

From which have

$$\int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))]^2}{\mid x-y \mid^{N+2s}} = 0.$$

$$\int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))]^2}{\mid x-y \mid^{N+2s}}$$
$$= \int_{\mathbb{R}^{2N}} \frac{(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))}{\mid x-y \mid^{N+2s}} = 0$$
$$\int_{\Omega_{n_k}} f\left(u_{\Omega_{n_k}} - u_\Omega\right) = 0.$$

Then

$$u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \xrightarrow{L^2} u_\Omega(x) - u_\Omega(y)$$
$$u_{\Omega_{n_k}} \xrightarrow{L^2} u_\Omega.$$

So,

$$u_{\Omega_{n_k}} \xrightarrow{D^{s,2}} u_\Omega$$

Finally

$$\lim_{k\longrightarrow\infty} J(\Omega_{n_k}) = C(N,s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \mid^2}{\mid x-y \mid^{N+2s}} dxdy$$
$$= J(\Omega) = C(N,s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^2}{\mid x-y \mid^{N+2s}} dxdy = m.$$

We can conclude that there is an open $\Omega^*$ which minimizes $J$ and $\Omega^* \in \mathcal{O}_\epsilon$. □

## 4. **Shape and Topological Derivative via Minmax Method**

4.1. **Shape derivative.** Shape optimization involves deforming an object in an ideal manner to minimize or maximize a cost function. Therefore, to determine a suitable deformation method, we focus on a shape analogy with classical derivatives. Here we use a method widely used in the litterature, see for instance [16, 25, 35]. The use of derivatives is essential to minimize the function $J(\Omega)$ depending on the domain $\Omega$. Then for $\epsilon > 0$ and a vector field $V$, we consider the following transformation $\Phi_\epsilon$, known as perturbation of identity

$$\Phi_\epsilon(x) := x + \epsilon V(x).$$

We define the perturbed domain as

$$\Omega_\epsilon = \Phi_\epsilon(\Omega) = \{\Phi_\epsilon(x) := x + \epsilon V(x),\ x \in \Omega\}.$$

We therefore define the shape derivative as follows:

$$DJ(\Omega, V) := \lim_{\epsilon\to 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon}.$$

We now give a provide the proof of Theorem 1.3 by applying Theorem 2.19.

**Proof of Theorem 1.3:** First, let us consider the following variational formulation:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy = \int_\Omega f(x)v(x)dx. \tag{4.1}$$

We define the transformation $\Phi_\epsilon$ such that $\frac{d}{d\epsilon}\Phi_\epsilon(x)_{|\epsilon=0} = V(x)$. The perturbed domain is defined by $\Omega_\epsilon = \Phi_\epsilon(\Omega)$. Then, in the perturbed domain we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\epsilon(x)-u_\epsilon(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \int_{\Omega_\epsilon} f(x)v(x)\,dx. \tag{4.2}$$

We return to the initial domain using the transfomation due to the fact that $\Omega_\epsilon = \Phi_\epsilon(\Omega)$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_\epsilon(x)-u_\epsilon(y))(v(x)-v(y))}{|x-y|^{N+2s}}\circ\Phi_\epsilon Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)\,dxdy = \int_\Omega (fv)\circ\Phi_\epsilon Jac_{\Phi_t}(x)\,dx$$

or even

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{(u^\epsilon(x)-u^\epsilon(y))(v\circ\Phi_\epsilon(x)-v\circ\Phi_\epsilon(y))}{|\Phi_\epsilon(x)-\Phi_\epsilon(y)|^{N+2s}} Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)\,dxdy\\ &= \int_\Omega f\circ\Phi_\epsilon v\circ\Phi_\epsilon Jac_{\Phi_\epsilon}(x)\,dx.\end{aligned}$$

With the change of variables $\phi = v\circ\Phi_\epsilon$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^\epsilon(x)-u^\epsilon(y))(\phi(x)-\phi(y))}{|\Phi_\epsilon v(x)-\Phi_\epsilon(y)|^{N+2s}} Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)\,dxdy = \int_\Omega f\circ\Phi_\epsilon\phi Jac_{\Phi_\epsilon}(x)\,dx.$$

Then, by setting $K_\epsilon(x,y) = C(N,s)\frac{Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)}{|\Phi_\epsilon(x)-\Phi_\epsilon(y)|^{N+2s}}$, the previous expresion becomes

$$\frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u^\epsilon(x)-u^\epsilon(y))(\phi(x)-\phi(y))K_\epsilon(x,y)\,dxdy = \int_\Omega f\circ\Phi_\epsilon\phi Jac_{\Phi_\epsilon}(x)\,dx.$$

The objective function in the perturbed domain is defined by

$$J(\Omega_\epsilon) = C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dxdy. \tag{4.3}$$

In the same way, we come back to the unperturbed domain via the transformation $\Phi_\epsilon$. Doing this, we obtain

$$\begin{aligned}J(\Omega_\epsilon) &= C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{N+2s}}\circ\Phi_\epsilon Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)\,dxdy\\ &= C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^\epsilon(x)-u^\epsilon(y)|^2}{|\Phi(x)-\Phi(y)|^{N+2s}}Jac_{\Phi_\epsilon}(x)Jac_{\Phi_\epsilon}(y)\,dxdy\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u^\epsilon(x)-u^\epsilon(y)|^2K_\epsilon(x,y)\,dxdy.\end{aligned}$$

Using the variational formulation and the objective function in the perturbed domain, we construct the perturbed Lagrangian as follows:

$$\begin{aligned}L(\epsilon,\varphi,\phi) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x)-\varphi(y)|^2K_\epsilon(x,y)\,dxdy\\ &\quad+\frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\varphi(x)-\varphi(y))(\phi(x)-\phi(y))K_\epsilon(x,y)\,dxdy - \int_\Omega(f\circ\Phi_\epsilon)\phi Jac_{\Phi_\epsilon}(x)\,dx.\end{aligned}$$

The derivative of the Lagrangian with respect to $\epsilon$ is:

$$
\begin{aligned}
d_\epsilon L(\epsilon,\varphi,\phi) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(y)|^2 K'_\epsilon(x,y)\,dxdy \\
&+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\varphi(x)-\varphi(y))(\phi(x)-\phi(y))K'_\epsilon(x,y)\,dxdy \\
&- \int_\Omega (\nabla f\cdot V(\epsilon))\phi Jac_{\Phi_\epsilon}(x)\,dx - \int_\Omega (f\circ\Phi_\epsilon)\phi Jac_{\Phi_\epsilon}(x) divV(\epsilon)\circ\Phi_\epsilon\,dx
\end{aligned}
$$

where

$$
K'_\epsilon(x,y)\big|_{\epsilon=0} = -\left[(N+2s)\frac{x-y}{|x-y|}\cdot P_V(x,y) - (divV(x)+divV(y))\right]K_0(x,y)
$$

and $P_V \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ is given by

$$
P_V(x,y) = \frac{V(x)-V(y)}{|x-y|}.
$$

To define the function $\mathcal{R}(\epsilon)$, we must calculate the derivative of the Lagrangian with respect to $\varphi$ in one direction $\varphi'$. In doing so, we

$$
\begin{aligned}
d_\varphi L(\epsilon,\varphi,\phi;\varphi') &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\varphi(x)-\varphi(y))(\varphi'(x)-\varphi'(y))K_\epsilon(x,y)\,dxdy \\
&+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\varphi'(x)-\varphi'(y))(\phi(x)-\phi(y))K_\epsilon(x,y)\,dxdy.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}(\epsilon) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)+u(x)}{2}\right)-\left(\frac{u_\epsilon(y)+u(y)}{2}\right)\right]\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)+u(x)}{2}\right)-\left(\frac{u_\epsilon(y)+u(y)}{2}\right)\right]\left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy.
\end{aligned}
$$

Substituting $\varphi' = \frac{u_\epsilon - u}{\epsilon}$ into the adjoint equation for $p$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy = 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}(\epsilon) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2}\right)\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy.
\end{aligned}
$$

Or

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy$$

$$=-\frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy$$

In this case $\mathcal{R}(\epsilon)$ becomes

$$\mathcal{R}(\epsilon)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2}\right)\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy.$$

This completes the proof. □

4.2. **Topological Derivative.** Let $\Omega$ be a regular domain of $\mathbb{R}^N$ and $\omega_t\subset\Omega$. Given $\Omega_t$ the perturbed domain obtained by removing $\omega_t$, the topological derivative of objective function $J(\Omega)$ at point $x_0$ is defined by

$$D_T J(\Omega)(x_0)=\lim_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{|\omega_t|}, \tag{4.4}$$

verifying the following asymptotic topological expansion

$$J(\Omega_t)=J(\Omega)+|\omega_t|D_T J(\Omega)(x_0)+o(t), \quad \text{with } t\to 0. \tag{4.5}$$

The aim of this section is to provide a proof of Theorem 1.4, based on the result for the derivation of the topological derivative or the shape derivative given by Theorem 2.19.

**Proof of Theorem 1.4:** Let us consider the function defined in $\Omega_t$ as

$$J(\Omega_t)=C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\, u_{\Omega_t}(x)-u_{\Omega_t}(y)\,|^2}{|\, x-y\,|^{N+2s}}dxdy. \tag{4.6}$$

where $u_{\Omega_t}$ be the solution to the following problem

$$\begin{cases}(-\Delta)^s u_{\Omega_t}=f \text{ in } \ \Omega_t\\ u_{\Omega_t}=0 \text{ on } \mathbb{R}^N\backslash\Omega_t.\end{cases} \tag{4.7}$$

We recall that $t=\alpha_{N-d}r^{N-d}$. Let us consider as shape functional $J$ define by

$$J(\Omega)=C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\Omega(x)-u_\Omega(y)|^2}{|x-y|^{N+2s}}dxdy \tag{4.8}$$

and $u_\Omega\in D^{s,2}(\Omega)$ is solution to the variational problem

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_\Omega(x)-u_\Omega(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy=\int_\Omega f(x)v(x)dx \ \ \forall v\in D^{s,2}(\Omega). \tag{4.9}$$

We aim to compute the topological derivative of the functional $J(\Omega_t)$

$$dJ=\lim_{t\longrightarrow 0}\frac{J(\Omega_t)-J(\Omega)}{\alpha_{N-d}r^{N-d}}.$$

Thus, the Lagrangian dependent on $t$ will be written in the form :

$$L(t,\phi,\Phi) = C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid \phi(x)-\phi(y)\mid^2}{\mid x-y\mid^{N+2s}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+2s}}dxdy - \int_{\Omega_t} f(x)\Phi(x)dx.$$

$$J(\Omega_t) = \inf_{\phi\in D^{s,2}(\Omega)}\sup_{\Phi\in D^{s,2}(\Omega)} L(t,\phi,\Phi).$$

From this, we can evaluate the derivative of the Lagrangian, dependent on $t$, with respect to $\phi$.

$$d_\phi L(t,\phi,\Phi,\phi') = C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} 2\frac{(\phi'(x)-\phi'(y))\,(\phi(x)-\phi(y))}{\mid x-y\mid^{N+2s}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi'(x)-\phi'(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+2s}}dxdy.$$

The initial adjoint state $p_{\Omega_0}$ is a solution of $d_\phi L(0,u_{\Omega_0},p_{\Omega_0},\phi') = 0$ for all $\phi'$ for $t=0$. Thus the variational formulation of the adjoint equation of state is given by

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi'(x)-\phi'(y))\,[2\,C(N,s)\,(u_{\Omega_0}(x)-u_{\Omega_0}(y))+(p_{\Omega_0}(x)-p_{\Omega_0}(y))]}{\mid x-y\mid^{N+2s}}dxdy = 0.$$

Next, we derive the Lagrangian with respect to $\Phi$.

$$d_\Phi L(t,\phi,\Phi,\Phi') = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\phi(x)-\phi(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+2s}}dxdy - \int_{\Omega_t} f(x)\Phi'(x)dx.$$

The initial state $u_{\Omega_0}$ is a solution of $d_\Phi L(0,u_{\Omega_0},0,\Phi'_{\Omega_0}) = 0\ \forall\ \Phi'_{\Omega_0}\in D^{s,2}(\Omega)$ and in this case, we have:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_{\Omega_0}(x)-u_{\Omega_0}(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+2s}}dxdy - \int_\Omega f(x)\Phi'(x)dx = 0.$$

And we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_{\Omega_0}(x)-u_{\Omega_0}(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+2s}}dxdy = \int_\Omega f(x)\Phi'(x)dx.$$

$$L(t,\phi,\Phi) - L(0,\phi,\Phi) = \int_{\Omega_t} f(x)\Phi(x)dx - \int_\Omega f(x)\Phi(x)dx$$

$$L(t,\phi,\Phi) - L(0,\phi,\Phi) = \int_{\Omega_t} f(x)\Phi(x)dx - \int_{\omega_t} f(x)\Phi(x)dx - \int_{\Omega_t} f(x)\Phi(x)dx$$

$$L(t,\phi,\Phi) - L(0,\phi,\Phi) = -\int_{\omega_t} f(x)\Phi(x)dx$$

$$d_t L(0,\phi,\Phi) = -\lim_{r\longrightarrow 0}\frac{1}{|B(x_0,r)|}\left[\int_{B(x_0,r)} f(x)\Phi(x)\right]dx$$

$$d_t L(0,\phi,\Phi) = -f(x_0)\Phi(x_0).$$

We will now define $R(t)$ by

$$\mathcal{R}(t) = \int_0^1 d_\phi L\left(t, u_{\Omega_0} + \Psi\left(u_{\Omega_t} - u_{\Omega_0}\right), p_{\Omega_0}, \left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)\right) d\Psi.$$

By substituting $\phi' = \frac{u_{\Omega_t} - u_{\Omega_0}}{t}$ and $\Psi = \frac{u_{\Omega_t} - u_{\Omega_0}}{2}$ into the adjoint equation for $p_{\Omega_0}$, we obtain:

$$\begin{aligned}
&\mathcal{R}(t) = \\
&\quad C(N,s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 2 \frac{\left(\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)(x) - \left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)(y)\right)\left(\left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right)(x) - \left(\frac{u_{\Omega_t} + u_{\Omega_0}}{2}\right)(y)\right)}{\mid x-y \mid^{N+2s}} \\
&+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)(x) - \left(\frac{u_{\Omega_t} - u_{\Omega_0}}{t}\right)(y)\left(p_{\Omega_0}(x) - p_{\Omega_0}(y)\right)}{|x-y|^{N+2s}} dxdy. \\
&= \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right) - \left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)\left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right) + p_{\Omega_0}(x)\right]}{t \mid x-y \mid^{N+2s}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right) - \left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)\left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right) - p_{\Omega_0}(y)\right]}{t \mid x-y \mid^{N+2s}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right) - \left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)(\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right)\right]}{t \mid x-y \mid^{N+2s}} dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_0}(x)\right) - \left(u_{\Omega_t}(y) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)\left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right) + p_{\Omega_0}(x)\right]}{t \mid x-y \mid^{N+2s}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_0}(x)\right) - \left(u_{\Omega_t}(y) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)\left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right) - p_{\Omega_0}(y)\right]}{t \mid x-y \mid^{N+2s}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right) - \left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)(\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right)\right]}{t \mid x-y \mid^{N+2s}} dxdy
\end{aligned}$$

$$\mathcal{R}(t) = \int_{\mathbb{R}^{2N}} \frac{\left[\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right) - \left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)\right]\left[C(N,s)(\left(u_{\Omega_t}(x) - u_{\Omega_t}(y)\right)\right]}{t \mid x-y \mid^{N+2s}} dxdy$$

$$\mathcal{R}(t) = \frac{C(N,s)}{\sqrt{t}} \int_{\mathbb{R}^{2N}} \frac{|u_{\Omega_t}(x) - u_{\Omega_t}(y)|^2}{\sqrt{t} \mid x-y \mid^{N+2s}} dxdy - \frac{C(N,s)}{\sqrt{t}} \int_{\mathbb{R}^{2N}} \frac{\left(u_{\Omega_0}(x) - u_{\Omega_0}(y)\right)}{\sqrt{t} \mid x-y \mid^{N+2s}} dxdy.$$

Then, we get the proof. □

**Conclusion.** In this study, we investigate the problems of the existence of an optimal form for a given cost function using $(s-\gamma)-$ convergence. Thus we have shown the existence of an optimal form in the s-quasi open class with compactness of the admissible set. Then this existence was also shown using the $\epsilon$-cône property. Next, we computed the shape derivative of the cost function under consideration, but use the minmax method. Finally, the topological derivative of the same functional is established. In future work, we intend to generalize this problem and propose a numerical method for calculating the cost function.

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