Journal of Finsler Geometry and its Applications
Vol. 7, No. 1 (2026), pp 31-44
https://doi.org/10.22098/jfga.2025.17333.1158

Study of W_7- curvature tensor on $(LPK)_n$ manifolds

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Abstract. In this paper, we are going to study the characteristics of n-dimensional Lorentzian para-Kenmotsu manifolds (briefly, $(LPK)_n$) endowed with the W_7-curvature tensor. First, we analyzed $(LPK)_n$ manifolds under the condition $W_7(X, Y, Z, \xi) = 0$. Next, we explore $(LPK)_n$ manifolds satisfying the W_7-semisymmetric condition, ϕ-W_7-symmetric condition, and ϕ-W_7-flat condition. Moreover, we discuss Lorentzian para-Kenmotsu manifolds under the condition $W_7(U, V) \cdot R = 0$, and prove that such manifolds reduce to Einstein manifolds. Finally, all the relevant results have been verified through an example.

Keywords: Lorentzian para-Kenmotsu manifold, W_7-curvature tensor, Ricci flat, Einstein manifold.

1. Introduction

The study of curvature tensors plays a fundamental role in differential geometry, particularly in the context of various specialized manifolds. The concept of curvature tensors is central to understanding the geometric and physical

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AMS 2020 Mathematics Subject Classification: 53C25, 53C50

properties of differentiable manifolds. In this regard, the W_7-curvature tensor, introduced by G.P. Pokhariyal [1] in 1982, has been extensively explored in the literature. This tensor, defined with the support of the Weyl curvature tensor, has found significant applications in the study of Lorentzian para-Kenmotsu manifolds.

Para-Kenmotsu manifolds were first introduced by B.B. Sinha and K.L. Sai Prasad [8] in 1989, and since then, these manifolds have been a subject of continued research due to their intriguing geometric properties. In recent years, Lorentzian para-Kenmotsu manifolds have garnered attention, particularly in the study of invariant submanifolds and Ricci solitons. The seminal work by Haseeb and Prasad [9] initiated the study of Lorentzian para-Kenmotsu manifolds, and subsequent contributions by Atceken [2](2022) provided conditions for invariant submanifolds to be totally geodesic. Ricci solitons, which represent self-similar solutions to the Ricci flow, have also been examined in the context of these manifolds by Bagewadi [14, 15], Bejan and Crasmareanu [3], Blaga [4] and many others (see also [16, 17, 18, 19]).

This paper is organized as follows: Section 1 provides the necessary background and historical developments related to the para-Kenmotsu manifolds, Lorentzian para-Kenmotsu manifolds, and curvature tensors. Section 2 outlines the fundamental preliminaries and essential results required for subsequent discussions. Section 3 delves into the condition $W_7(X, Y, Z, \xi) = 0$ of Lorentzian para-Kenmotsu manifolds. Section 4 examines the W_7-semisymmetric condition of Lorentzian para-Kenmotsu manifolds. In section 5, we analyze the ϕ-W_7-symmetry condition in $(LPK)_n$ manifolds. Section 6 is devoted to the study of ϕ-W_7-flatness in Lorentzian para-Kenmotsu manifolds. Section 7, considers Lorentzian para-Kenmotsu manifolds satisfying the condition, $W_7(U, V) \cdot R = 0$ and shows that such manifolds reduce to an Einstein manifolds. Finally, in section 8 we construct an example to verify the results.

Through this work, we aim to establish a foundational framework for the W_7-curvature tensor, offering new directions for research in differential geometry and its applications in the study of special manifolds.

2. **Preliminaries**

2.1. **Lorentzian almost paracontact metric manifold.**

Definition 2.1. *An n-dimensional differentiable manifold M equipped with a structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact metric manifold if it satisfies the following properties* [21] *:*

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$\Phi(X, Y) = \Phi(Y, X) = g(X, \phi Y). \tag{2.6}$$

for any vector field X, Y on M, where ϕ is a (1,1) tensor field, ξ is a contravariant vector field also known as Reeb vector field, η is a 1-form, and g is a Riemannian metric.

A Lorentzian para-Sasakian manifold is a Lorentzian almost paracontact manifold if

$$(\nabla_X \phi)Y = g(X, Y) + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.7}$$

2.2. **Lorentzian Para-Kenmotsu Manifolds.**

Definition 2.2. *A Lorentzian almost paracontact metric manifold M is called a Lorentzian para-Kenmotsu manifold if* [9]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.8}$$

for any vector fields $X, Y \in M$.

In a Lorentzian para-Kenmotsu manifold, denoted by $(LPK)_n$, we have the following fundamental relations:

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.9}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.10}$$

where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g.

Curvature Properties :
On a Lorentzian para-Kenmotsu manifold M, the Riemannian Curvature tensor R satisfies the following fundamental relations [9]:

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

Ricci Tensor Properties :
The Ricci tensor S and Ricci operator Q on an $(LPK)_n$ manifold satisfy:

$$S(X, \xi) = (n-1)\eta(X), \tag{2.15}$$

$$Q\xi = (n-1)\xi, \tag{2.16}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.17}$$

Furthermore, by the second Bianchi identity, we obtain:

$$(divR)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z), \tag{2.18}$$

$$(\nabla_U S)(Z,\xi) = S(U,Z) - (n-1)g(U,Z), \tag{2.19}$$

where R, S, and Q denote the Riemannian curvature tensor, Ricci tensor, and Ricci operator on $(LPK)_n$ manifold M, respectively.

2.3. **The W_7-Curvature Tensor.**

Definition 2.3. *The W_7-curvature tensor on a Lorentzian para-Kenmotsu manifold is defined by:*[20]

$$W_7(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\Big\{g(Y,Z)QX - S(Y,Z)X\Big\}. \tag{2.20}$$

By substituting specific vector fields $X = \xi$ into equation (2.20), we obtain the following important relations:

$$W_7(\xi,Y)Z = 2g(Y,Z)\xi - \eta(Z)Y - \frac{1}{(n-1)}S(Y,Z)\xi. \tag{2.21}$$

Setting $Y = \xi$:

$$W_7(X,\xi)Z = -g(X,Z)\xi + \frac{1}{(n-1)}\eta(Z)QX, \tag{2.22}$$

and setting $Z = \xi$:

$$W_7(X,Y)\xi = -\eta(X)Y + \frac{1}{(n-1)}\eta(Y)QX. \tag{2.23}$$

2.4. **η-Einstein Manifold.**

Definition 2.4. *A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor satisfies the following condition:*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.24}$$

where a and b are scalar functions on M. When $b = 0$, the manifold reduces to an Einstein manifold.

3. **$(LPK)_n$ with condition $W_7(X,Y,Z,\xi) = 0$**

Theorem 3.1. *A ξ-W_7-Flat $(LPK)_n$ is an η-Einstein manifold.*

Proof. From (2.20), we have

$$W_7(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\Big\{g(Y,Z)S(X,U) - S(Y,Z)g(X,U)\Big\}. \tag{3.1}$$

Putting $U = \xi$ in (3.1) we have

$$W_7(X,Y,Z,\xi) = R(X,Y,Z,\xi) + \frac{1}{n-1}\Big\{g(Y,Z)S(X,\xi) - S(Y,Z)g(X,\xi)\Big\}. \tag{3.2}$$

By considering $W_7(X, Y, Z, \xi) = 0$ in the above relation, we have

$$R(X, Y, Z, \xi) = \frac{1}{n-1}\Big\{S(Y, Z)g(X, \xi) - g(Y, Z)S(X, \xi)\Big\}. \tag{3.3}$$

Using the equations (2.13) and (2.15) in the equation (3.3), we have

$$\frac{1}{n-1}\eta(X)S(Y, Z) = 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z). \tag{3.4}$$

Taking $X = \xi$ in the (3.4), we get

$$S(Y, Z) = 2(n-1)g(Y, Z) + (n-1)\eta(Y)\eta(Z). \tag{3.5}$$

Thus, manifold $\mathcal{M}$ is an η-Einstein manifold. □

4. W_7-semisymmetric $(LPK)_n$

Definition 4.1. *A Lorentzian para-Kenmotsu manifold $(LPK)_n$ M is said to be W_7-semisymmetric if it satisfies the following condition [21]:*

$$(R(X, Y) \cdot W_7)(Z, V)W = 0, \tag{4.1}$$

for every vector field $X, Y, Z, V, W \in \chi(M)$.

Theorem 4.2. *If M is an W_7-semisymmetric $(LPK)_n$ manifold, then it is Ricci-flat.*

Proof. The above relation can be written as

$$\begin{aligned} R(X, Y)W_7(Z, V)W - W_7(R(X, Y)Z, V)W \\ -W_7(Z, R(X, Y)V)W - W_7(Z, V)R(X, Y)W = 0. \end{aligned} \tag{4.2}$$

Now, putting $X = \xi$ in (4.2), we have

$$\begin{aligned} R(\xi, Y)W_7(Z, V)W - W_7(R(\xi, Y)Z, V)W \\ -W_7(Z, R(\xi, Y)V)W - W_7(Z, V)R(\xi, Y)W = 0. \end{aligned} \tag{4.3}$$

Using the equations (2.12), (2.21), (2.22), and (2.23) in (4.3), we have

$$\begin{aligned} g(Y, W_7(Z, V)W)\xi - \eta(W_7(Z, V)W)Y - 2g(Y, Z)g(V, W)\xi \\ +\eta(W)g(Y, Z)V + \frac{1}{(n-1)}g(Y, Z)S(V, W)\xi + \eta(Z)W_7(Y, V)W \\ +g(Z, W)g(Y, V)\xi - \frac{1}{(n-1)}\eta(W)g(Y, V)QZ + \eta(V)g(Z, Y)W \\ +\eta(Z)Vg(Y, W) - \frac{1}{(n-1)}\eta(V)g(Y, W)QZ + \eta(W)W_7(Z, V)Y = 0. \end{aligned} \tag{4.4}$$

Setting $V = \xi$ and using the equation (2.22) in (4.4), we get

$$\begin{aligned} -g(Z, W)Y - g(Z, Y)W - \eta(W)g(Y, Z)\xi - \eta(Z)\eta(W)Y \\ +\frac{1}{(n-1)}\{\eta(W)S(Y, Z)\xi + \eta(Z)\eta(W)QY + g(Y, W)QZ\} = 0. \end{aligned} \tag{4.5}$$

Setting $Z = \xi$ into the above equation, we get

$$\eta(W)QY = (n-1)\{g(Y,W)\xi - \eta(Y)W\}. \tag{4.6}$$

Taking the inner product of (4.6) with T, we have

$$\eta(W)S(Y,T) = (n-1)\{\eta(T)g(Y,W) - \eta(Y)g(W,T)\}. \tag{4.7}$$

Replacing W by ξ into the above equation, we get

$$S(X,Y) = 0, \tag{4.8}$$

for every $X, Y \in \chi(M)$. Hence, we have the result. □

5. ϕ-W_7-symmetric $(LPK)_n$ Manifolds

Definition 5.1. *A Lorentzian para-Kenmotsu manifold is said to be ϕ-W_7-symmetric if it satisfies the following condition[21]:*

$$\phi^2\Big((\nabla_U W_7)(X,Y)Z\Big) = 0. \tag{5.1}$$

for every $X, Y, Z, U \in \chi(M)$.

Theorem 5.2. *If a $(LPK)_n$ is ϕ-W_7-symmetric then the manifold reduces to an Einstein manifold.*

Proof. Differentiating (2.20) covariantly along U, we get

$$(\nabla_U W_7)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{(n-1)}\Big\{g(Y,Z)(\nabla_U Q)X - (\nabla_U S)(Y,Z)X\Big\}. \tag{5.2}$$

Applying ϕ^2 on both sides and using (2.2), we have

$$\begin{aligned}&(\nabla_U R)(X,Y)Z + \eta((\nabla_U R)(X,Y)Z)\xi + \frac{1}{(n-1)}\Big\{g(Y,Z)(\nabla_U Q)X \\ &+\eta(g(Y,Z)(\nabla_U Q)X)\xi - (\nabla_U S)(Y,Z)X - \eta((\nabla_U S)(Y,Z)X)\xi\Big\} = 0,\end{aligned} \tag{5.3}$$

Using the equation (2.19) into the above equation, we have

$$\begin{aligned}&(\nabla_U R)(X,Y)Z + \eta((\nabla_U R)(X,Y)Z)\xi + \frac{1}{(n-1)}\Big\{g(Y,Z)(\nabla_U Q)X \\ &+ g(Y,Z)S(X,U)\xi - (n-1)g(Y,Z)g(X,U)\xi - (\nabla_U S)(Y,Z)X \\ &- (\nabla_U S)(Y,Z)\eta(X)\xi\Big\} = 0.\end{aligned} \tag{5.4}$$

Differentiating equation (2.11) covariantly along U, we get

$$\eta((\nabla_U R)(X,Y)Z) = g(R(X,Y)Z,U) + g(X,Z)g(Y,U) - g(Y,Z)g(X,U). \tag{5.5}$$

So, from (5.4), we have

$$\begin{aligned}
&(\nabla_U R)(X,Y)Z + g(R(X,Y)Z,U)\xi + g(X,Z)g(Y,U)\xi \\
&-g(Y,Z)g(X,U)\xi + \frac{1}{(n-1)}\Big\{g(Y,Z)(\nabla_U Q)X + g(Y,Z)S(X,U)\xi \\
&-(n-1)g(Y,Z)g(X,U)\xi - (\nabla_U S)(Y,Z)X - (\nabla_U S)(Y,Z)\eta(X)\xi\Big\} = 0.
\end{aligned} \tag{5.6}$$

Contracting the above equation along U, we get

$$\begin{aligned}
&\sum_{i=1}^{n}\epsilon_i g((\nabla_{e_i}R)(X,Y)Z,e_i) + \sum_{i=1}^{n}\epsilon_i R(X,Y,Z,e_i)g(e_i,\xi) \\
&+\sum_{i=1}^{n}\epsilon_i g(X,Z)g(Y,e_i)g(\xi,e_i) - \sum_{i=1}^{n}\epsilon_i g(Y,Z)g(X,e_i)g(e_i,\xi) \\
&+\frac{1}{(n-1)}\sum_{i=1}^{n}\epsilon_i g(Y,Z)g((\nabla_{e_i}Q)X,e_i) + \frac{1}{(n-1)}\sum_{i=1}^{n}\epsilon_i g(Y,Z)S(X,e_i)g(e_i,\xi) \\
&-g(Y,Z)\sum_{i=1}^{n}\epsilon_i g(X,e_i)g(e_i,\xi) - \frac{1}{(n-1)}\sum_{i=1}^{n}\epsilon_i g((\nabla_{e_i}S)(Y,Z)X,e_i) \\
&-\frac{1}{(n-1)}\sum_{i=1}^{n}\epsilon_i(\nabla_{e_i}S)(Y,Z)\eta(X)g(e_i,\xi) = 0.
\end{aligned}$$

So, from the above equation, we have

$$\begin{aligned}
&(divR)(X,Y)Z + R(X,Y,Z,\xi) + g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \\
&+\frac{1}{(n-1)}\Big\{\frac{X(r)}{2}g(Y,Z) - (\nabla_X S)(Y,Z) - \eta(X)(\nabla_\xi S)(Y,Z)\Big\} = 0.
\end{aligned} \tag{5.7}$$

Using the equation (2.18) into the equation (5.7), we have

$$\begin{aligned}
\frac{(n-2)}{(n-1)}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + &\frac{1}{2(n-1)}g(Y,Z)X(r) \\
&- \frac{1}{(n-1)}\eta(X)(\nabla_\xi S)(Y,Z) = 0.
\end{aligned} \tag{5.8}$$

Using the equation (2.19) into (5.8), we get

$$S(X,Y) = (n-1)g(X,Y) + \frac{1}{2}X(r)\eta(Y). \tag{5.9}$$

Putting $Y=\xi$ into (5.9), we get $X(r)=0$.
Hence, from (5.9), we get

$$S(X,Y) = (n-1)g(X,Y). \tag{5.10}$$

Hence, manifold $\mathcal{M}$ is an Einstein manifold. □

6. ϕ-W_7-flat $(LPK)_n$

Definition 6.1. *A Lorentzian para-Kenmotsu manifold is said to be ϕ-W_7-flat if,*

$$W_7(\phi X, \phi Y, \phi Z, \phi W) = 0. \tag{6.1}$$

for all $X, Y, Z \in \chi(M)$.

Theorem 6.2. *If an n-dimensional Lorentzian para-Kenmotsu manifold is ϕ-W_7-flat then the distribution defined by ϕ is null.*

Proof.

$$R(X, Y, \phi Z, \phi W) = g(\nabla_X \nabla_Y \phi Z, \phi W) - g(\nabla_Y \nabla_X \phi Z, \phi W) - g(\nabla_{[X,Y]} \phi Z, \phi W) = 0. \tag{6.2}$$

Now, using (2.8), we have

$$\nabla_X \nabla_Y \phi Z = \nabla_X \{-g(\phi Y, Z)\xi - \eta(Z)\phi Y + \phi(\nabla_Y Z)\}. \tag{6.3}$$

Using the equations (2.8), (2.9), and (2.10) into (6.3), we get

$$\begin{aligned} \nabla_X \nabla_Y \phi Z = & - g(\nabla_X(\phi Y), Z)\xi - g(\phi Y, \nabla_X Z)\xi + g(\phi Y, Z)X \\ & + g(\phi Y, Z)\eta(X)\xi + g(\phi X, Y)\eta(Z)\xi + \eta(Y)\eta(Z)\phi X \\ & - (\nabla_X \eta)(Z)\phi Y - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi(\nabla_X Y) \\ & - g(\phi X, \nabla_Y Z)\xi - \eta(\nabla_Y Z)\phi X + \phi(\nabla_X \nabla_Y Z). \end{aligned} \tag{6.4}$$

Taking inner product of (6.4) with ϕW, we have

$$\begin{aligned} g(\nabla_X \nabla_Y \phi Z, \phi W) = & \, g(\phi Y, Z)g(X, \phi W) - (\nabla_Y \eta)(Z)g(\phi Y, \phi W) \\ & - \eta(\nabla_X Z)g(\phi Y, \phi W) + \eta(Y)\eta(Z)g(\phi X, \phi W) \\ & - \eta(Z)g(\phi(\nabla_X Y), \phi W) - \eta(\nabla_Y Z)g(\phi X, \phi W) \\ & + (g(\phi(\nabla_X \nabla_Y Z), \phi W). \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} g(\nabla_{[X,Y]}(\phi Z), \phi W) = & -\eta(Z)g(\phi(\nabla_X Y), \phi W) + \eta(Z)g(\phi(\nabla_Y X), \phi W) \\ & + g(\phi(\nabla_{[X,Y]} Z), \phi W). \end{aligned} \tag{6.6}$$

So, from equation (6.2), we have

$$\begin{aligned} R(X, Y, \phi Z, \phi W) = & \, g(\phi Y, Z)g(X, \phi W) - g(\phi X, Z)g(Y, \phi W) \\ & + (\nabla_Y \eta)(Z)g(\phi X, \phi W - (\nabla_X \eta)(Z)g(\phi Y, \phi W) \\ & + \eta(Y)\eta(Z)g(\phi X, \phi W) - \eta(X)\eta(Z)g(\phi Y, \phi W) \\ & + g(\phi(R(X, Y)Z, \phi W). \end{aligned} \tag{6.7}$$

Using (2.2) and (2.6) in (6.7), we get

$$\begin{aligned} R(X, Y, \phi Z, \phi W) - R(X, Y, Z, W) = & \, g(\phi Y, Z)g(X, \phi W) - g(\phi X, Z)g(Y, \phi W) \\ & - g(Y, Z)g(X, W) + g(X, Z)g(Y, W). \end{aligned} \tag{6.8}$$

Interchanging X by Z and Y by W, in the equation (6.8), we have

$$\begin{aligned} R(Z,W,\phi X,\phi Y) - R(Z,W,X,Y) &= g(\phi W,X)g(Z,\phi Y) - g(\phi Z,X)g(W,\phi Y) \\ &\quad - g(W,X)g(Z,Y) + g(X,Z)g(Y,W). \end{aligned} \tag{6.9}$$

Subtracting (6.9) from (6.8), we get

$$R(X,Y,\phi Z,\phi W) = R(Z,W,\phi X,\phi Y). \tag{6.10}$$

Replacing X by ϕX and Y by ϕY in (6.10), we have

$$\begin{aligned} R(\phi X,\phi Y,\phi Z,\phi W) &= R(X,Y,Z,W) - \eta(X)\eta(Z)g(Y,W) \\ &\quad + \eta(X)\eta(W)g(Y,Z) + \eta(Y)\eta(Z)g(X,W) \\ &\quad - \eta(Y)\eta(W)g(X,Z). \end{aligned} \tag{6.11}$$

From (2.20), we have

$$\begin{aligned} W_7(\phi X,\phi Y,\phi Z,\phi W) &= R(\phi X,\phi Y,\phi Z,\phi W) \\ +\frac{1}{(n-1)}\Big\{g(\phi Y,\phi Z)S(\phi X,\phi W) &- S(\phi Y,\phi Z)g(\phi X,\phi W)\Big\} = 0. \end{aligned} \tag{6.12}$$

Using (2.4) and (2.17) in (6.12), we get

$$\begin{aligned} &R(X,Y,Z,W) - \eta(X)\eta(Z)g(Y,W) + \eta(X)\eta(W)g(Y,Z) \\ &- \eta(Y)\eta(W)g(X,Z) + \frac{1}{(n-1)}\{g(Y,Z)S(X,W) + (n-1)\eta(X)\eta(W)g(Y,Z) \\ &+ \eta(Y)\eta(Z)S(X,W) - g(X,W)S(Y,Z) - \eta(X)\eta(W)S(Y,Z)\} = 0, \end{aligned} \tag{6.13}$$

Contracting the above equation along X and W, we get

$$g(\phi Y,\phi Z) = 0. \tag{6.14}$$

for all vector fields Y and Z on M. Hence, we have the result. □

7. A $(LPK)_n$ admitting the condition $W_7(U,V)\cdot R = 0$

Theorem 7.1. *If a Lorentzian para-Kenmotsu manifold $(LPK)_n$ satisfies condition $W_7(U,V)\cdot R = 0$, then manifold reduces to an Einstein manifold.*

Proof. Let $(LPK)_n$ admits the condition

$$W_7(U,V)\cdot R = 0. \tag{7.1}$$

From the relation (7.1), we have

$$\begin{aligned} &W_7(U,V)(R(X,Y)Z) - R(W_7(U,V)X,Y)Z \\ &- R(X,W_7(U,V)Y)Z - R(X,Y)W_7(U,V)Z = 0. \end{aligned} \tag{7.2}$$

Putting $Y=\xi$ into the relation (7.2), we have

$$\begin{aligned} &W_7(U,V)(R(X,\xi)Z) - R(W_7(U,V)X,\xi)Z \\ &- R(X,W_7(U,V)\xi)Z - R(X,\xi)W_7(U,V)Z = 0. \end{aligned} \tag{7.3}$$

Now, we evaluate each term of (7.3).
Using (2.12) into (2.20), we get

$$\begin{aligned}W_7(U,V)(R(X,\xi)Z) = {} & g(X,Z)\eta(U)V - \frac{1}{(n-1)}\eta(V)g(X,Z)QU \\ & + \eta(Z)R(U,V)X + \frac{1}{(n-1)}\eta(Z)g(V,X)QU \\ & - \frac{1}{(n-1)}\eta(Z)S(V,X)U.\end{aligned} \tag{7.4}$$

The second, third and fourth terms are given by the equations (7.5), (7.6) and (7.7) respectively.

$$\begin{aligned}R(W_7(U,V)X,\xi)Z = {} & -g(R(U,V)X,Z)\xi - \frac{1}{(n-1)}g(V,X)S(U,Z)\xi \\ & + \frac{1}{(n-1)}S(V,X)g(U,Z)\xi + \eta(Z)R(U,V)X \\ & + \frac{1}{(n-1)}\eta(Z)g(V,X)QU - \frac{1}{(n-1)}\eta(Z)S(V,X)U,\end{aligned} \tag{7.5}$$

$$R(X,W_7(U,V)\xi)Z = -\eta(U)R(X,V)Z + \frac{1}{(n-1)}\eta(V)R(X,QU)Z, \tag{7.6}$$

$$\begin{aligned}R(X,\xi)W_7(U,V)Z = {} & -g(X,R(U,V)Z)\xi - \frac{1}{(n-1)}g(V,Z)S(X,U)\xi \\ & + \frac{1}{(n-1)}S(V,Z)g(X,U)\xi + 2g(V,Z)\eta(U)X \\ & - g(U,Z)\eta(V)X - \frac{1}{(n-1)}S(V,Z)\eta(U)X.\end{aligned} \tag{7.7}$$

Using equations (7.4),(7.5),(7.6) and (7.7) in (7.3), we have

$$\begin{aligned} & g(X,Z)\eta(U)V - \frac{1}{(n-1)}\eta(V)g(X,Z)QU + g(R(U,V)X,Z)\xi \\ & + \frac{1}{(n-1)}g(V,X)S(U,Z)\xi - \frac{1}{(n-1)}S(V,X)g(U,Z)\xi + \eta(U)R(X,V)Z \\ & - \frac{1}{(n-1)}\eta(V)S(X,QU)Z + g(X,R(U,V)Z)\xi + \frac{1}{(n-1)}g(V,Z)S(X,U)\xi \\ & - \frac{1}{(n-1)}S(V,Z)g(X,U)\xi - 2g(V,Z)\eta(U)X + g(U,Z)\eta(V)X \\ & + \frac{1}{(n-1)}S(V,Z)\eta(U)X = 0.\end{aligned} \tag{7.8}$$

Taking the inner product of (7.8) with W and contracting along X and W, we get

$$\begin{aligned}&\eta(U)g(V,Z)\xi-\eta(V)g(U,Z)+\eta(U)S(V,Z)\\&-\frac{1}{(n-1)}\eta(V)S(QU,Z)+\eta(U)g(V,Z)-\frac{1}{(n-1)}\eta(U)S(V,Z)\\&-2n\eta(U)g(V,Z)+ng(U,Z)\eta(V)+\frac{n}{(n-1)}S(V,Z)\eta(U)=0.\end{aligned} \tag{7.9}$$

Putting $U=\xi$ into (7.9), we have

$$S(V,Z)=ng(V,Z), \tag{7.10}$$

for all vector fields $V,Z\in\chi(M)$. Hence, the M is an Einstein manifold. $\square$

8. **Example**

Let us consider a smooth manifold $M=\{(u,v,w,t)\in R^4 : u,v,w$ is non-zero, $t>0\}$ of dimension 4, here (u,v,w,t) is the standard coordinates in R^4. Consider a set of linearly independent vector fields $\{\xi_1,\xi_2,\xi_3,\xi_4\}$ at every point of the manifold M.

We define

$$\xi_1=e^{u+t}\frac{\partial}{\partial u},\xi_2=e^{v+t}\frac{\partial}{\partial v},\xi_3=e^{w+t}\frac{\partial}{\partial w},\xi_4=\frac{\partial}{\partial t}.$$

Let g be the Lorentzian metric defined by

$$g_{ij}=\begin{cases}1, \text{ if } i=j\neq 4\\0, \text{ if } i\neq j\\-1, \text{ if } i=j=4,\end{cases}$$

Let η be the 1-form on M defined as $\eta(X)=g(X,\xi_4)=g(X,\xi)$ for all $X\in\chi(M)$ and let ϕ be the $(1,1)$-tensor field on M defined as

$$\phi\xi_1=\xi_1,\phi\xi_2=\xi_2,\phi\xi_3=\xi_3,\phi\xi_4=0. \tag{8.1}$$

using the linear property of ϕ and g, we have

$$\begin{aligned}&\eta(\xi)=-1,\phi^2X=X+\eta(X)\xi,\eta(\phi X)=0\\&g(X,\xi)=\eta(X),g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y)\end{aligned} \tag{8.2}$$

for all $X,Y\in\chi(M)$. This shows that the manifold M is equipped with a Lorentzian paracontact structure. Hence the chosen manifold is a Lorentzian para-contact manifold of dimension 4. The non-zero constituents of Lie brackets are evaluated as

$$[\xi_1,\xi_4]=-\xi_1,[\xi_2,\xi_4]=-\xi_2,[\xi_3,\xi_4]=-\xi_3.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_U V, W) &= Ug(V, W) + Vg(W, U) - Wg(U, V) - g(U, [V, W]) \\ &+ g(V, [W, U]) + g(W, [U, V]), \end{aligned} \tag{8.3}$$

which is known as *Koszul's formula.* we can easily calculate

$$\begin{gathered} \nabla_{\xi_1}\xi_1 = -\xi_4, \nabla_{\xi_1}\xi_2 = 0, \nabla_{\xi_1}\xi_3 = 0, \nabla_{\xi_1}\xi_4 = -\xi_1, \\ \nabla_{\xi_2}\xi_1 = 0, \nabla_{\xi_2}\xi_2 = -\xi_4, \nabla_{\xi_2}\xi_3 = 0, \nabla_{\xi_2}\xi_4 = -\xi_2, \\ \nabla_{\xi_3}\xi_1 = 0, \nabla_{\xi_3}\xi_2 = 0, \nabla_{\xi_3}\xi_3 = -\xi_4, \nabla_{\xi_3}\xi_4 = -\xi_3, \\ \nabla_{\xi_4}\xi_1 = 0, \nabla_{\xi_4}\xi_2 = 0, \nabla_{\xi_4}\xi_3 = 0, \nabla_{\xi_4}\xi_4 = 0, \end{gathered}$$

Let X be any arbitrary vector field on M, Then

$$X = c^i\xi_i = c^1\xi_1 + c^2\xi_2 + c^3\xi_3 + c^4\xi_4$$

for some scalars c^1, c^2, c^3, c^4.

With the help of the above relation and using the linearity property of the connection, we can easily verify that $\nabla_X\xi_4 = -X - \eta(X)\xi_4$.
Hence, M is a Lorentzian para-Kenmotsu manifold of dimension 4.
The non-vanishing components of the curvature tensor are evaluated as follows:

$$\begin{gathered} R(\xi_1, \xi_2)\xi_1 = -\xi_2, R(\xi_1, \xi_3)\xi_1 = -\xi_3, R(\xi_1, \xi_4)\xi_1 = -\xi_4, \\ R(\xi_1, \xi_2)\xi_2 = \xi_1, R(\xi_2, \xi_3)\xi_2 = -\xi_3, R(\xi_2, \xi_4)\xi_2 = -\xi_4, \\ R(\xi_1, \xi_3)\xi_3 = \xi_1, R(\xi_2, \xi_3)\xi_3 = \xi_2, R(\xi_3, \xi_4)\xi_3 = -\xi_4, \\ R(\xi_1, \xi_4)\xi_4 = -\xi_1, R(\xi_2, \xi_4)\xi_4 = -\xi_2, R(\xi_3, \xi_4)\xi_4 = -\xi_3, \end{gathered} \tag{8.4}$$

It can be easily seen that $R(X, Y)Z = g(Y, Z)X - g(X, Z)Y$.
Since,

$$S(X, Y) = g(R(\xi_1, X)Y, \xi_1) + g(R(\xi_2, X)Y, \xi_2) + g(R(\xi_3, X)Y, \xi_3) - g(R(\xi_4, X)Y, \xi_4)$$

Using the equation (8.4), we can easily see that $S(X, Y) = 3g(X, Y)$, It clearly implies that $(LPK)_4$ is an Einstein manifold, and using these relations, we see that the relation $\phi^2((\nabla_U W_7)(X, Y)Z) = 0$ holds good.

9. **Conclusion**

In this work, we investigated various geometric conditions on $(LPK)_n$ manifolds involving the W_7-curvature tensor and analyzed their consequences. Although the results may appear individually structured, they collectively highlight a deeper relationship between curvature constraints and the Ricci properties of Lorentzian para-Kenmotsu manifolds.

We began by establishing in **Theorem 1** that a ξ-W_7-flat $(LPK)_n$ manifold is necessarily an η-Einstein manifold. **Theorem 2** revealed that if the manifold is W_7-semisymmetric, it becomes Ricci-flat. In **Theorem 3**, we proved that under ϕ-W_7-symmetry, the manifold reduces to an Einstein manifold, aligning with the curvature rigidity introduced by the symmetry. **Theorem 4** adds a structural interpretation by showing that the ϕ-W_7-flat condition implies that the distribution defined by ϕ is null. Finally, in **Theorem 5**, we demonstrated that if the manifold satisfies the curvature condition $W_7(U,V) \cdot R = 0$, then it again reduces to an Einstein manifold.

These results, while framed under different curvature assumptions, converge toward a central theme: under various W_7-curvature constraints, Lorentzian para-Kenmotsu manifolds tend to exhibit Einstein or Ricci-flat properties. This observation not only strengthens the geometric significance of the W_7-tensor in such structures but also motivates further exploration of the curvature-induced rigidity in paracontact geometry.

Acknowledgment: The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

References

1. G. P. Pokhariyal, *Relativistic significance of curvature tensors*, Int. J. Math. Math. Sci. **5**(1) (1982), 133-139.
2. M. Atceken, *Some results on invariant submanifolds of Lorentzian para-Kenmotsu manifolds* Korean J. Math. **30**(1) (2022), 175-185.
3. C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi-constant curvature.* Publ. Math. Debrecen, **78**(1) (2011), 235-243.
4. A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds.* ArXiv (Cornell University), 2014.
5. A. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds.* Filomat, **30**(2) (2016), 489-496.
6. G.P. Pokhariyal. *Study of new curvature tensor in Sasakian manifold.* Tensor, N.S., **36**(1982), 222-226.
7. A. Magnon, *Semi-Riemannian Geometry with Applications to Relativity* (Barrett O'Neill). SIAM Review, **28**(2) (1986), 269-270.
8. B.B. Sinha, and K.L.S. Sai Prasad,*A class of almost para contact metric manifold*, Bull. Calcutta. Math. Soc. **87**(1995), 307-312.
9. A. Haseeb and R. Prasad,*Certain results on Lorentzian para-Kenmotsu manifolds.* Bull. Paraan. Math. Soc. **39**(3) (2021), 201-220.
10. A. Singh and Shyam Kishor. *Some Types of η-Ricci Solitons on Lorentzian Para-Sasakian manifolds.* Facta. Univ. Ser. Math. Inf. **33**(2) (2018), 217-217.
11. Shyam Kishor and A. Singh, *η-Ricci solitons on 3-dimensional Kenmotsu manifolds*, Bull. Transilvania. Univ. Brasov. Ser. III Math. Comp. Sci. **13**(62)(1) (2020), 209-218..
12. C. S. Bagewadi, and G. Ingalahalli, *Ricci solitons in Lorentzian-Sasakian manifolds.* Acta Math. Acad. Paeda. Nyire. **28**(2012), 59-68.
13. G. Ingalahalli and C.S. Bagewadi,*Ricci Solitons in α-Sasakian manifolds.* ISRN Geometry, 13.421384, 2012.
14. S. Kishor and P. Verma, *Some Results On W_1-Curvature tensor On (k,μ)- Contact Space forms*, 2018.

15. R. Prasad, A. Haseeb, A. Verma, and V.S. Yadav, *A study of ϕ-Ricci symmetric LP-Kenmotsu manifolds.* Int. J. Maps. Math. Volume 7, Issue 1, Pages:33-44, 2024.
16. A. Singh, S. Kishor,*Curvature Properties of η-Riccci Solitons on Para-Kenmotsu Manifolds.* Kyungpook. Math. Journal. **59**(2019), 149-161.
17. S. Kishor and P. Verma, *On W_7-Curvature tensor of Generalized Sasakian Space Forms,* Int. J. Math. Tren. Tech. **49**(2017).
18. S. Kishor and P. Verma, *On W_0 Curvature Tensor of Generalized Sasakian-Space-Forms,* JUSPS-A Vol. **29**(10) (2017), 427-439.
19. P. Alegre, *Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds.* Taiwanese. J. Math. **17**(3) (2013), 897-910.
20. M. M. Tripathi, P. Gupta, *T-curvature tensor on a semi-Riemannian manifold*, J. Adv. Math. Stud, **4**(1) (2011), 117-129.
21. U. C. De, R. N. Singh and S. K. Pandey, *On the Conharmonic Curvature Tensor of Generalized Sasakian-Space-Forms*, Int. Sch. Res. Notices, **1**(2012), 876276.

Received: 29.04.2025
Accepted: 14.08.2025