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Some characterization of α-cosymplectic manifolds admitting hyperbolic Ricci solitons (HRS)

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Abstract. This work investigates α-cosymplectic and $N(k)$-contact metric (CM) manifolds equipped with an HRS. We derive some characterization properties for these manifolds.

Keywords: Ricci Soliton, α-Cosymplectic Manifold, $N(k)$-CM Manifold, HRS.

1. Introduction

Hamilton brought forth the Ricci solitons (RS) concept, expanding on Einstein manifolds. Additionally, RS provides an analogous resolution for Ricci flow (RF), an evolution formula governing the Riemannian manifold metrics (M, g_0), given by

$$\frac{\partial \mathrm{g}_{ij}(t)}{\partial t} = -2R_{ij}(\mathrm{g}(t)), \qquad g(0) = g_0.$$

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In this case, it R_{ij} stands for the Ricci curvature tensor (RC) associated with $\mathrm{g} = \mathrm{g}(t)$. This topic has been the main focus of extensive research in geometric analysis and differential geometry [13, 14]. When Perelman solved the Poincaré conjecture and discovered that RS that acts as solutions to the RF are gradient RS in compact, connected RMs, its significance increased significantly [16]. Numerous nontrivial examples of compact and noncompact RS have been identified [8]. The RF describes the thermal nature of the manifold curvatures and metrics. Formally stated in [1], the hyperbolic Ricci flow (HRF) is a framework of 2^{nd}-order non-linear growth PDE, comparable to the wave equation flow metrics:

$$\frac{\partial^2}{\partial t^2}\mathrm{g} = -2\,\mathrm{Ric}, \quad \mathrm{g}(0) = \mathrm{g}_0, \quad \frac{\partial \mathrm{g}}{\partial t}(0) = k_0, \tag{1.1}$$

where k_0 is type-2 tensor field that is symmetric on M. The presence and exclusivity of the equation (1.1) have been investigated within the framework of closed RM in the research carried out by [9].

The pioneering work on HRF can be attributed to Kong and Liu, whose research is detailed in [15]. As mentioned above, second-order nonlinear evolution PDEs determine this flow. HRF is a geometric flow that models the evolution of metrics and curvatures in manifolds, exhibiting wavelike properties.

Azami, in his work [2], investigated the hyperbolic Ricci-Bourguignon flow in compact manifolds and established the uniqueness and existence of solutions for this flow within a short time interval, subject to specific initial conditions. He derived equations that describe how the Riemannian curvature tensor (RC) and the scalar curvature (SC) of the manifold evolve under the influence of this flow.

In a published study [7], Chaubey, Siddiqi, and Prakasha examined the characteristics of η-Ricci-Bourguignon solitons on invariant submanifolds located within hyperbolic Sasakian manifolds. They also verified several of their theoretical results and gave a non-trivial instance of a 3-dim invariant submanifold inside the same ambient space of dimension 5. In the setting of HRF, this work explores the domain of self-similar solutions. Specifically, HRS. Building upon our exploration of HRS, we focus our attention on their properties within the specific geometric frameworks of α-cosymplectic and $N(k)$-CM manifolds. In addition, certain characterization results will be demonstrated within these manifolds.

2. **Perliminaries**

This section provides a concise overview of essential definitions and formulas that will be utilized in subsequent sections.

Definition 2.1. *A Ricci soliton (RS) on (M^m, g) is characterized by the existence of a smooth v.f. X on M^m fulfilling the criteria*

$$\mathcal{L}_X \mathrm{g} + 2Ric = 2\mu \mathrm{g}, \tag{2.1}$$

here, μ represents a constant real number, while Ric stands for Ricci tensor and $\mathcal{L}_X$ denotes the Lie derivative operator along X.

With (M^m, g, X, μ), we represent an RS. The aforementioned X is termed a potential field for RS. An RS is classified as steady, shrinking, or expanding according to whether the constant μ is zero, positive, or negative, respectively. Additionally, a RS is classified as a gradient soliton if

$$X = \nabla f.$$

Here, f serves as the potential function of RS and X is a field. This allows (2.1) to be restated as

$$Ric + \nabla^2 f = \mu \mathrm{g}. \tag{2.2}$$

Here, the Hessian of f is represented by $\nabla^2 f$ [11][12]. Given a positive function $\sigma(t)$ and a one-parameter family of diffeomorphisms $\varphi(t)$ on M to itself that satisfies the following criteria, then a function $\mathrm{g}(t)$ that solves the HRF on M^n is called a HRS (or self-similar solution):

$$\mathrm{g}(t) = \sigma(t)\varphi(t)^* \mathrm{g}(0). \tag{2.3}$$

Definition 2.2. *According to [3], a RM (M^n, g) is termed as HRS structure given X in M, along with real constants μ and λ, that meet the following criteria:*

$$\mathrm{Ric} + \lambda \mathcal{L}_X \mathrm{g} + \frac{1}{2}(\mathcal{L}_X \circ \mathcal{L}_X)(\mathrm{g}) = \mu \mathrm{g}. \tag{2.4}$$

We represent an HRS by $(M, \mathrm{g}, X, \lambda, \mu)$. An HRS can be categorized for μ:

- $\lambda = 0$ (steady)
- $\lambda < 0$ (shrinking)
- $\lambda > 0$ (expanding).

Next, we will review some definitions and fundamental equations related to contact manifolds.

A smooth manifold M^{2n+1} equipped by a nearly CM structure $(\phi, \xi, \eta, \mathrm{g})$ becomes a nearly CM manifold, where g adheres to this specific condition:

$$\phi^2 X = \eta(X)\xi - X, \qquad \eta(X) = \mathrm{g}(X, \xi), \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \qquad \eta(\xi) = 1. \tag{2.5}$$

$$\begin{aligned} \mathrm{g}(\phi X, Y) &= -\mathrm{g}(X, \phi Y), \\ \mathrm{g}(\phi X, \phi X) &= \mathrm{g}(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.6}$$

$\forall X, Y \in \Gamma(TM)$, here ξ is the v.f., and η is the 1-form that corresponds to the g-dual of ξ, and ϕ represents $(1,1)$-tensor field on M. Moreover, the fundamental 2-form Φ on M is described by [4]:

$$\Phi(X,Y) = g(X,\phi Y). \tag{2.7}$$

Additionally, if the subsequent requirement is met

$$\Phi(X,Y) = d\eta(X,Y), \tag{2.8}$$

a nearly CM manifold (M,ϕ,ξ,η,g) is recognized as a CM manifold provided

$$d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - \eta([X,Y]) - Y\eta(X)\}. \tag{2.9}$$

Define the Nijenhuis tensor field N_ϕ as follows:

$$N_\phi(X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]. \tag{2.10}$$

As M is a nearly CM manifold and N_ϕ satisfies

$$2d\eta \otimes \xi + N_\phi = 0, \tag{2.11}$$

in this context, M is identified as a normal CM manifold. A standard CM manifold M is described as Sasakian. A nearly CM manifold M is Sasakian iff

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.12}$$

Additionally, it follows for a Sasakian manifold,

$$\nabla_X \xi = -\phi X, \tag{2.13}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(x)Y, \tag{2.14}$$

with ∇ representing the L.C. connection and R symbolizing $\mathcal{RC}$ on M. In [6], the (k,ρ)-nullity distribution on CM manifolds was put forward, defined as [6]

$$N(k,\rho): p \to N_p(k,\rho) = \Big\{ Z \in T_pM | R(X,Y)Z = (kI + \rho h)((g(Y,Z)X \\ -g(X,Z)Y) \Big\} \tag{2.15}$$

with (k,ρ) belonging to $\mathbb{R}^2$, I representing the identity map, and h being the $(1,1)$-tensor field described with

$$h = \frac{1}{2}\mathcal{L}_\xi \phi.$$

The h adheres to the next relationships:

$$\begin{aligned} h\phi + \phi h &= 0, \\ \nabla_X \xi &= -\phi X - \phi h X, \\ h\xi &= 0. \end{aligned} \tag{2.16}$$

and

$$g(hX,Y) = g(X,hY), \tag{2.17}$$
$$\eta(hX) = 0. \tag{2.18}$$

M equipped with a CM, is considered a (k,ρ)-CM manifold if ξ is included in the $N(k,\rho)$. When the coefficient ρ is identically zero as in (2.15), $N(k,\rho)$ simplifies to the k-nullity distribution $N(k)$, which is specified as [17]

$$N(k) : p \to N_p(k) = \Big\{Z \in T_pM | R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\Big\}. \tag{2.19}$$

Moreover, when $\xi \in N(k)$, M possessing a CM structure, it is named an $N(k)$-CM manifold [17]. This $N(k)$-CM manifold transforms into a Sasakian manifold when $k = 1$. When $k = 0$, and given $n > 1$, the manifold is locally isometric to $E^{n+1} \times S^4$, while it is flat when $n = 1$ [5].
The following conditions were met for a $N(k)$-CM manifold:

$$Ric(X,Y) = [2nk - 2(n-1)]\eta(X)\eta(Y) + 2(n-1)g(X,Y) + 2(n-1)g(hX,Y), \quad n \geq 1, \tag{2.20}$$
$$Ric(X,\xi) = 2nk\eta(X). \tag{2.21}$$

Example 2.3. [10] *We take into account the 3-dimensional manifold*

$$M = \{(x,y,z) \in \mathbb{R}^3, (x,y,z) \neq (0,0,0)\}. \tag{2.22}$$

Consider e_1, e_2, and e_3 as linearly independent vector fields in $\mathbb{R}^3$ that fulfill

$$[e_1,e_2] = (1+a)e_3, \qquad [e_1,e_3] = -(1-a)e_2 \qquad \textit{and} \qquad [e_2,e_3] = 2e_1, \tag{2.23}$$

where a is any real number. Assume that the Riemannian metric g is given as:

$$g(e_i,e_i) = 1,$$
$$g(e_i,e_j) = 0 \qquad for \quad i \neq j.$$

We define η as a 1-form and ϕ as a $(1,1)$-tensor field, with the following properties:

$$\eta(Z) = g(Z,e_1),$$
$$\phi(e_2) = e_3, \qquad \phi(e_3) = -e_2, \qquad \phi(e_1) = 0,$$
$$he_1 = 0, \qquad he_2 = ae_2, \qquad he_3 = -ae_3. \tag{2.24}$$

Employing Koszul's formula for g yields:

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = \nabla_{e_1}e_3 = \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = 0,$$
$$\nabla_{e_3}e_2 = -(1-a)e_1, \qquad \nabla_{e_3}e_1 = (1-a)e_2,$$
$$\nabla_{e_2}e_1 = -(1+a)e_3, \qquad \nabla_{e_2}e_3 = (1+a)e_1. \tag{2.25}$$

Hence, $(M, \phi, \xi, \eta, \mathrm{g})$ *represents a 3-dim CM manifold.*
In view of (2.25),

$$R(a_1)e_3 = 0, \qquad R(a_1)e_2 = 0, \qquad R(a_3)e_1 = 0,$$

$$R(a_1)e_2 = (1 - a^2)e_1, \qquad R(a_1)e_1 = -(1 - a^2)e_2,$$

$$R(a_2)e_3 = (1 - a^2)e_1, \qquad R(a_2)e_1 = -(1 - a^2)e_3,$$

$$R(a_3)e_3 = -(1 - a^2)e_2, \qquad R(a_3)e_2 = (1 - a^2)e_3.$$

here $a_1 = (e_1, e_2), a_2 = (e_1, e_3)$ *and* $a_3 = (e_2, e_3)$. *Hence, the above relations show that* M *is a* 3*-dim* $N(1 - a^2)$*-CM manifold.*

(M, ϕ, ξ, η, g) with a normal CM is termed as cosymplectic if the subsequent is true:

$$d\eta = 0, \qquad d\Phi = 0. \tag{2.26}$$

Equivalently,

$$(\nabla_X \phi)Y = 0, \qquad \nabla_X \xi = 0, \qquad \forall X, Y \in \Gamma(TM). \tag{2.27}$$

If

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi, \tag{2.28}$$

are met, it M is referred to as a α-cosypmlectic manifold, in which α represents a real number. Equivalently,

$$(\nabla_X \phi)Y = \alpha(\mathrm{g}(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.29}$$

$$\nabla_X \xi = -\alpha\phi^2 X. \tag{2.30}$$

If $\alpha = 0$, it becomes evident that M qualifies as a cosymplectic manifold. For $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, M is called an α-Kenmotsu manifold. Moreover, an α-cosymplectic manifold adheres to

$$R(X, Y)\xi = \alpha^2(\eta(X)Y - \eta(Y)X), \tag{2.31}$$

$$R(X, \xi)Y = \alpha^2(\mathrm{g}(X, Y)\xi - \eta(Y)X), \tag{2.32}$$

$$R(X, Y)\xi = \alpha^2(\eta(X)\xi - X), \tag{2.33}$$

$$Ric(X, \xi) = -2n\alpha^2\eta(X). \tag{2.34}$$

Consider (M^n, g) represent RM, here $X(M)$ represents the Lie algebra of smooth vector fields on M. A $V \in X(M)$ is referred to as a conformal v.f. if

$$\mathcal{L}_V = 2f\mathrm{g}. \tag{2.35}$$

Let $\mathcal{L}_V$ is denoted by the Lie derivative associated with V. Suppose there are functions a and b satisfying

$$(Ric = a\mathrm{g} + bE),$$

where E is a non-zero symmetric $(0,2)$-tensor on M. In this case, (M, g) is termed a nearly quasi-Einstein manifold [18]. Additionally, (M, g) is known as an η-Einstein manifold if

$$(Ric = c\mathrm{g} + d\eta \otimes \eta)$$

for some constants c and d.

Finally, recall that V on an RM is classified as concircular if it meets the criteria:

$$\nabla_X V = fX \tag{2.36}$$

$\forall X \in \Gamma(TM)$. If $f = 1$, V is referred to as a concurrent vector field.

3. α-Cosymplectic Manifolds Admitting a Homogeneous Riemannian Soliton (HRS)

This portion examines α-cosymplectic manifold (M) endowed with **HRS**.

Theorem 3.1. *Consider M admits a* **HRS**. *M qualifies as a virtually quasi-Einstein manifold if the potential vector field V is pointwise aligned with ξ.*

Proof. Consider $(M, \mathrm{g}, V, \lambda, \mu)$ as an HRS in which V is pointwise aligned with ξ, implying $V = b\xi$ for a smooth function b. Applying equations (2.5) and (2.30), we find

$$\begin{aligned}\mathcal{L}_V \mathrm{g}(X,Y) &= g(\nabla_X V, Y) + \mathrm{g}(\nabla_Y V, X) \\ &= \mathrm{g}(X(b)\xi + b\nabla_X \xi, Y) + \mathrm{g}(Y(b)\xi + b\nabla_Y \xi, X) \\ &= X(b)\eta(Y) + 2b\alpha(\mathrm{g}(X,Y) - \eta(X)\eta(Y) + Y(b)\eta(X) \\ &= \mathrm{g}(\nabla b, X)\eta(Y) + 2b\alpha(\mathrm{g}(X,Y) - \eta(X)\eta(Y)) + \mathrm{g}(\nabla b, Y)\eta(X)\end{aligned} \tag{3.1}$$

$\forall X, Y \in \Gamma(TM)$,in which ∇b stands for the gradient of b.

From equation (3.1) and the definition of $\mathcal{L}$, we can write

$$
\begin{aligned}
(\mathcal{L}_V(\mathcal{L}_V \mathrm{g}))(X,Y) &= V.\mathcal{L}_V\mathrm{g}(X,Y) - \mathcal{L}_V\mathrm{g}(\mathcal{L}_V X, Y) - \mathcal{L}_V\mathrm{g}(X, \mathcal{L}_V Y) \\
&= V.\Big[\mathrm{g}(\nabla b, X)\eta(Y) + \mathrm{g}(\nabla b, Y)\eta(X) + 2b\alpha(\mathrm{g}(X,Y) \\
&\quad -\eta(X)\eta(Y))\Big] - \Big[\mathrm{g}(\nabla b, \mathcal{L}_V X)\eta(Y) + \mathrm{g}(\nabla b, Y)\eta(\mathcal{L}_V X) \\
&\quad +2b\alpha(\mathrm{g}(\mathcal{L}_V X, Y) - \eta(\mathcal{L}_V X)\eta(Y))\Big] - \Big[\mathrm{g}(\nabla b, X)\eta(\mathcal{L}_V Y) \\
&\quad +\mathrm{g}(\nabla b, \mathcal{L}_V Y)\eta(X) + 2b\alpha(\mathrm{g}(X, \mathcal{L}_V Y) - \eta(X)\eta(\mathcal{L}_V Y))\Big] \\
&= V.\mathrm{g}(\nabla b, X)\eta(Y) + \mathrm{g}(\nabla b, X)V.\eta(Y) + V.\mathrm{g}(\nabla b, Y)\eta(X) \\
&\quad +\mathrm{g}(\nabla b, Y)V.\eta(X) + 2V(b)\alpha(\mathrm{g}(X,Y) - \eta(X)\eta(Y)) \\
&\quad +2b\alpha V.(\mathrm{g}(X,Y) - \eta(X)\eta(Y)) - \mathrm{g}(\nabla b, \mathcal{L}_V X)\eta(Y) \\
&\quad -\mathrm{g}(\nabla b, Y)\eta(\mathcal{L}_V Y) - 2b\alpha\mathrm{g}(\mathcal{L}_V X, Y) + 2b\alpha\eta(\mathcal{L}_V X)\eta(Y) \\
&\quad -\mathrm{g}(\nabla b, X)\eta(\mathcal{L}_V Y) - \mathrm{g}(\nabla b, \mathcal{L}_V Y)\eta(X) - 2b\alpha\mathrm{g}(X, \mathcal{L}_V Y) \\
&\quad +2b\alpha\eta(X)\eta(\mathcal{L}_V Y). \qquad (3.2)
\end{aligned}
$$

By combining (2.4) and (3.2), we obtain

$$
\begin{aligned}
Ric(X,Y) &= \mu\mathrm{g}(X,Y) - \frac{1}{2}(\mathcal{L}_V(\mathcal{L}_V\mathrm{g}))(X,Y) - \lambda\mathcal{L}_V\mathrm{g}(X,Y) \\
&= (\mu - \alpha V(b) - 2\lambda b\alpha)\mathrm{g}(X,Y) - \frac{1}{2}V.\mathrm{g}(\nabla b, X)\eta(Y) \\
&\quad -\frac{1}{2}\mathrm{g}(\nabla b, X)V.\eta(Y) - \frac{1}{2}V.\mathrm{g}(\nabla b, Y)\eta(X) - \frac{1}{2}\mathrm{g}(\nabla b, Y)V.\eta(X)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha V(b)\eta(X)\eta(Y) - b\alpha V.(\mathrm{g}(X,Y) - \eta(X)\eta(Y)) + \frac{1}{2}\mathrm{g}(\nabla b, \mathcal{L}_V X)\eta(Y) \\
&+\frac{1}{2}\mathrm{g}(\nabla b, Y)\eta(\mathcal{L}_V X) + b\alpha\mathrm{g}(\mathcal{L}_V X, Y) - b\alpha\eta(\mathcal{L}_V X)\eta(Y) \\
&+\frac{1}{2}\mathrm{g}(\nabla b, X)\eta(\mathcal{L}_V Y) + \frac{1}{2}(\mathrm{g}(\nabla b, \mathcal{L}_V Y)\eta(X) \\
&+b\alpha\mathrm{g}(X, \mathcal{L}_V Y) - b\alpha\eta(X)\eta(\mathcal{L}_V Y) - \lambda\mathrm{g}(\nabla b, X)\eta(Y) \\
&-\lambda\mathrm{g}(\nabla b, Y)\eta(X) + 2\lambda b\alpha\eta(X)\eta(Y). \qquad (3.3)
\end{aligned}
$$

Also, specify a symmetric, non-vanishing tensor E of type $(0,2)$ by

$$\begin{aligned} E(X,Y) = &-\frac{1}{2}V.\mathrm{g}(\nabla b, X)\eta(Y) - \frac{1}{2}\mathrm{g}(\nabla b, X)V.\eta(Y) \\ &- \frac{1}{2}V.\mathrm{g}(\nabla b, Y)\eta(X) - \frac{1}{2}\mathrm{g}(\nabla b, Y)V.\eta(X) + \alpha V(b)\eta(X)\eta(Y) \\ &- b\alpha V.(\mathrm{g}(X,Y) - \eta(X)\eta(Y)) + \frac{1}{2}\mathrm{g}(\nabla b, \mathcal{L}_V X)\eta(Y) \\ &+ \frac{1}{2}\mathrm{g}(\nabla b, Y)\eta(\mathcal{L}_V X) + b\alpha\mathrm{g}(\mathcal{L}_V X, Y) - b\alpha\eta(\mathcal{L}_V X)\eta(Y) \\ &+ \frac{1}{2}\mathrm{g}(\nabla b, X)\eta(\mathcal{L}_V Y) + \frac{1}{2}(\mathrm{g}(\nabla b, \mathcal{L}_V Y)\eta(X) \\ &+ b\alpha\mathrm{g}(X, \mathcal{L}_V Y) - b\alpha\eta(X)\eta(\mathcal{L}_V Y) - \lambda\mathrm{g}(\nabla b, X)\eta(Y) \\ &- \lambda\mathrm{g}(\nabla b, Y)\eta(X) + 2\lambda b\alpha\eta(X)\eta(Y). \end{aligned} \tag{3.4}$$

Equation (3.3) then takes the form

$$Ric(X,Y) = (\mu - \alpha V(b) - 2\lambda b\alpha)\mathrm{g}(X,Y) + E. \tag{3.5}$$

Consequently, M is a manifold that is almost quasi-Einsteinian. □

Theorem 3.2. *If M possesses a concircular vector field V, and if V acts as the potential vector field for a homothetic Ricci soliton (HRS) on M, then*

$$\mu = -2n\alpha^2 + V.f + 2f^2 + 2\lambda f. \tag{3.6}$$

Proof. It is evident that

$$\nabla_X V = fX \tag{3.7}$$

$\forall X \in \Gamma(TM)$. Utilizing (3.7) follows:

$$\begin{aligned} (\mathcal{L}_V\mathrm{g})(X,Y) &= \mathrm{g}(\nabla_X V, Y) + \mathrm{g}(X, \nabla_Y V) \\ &= \mathrm{g}(fX, Y) + \mathrm{g}(X, fY) \\ &= 2f\mathrm{g}(X,Y). \end{aligned} \tag{3.8}$$

From equation (3.8) and the explanation of $\mathcal{L}$, one can write

$$\begin{aligned} (\mathcal{L}_V(\mathcal{L}_V\mathrm{g}))(X,Y) &= V.\mathcal{L}_V\mathrm{g}(X,Y) - \mathcal{L}_V\mathrm{g}(\mathcal{L}_V X, Y) - \mathcal{L}_V\mathrm{g}(X, \mathcal{L}_V Y) \\ &= V.(2f\mathrm{g}(X,Y)) - 2f\mathrm{g}(\mathcal{L}_V X, Y) - 2f\mathrm{g}(X, \mathcal{L}_V Y) \\ &= 2V.f\mathrm{g}(X,Y) + 2fV.\mathrm{g}(X,Y) - 2f\mathrm{g}(\nabla_V X - \nabla_X V, Y) \\ &\qquad - 2f\mathrm{g}(X, \nabla_V Y - \nabla_Y V) \\ &= 2V.f\mathrm{g}(X,Y) + 2f\mathrm{g}(\nabla_V X, Y) + 2f\mathrm{g}(X, \nabla_V Y) \\ &\qquad - 2f\mathrm{g}(\nabla_V X, Y) + 2f\mathrm{g}(\nabla_X V, Y) - 2f\mathrm{g}(X, \nabla_V Y) \\ &\qquad + 2f\mathrm{g}(X, \nabla_Y V) \\ &= 2V.f\mathrm{g}(X,Y) + 2f\mathrm{g}(\nabla_X V, Y) + 2f\mathrm{g}(X, \nabla_Y X) \\ &= 2V.f\mathrm{g}(X,Y) + 4f^2\mathrm{g}(X,Y). \end{aligned} \tag{3.9}$$

By using (2.4) and (3.9):

$$Ric(X,Y) + V.f\mathrm{g}(X,Y)2f^2\mathrm{g}(X,Y) + 2\lambda f\mathrm{g}(X,Y) = \mu\mathrm{g}(X,Y). \tag{3.10}$$

Substituting $X = Y = \xi$ in (3.10) and from (2.5) and (2.34), we have

$$\mu = -2n\alpha^2 + V.f + 2f^2 + 2\lambda f. \tag{3.11}$$

□

Theorem 3.3. *Given M has an HRS with ξ as its potential vector field, then*

$$\mu = -2n\alpha^2. \tag{3.12}$$

Proof. The definition of $\mathcal{L}$ and the application of (2.30) yield the following $\forall X, Y \in \Gamma(TM)$:

$$\begin{aligned}(\mathcal{L}_\xi\mathrm{g})(X,Y) &= \mathrm{g}(\nabla_X\xi, Y) + \mathrm{g}(\nabla_Y\xi, X) \\ &= 2\alpha(\mathrm{g}(X,Y) - \eta(X)\eta(Y)).\end{aligned} \tag{3.13}$$

Furthermore, using the definition of $\mathcal{L}$, results,

$$(\mathcal{L}_\xi(\mathcal{L}_\xi\mathrm{g}))(X,Y) = \xi\mathcal{L}_\xi\mathrm{g}(X,Y) - \mathcal{L}_\xi\mathrm{g}(X,\mathcal{L}_\xi Y) - \mathcal{L}_\xi\mathrm{g}(Y,\mathcal{L}_\xi X), \tag{3.14}$$

and

$$\begin{aligned}\xi.\mathcal{L}_\xi\mathrm{g}(X,Y) &= \xi.(\mathrm{g}(\nabla_\xi X, Y) + \mathrm{g}(X, \nabla_\xi Y)) \\ &= \mathrm{g}(\nabla_\xi\nabla_\xi X, Y) + \mathrm{g}(\nabla_\xi X, \nabla_\xi Y) + \mathrm{g}(\nabla_\xi X, \nabla_\xi Y) + \mathrm{g}(X, \nabla_\xi\nabla_\xi Y).\end{aligned} \tag{3.15}$$

Put $X = Y = \xi$ then, by combining (3.13) and (3.14) and $\nabla_\xi\xi = \mathcal{L}_\xi\xi = 0$, we have

$$(\mathcal{L}_\xi(\mathcal{L}_\xi\mathrm{g}))(\xi,\xi) = \mathcal{L}\mathrm{g}(\xi,\xi) = 0, \tag{3.16}$$

Taking into account $X = Y = \xi$ in (2.4) and applying (3.16), we deduce

$$Ric(\xi,\xi) = \mu g(\xi,\xi). \tag{3.17}$$

Therefore, from (2.34) and (3.17), we get

$$\mu = -2n\alpha^2. \tag{3.18}$$

□

Theorem 3.4. *Consider an $N(k)$ CM manifold M that supports a HRS where V is both concircular and orthogonal to ξ. $(M, \mathrm{g}, V, \lambda, \mu)$ is steady iff M is locally isometric to $E^{n+1} \times S^4$ for any $n > 1$, and it is flat when $n = 1$.*

Proof. Using the definition of $\mathcal{L}$ and the property $\nabla_\xi \xi = 0$, it follows that

$$(\mathcal{L}_V \mathrm{g})(\xi, \xi) = 2\mathrm{g}(\nabla_\xi V, \xi) = 2\nabla_\xi(\mathrm{g}(V, \xi)) = 0. \tag{3.19}$$

As on M, V constitutes a concircular vector field

$$\nabla_X V = fX \tag{3.20}$$

for each X in $\Gamma(TM)$. By applying (3.20) together with the definition of $\mathcal{L}$, one can get

$$\begin{aligned}(\mathcal{L}_V \mathrm{g})(X, Y) &= 2f\mathrm{g}(X, Y),\\ (\mathcal{L}_V \mathrm{g})(\xi, \xi) &= 2f\mathrm{g}(\xi, \xi) = 2f.\end{aligned} \tag{3.21}$$

From (3.19) and (3.21) we have $f = 0$. Further,

$$(\mathcal{L}_V(\mathcal{L}_V \mathrm{g}))(X, Y) = V.\mathcal{L}_V \mathrm{g}(X, Y) - \mathcal{L}_V \mathrm{g}(\mathcal{L}_V X, Y) - \mathcal{L}_V \mathrm{g}(X, \mathcal{L}_V Y), \tag{3.22}$$

then $\mathcal{L}_V \mathrm{g} = (\mathcal{L}_V(\mathcal{L}_V \mathrm{g})) = 0$. Using (2.4) and replacing $X = Y = \xi$ in (2.21) results,

$$\mu = 2nk, \tag{3.23}$$

which concludes the proof. □

Remark 3.5. *The previous theorem remains true when V is a conformal vector field perpendicular to ξ.*

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