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Intrinsic Holmes-Thompson volumes and rigidity in Weil bundles

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Abstract. This paper develops a framework for defining intrinsic volumes on manifolds M by leveraging the structure of Weil bundles $M^{\mathbf{A}}$ associated with Weil algebras $\mathbf{A}$. We explore constructions for a Finsler-like structure $\mathcal{F}_{\mathbf{A}}$ primarily on the fibers of $M^{\mathbf{A}}$, aiming to derive it from the algebraic properties of $\mathbf{A}$ with minimal reliance on auxiliary metrics on M. The concept of $\mathbf{A}$-naturality is introduced to formalize the intrinsic nature of such structures. From this fiberwise $\mathcal{F}_{\mathbf{A}}$, an effective Finsler structure F_M on the tangent bundle TM is derived. The Busemann-Hausdorff measure dVol_{F_M} associated with F_M then provides a volume form on M. We establish foundational results concerning conditions under which a diffeomorphism $\phi : M \to M$ preserves dVol_{F_M}, linking this to the behavior of its prolongation $\phi^{\mathbf{A}}$ and exploring resulting rigidity phenomena, including a characterization theorem for dVol_{F_M} under affine symmetries. Furthermore, we propose several significant conjectures and future research directions concerning infinitesimal symmetries, axiomatic uniqueness of these volumes, interactions with curvature, sub-Riemannian limits, and holonomy restrictions.

Keywords: Weil Bundles, Finsler Geometry, Holmes-Thompson Volume, Geometric Rigidity, Affine Transformations.

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1. **Introduction**

The Holmes-Thompson volume offers a canonical approach to defining volume in Finsler geometry, intrinsically tied to a given Finsler structure $F : TM \to \mathbb{R}_{\geq 0}$ on a manifold M [6, 1]. This work seeks to extend and adapt these ideas to the richer context of Weil bundles $M^{\mathbf{A}}$ over M, associated with a Weil algebra $\mathbf{A}$ [2]. Our primary goal is to investigate how the algebraic structure of $\mathbf{A}$ can inform the geometry of M itself, particularly through the definition of volume forms.

2. **Preliminaries on Weil Bundles and Fibers**

Our primary goal is to define a Finsler-like structure (a norm) $\mathcal{F}_{\mathbf{A}} : V^{\mathbf{A}} \to \mathbb{R}_{\geq 0}$ on the fiber vector space $V^{\mathbf{A}} \cong \mathfrak{A}^n$ [2]. We aim for this norm to be "intrinsic," meaning it should be constructed primarily from the algebraic properties of $\mathbf{A}$ and functorial principles, ideally without direct reference to an auxiliary Riemannian metric g on M. The argument $x \in M$ in $\mathcal{F}_{\mathbf{A}}(x, v)$ is often suppressed when defining the structure on the fiber $V^{\mathbf{A}}$, as this part of the construction is intended to be independent of the base point.

Definition 2.1. [2] *A Weil algebra* $\mathbf{A}$ *is a finite-dimensional, commutative, associative, unital* $\mathbb{R}$*-algebra of the form* $\mathbf{A} = \mathbb{R} \oplus \mathfrak{A}$*, where* $\mathfrak{A}$ *is a maximal ideal satisfying* $\mathfrak{A}^{k+1} = 0$ *for some* $k \geq 1$*.*

Definition 2.2. [2] *Let* M *be a smooth manifold and* $\mathbf{A}$ *a Weil algebra. An infinitely near point to* $x \in M$ *of kind* $\mathbf{A}$ *is a smooth morphism of* $\mathbb{R}$*-algebras* $\phi : C^\infty(M) \to \mathbf{A}$ *such that the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{R} \oplus \mathfrak{A} \arrow[dr, "pr_{\mathbb{R}}"] & \\ C^\infty(M) \arrow[ur, "\phi"] \arrow[rr, "ev_x"] & & \mathbb{R}, \end{tikzcd}$$

where ev_x *is the evaluation map, defined by* $ev_x(f) := f(x)$*, and* $pr_{\mathbb{R}}$ *is the projection onto the real part of* $\mathbf{A}$*.*

We can construct the Weil bundle $M^{\mathbf{A}}$, which consists of all infinitely near points to M of kind $\mathbf{A}$. There exists a natural projection map $\pi_{\mathbf{A}} : M^{\mathbf{A}} \to M$, mapping each infinitely near point to its base point. The triple $(M^{\mathbf{A}}, \pi_{\mathbf{A}}, M)$ equipped with the bundle topology, is known as the bundle of $\mathbf{A}$-points near to points in M [2, 5].

2.1. **Intrinsic norms $N_{\mathfrak{A}}$ on the ideal $\mathfrak{A}$.**

Definition 2.3. *A norm* $N_{\mathfrak{A}}$ *is admissible if:*

(1) $\mathrm{Aut}_{gr}(\mathbf{A})$*-invariant.*
(2) *Compatible with filtration:* $N_{\mathfrak{A}}(\alpha) \geq c_s \|\alpha^{(s)}\|$*.*

(3) *Polynomial growth:* $N_{\mathfrak{A}}(t\alpha) \sim |t| N_{\mathfrak{A}}(\alpha)$ *as* $t \to 0$.

We first focus on defining a norm $N_{\mathfrak{A}} : \mathfrak{A} \to \mathbb{R}_{\geq 0}$ on the maximal ideal $\mathfrak{A}$ itself, drawing from its algebraic structure. Assume $\mathfrak{A}$ has a natural grading (or admits one through a canonical choice, e.g., from a presentation of $\mathbf{A}$): $\mathfrak{A} = \bigoplus_{s=1}^{k} \mathfrak{A}_s$, where $\mathfrak{A}_s$ are vector subspaces (the s-th order components), and $\mathfrak{A}^{k+1} = \{0\}$ but $\mathfrak{A}^k \neq \{0\}$. An element $\alpha \in \mathfrak{A}$ decomposes as $\alpha = \sum_{s=1}^{k} \alpha^{(s)}$, where $\alpha^{(s)} \in \mathfrak{A}_s$. We define the norm $N_{\mathfrak{A}}$ on $\mathfrak{A}$ as:

$$N_{\mathfrak{A}}(\alpha) = \left(\sum_{s=1}^{k} \lambda_s \left(N_s(\alpha^{(s)}) \right)^p \right)^{1/p}, \tag{2.1}$$

where $p \geq 1$, $\lambda_s > 0$ are positive weights, chosen to respect symmetries of $\mathbf{A}$ or by convention (e.g., $\lambda_s = 1/(s!)^p$), and $N_s : \mathfrak{A}_s \to \mathbb{R}_{\geq 0}$ is a norm on the finite-dimensional vector space $\mathfrak{A}_s$. For this to be "intrinsic," N_s should be defined canonically. Possibilities include:

- Canonical basis norms: If $\mathfrak{A}_s$ has a canonical basis, N_s can be an L_q-norm of coefficients.
- Norms from bilinear forms: If $\mathbf{A}$ (or $\mathfrak{A}_s$) admits a canonical non-degenerate symmetric bilinear form.
- Representation-Theoretic norms: If $\mathfrak{A}_s$ carries an irreducible representation of a relevant symmetry group (e.g., $\mathrm{Aut}_{gr}(\mathbf{A})$), a G-invariant inner product (unique up to scale) can define N_s.

Example 2.4. *For* $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$*, the maximal ideal is* $\mathfrak{A}_k = \bigoplus_{s=1}^{k} \mathfrak{A}_s$*, with* $\mathfrak{A}_s = \mathrm{span}\{\epsilon^s\}$*. An element* $\alpha \in \mathfrak{A}_k$ *is* $\alpha = \sum_{s=1}^{k} a_s \epsilon^s$*, so* $\alpha^{(s)} = a_s \epsilon^s$*. A canonical norm for* $\mathfrak{A}_s$ *is* $N_s(a_s \epsilon^s) = |a_s|$*. Using factorial weights* $\lambda_s = 1/(s!)^2$ *and* $p = 2$ *(for a quadratic sum):*

$$N_{\mathfrak{A}_k}(\alpha) = \left(\sum_{s=1}^{k} \frac{|a_s|^2}{(s!)^2} \right)^{1/2}.$$

This choice reflects that in jet bundles, the s-th derivative term (represented by ϵ^s*) is often associated with a factor of* $1/s!$ *from Taylor expansions.*

Remark 2.5 (Automorphism Invariance for $N_{\mathfrak{A}_k}$)**.** *Automorphisms of* $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ *that preserve the graded structure* $\mathfrak{A}_k = \bigoplus \mathfrak{A}_s$ *are of the form* $\phi_c(\epsilon) = c\epsilon$ *for* $c \in \mathbb{R}, c \neq 0$*. Under such an automorphism,*

$$\alpha = \sum a_s \epsilon^s \mapsto \phi_c(\alpha) = \sum a_s (c\epsilon)^s = \sum (a_s c^s) \epsilon^s.$$

Then,

$$N_s((\phi_c(\alpha))^{(s)}) = N_s(a_s c^s \epsilon^s) = |a_s c^s| = |c|^s |a_s|.$$

Thus, we have

$$N_{\mathfrak{A}_k}(\phi_c(\alpha)) = \left(\sum_{s=1}^{k} \frac{(|c|^s |a_s|)^2}{(s!)^2} \right)^{1/2} = \left(\sum_{s=1}^{k} |c|^{2s} \frac{|a_s|^2}{(s!)^2} \right)^{1/2}.$$

For $N_{\mathfrak{A}_k}$ to be strictly invariant ($N_{\mathfrak{A}_k}(\phi_c(\alpha)) = N_{\mathfrak{A}_k}(\alpha)$), we would generally need $|c| = 1$. If we desire homogeneity like $N_{\mathfrak{A}_k}(\phi_c(\alpha)) = |c|N_{\mathfrak{A}_k}(\alpha)$ (which means it scales like a first-order quantity), this particular factorial weighting does not satisfy it unless $k = 1$. This highlights that specific equivariance/invariance properties depend heavily on the choice of λ_s and N_s.

2.2. **From norm on $\mathfrak{A}$ to norm $\mathcal{F}_{\mathbf{A}}$ on fiber vector space $V_x^{\mathbf{A}} \cong \mathfrak{A}^n$.** Given an intrinsic norm $N_{\mathfrak{A}}$ on the ideal $\mathfrak{A}$, we define $\mathcal{F}_{\mathbf{A}}$ on the fiber vector space $V^{\mathbf{A}} \cong \mathfrak{A}^n$ by combining the norms of its n components. For $v = (\alpha^{(1)}, \dots, \alpha^{(n)}) \in \mathfrak{A}^n$, where each $\alpha^{(j)} \in \mathfrak{A}$, we define:

$$\mathcal{F}_{\mathbf{A}}(v) = \mathcal{F}_{\mathbf{A}}((\alpha^{(1)}, \dots, \alpha^{(n)})) = \left(\sum_{j=1}^{n} (N_{\mathfrak{A}}(\alpha^{(j)}))^q \right)^{1/q}, \tag{2.2}$$

where $q \geq 1$. This $\mathcal{F}_{\mathbf{A}}$ serves as a norm on $V^{\mathbf{A}}$.

Example 2.6. *Using $N_{\mathfrak{A}_2}(a\epsilon + b\epsilon^2) = \sqrt{|a|^2 + \frac{1}{4}|b|^2}$ from Example 2.4 (with $k = 2, \lambda_1 = 1, \lambda_2 = 1/4, p = 2$). An element $v \in V^{\mathbf{A}_2} \cong \mathfrak{A}_2^n$ is $v = (\alpha^{(1)}, \dots, \alpha^{(n)})$, where $\alpha^{(j)} = a_j\epsilon + b_j\epsilon^2$. With $q = 2$ in Eq. (2.2), we obtain:*

$$\mathcal{F}_{\mathbf{A}}(v) = \sqrt{\sum_{j=1}^{n} (N_{\mathfrak{A}_2}(a_j\epsilon + b_j\epsilon^2))^2} = \sqrt{\sum_{j=1}^{n} \left(|a_j|^2 + \frac{1}{4}|b_j|^2 \right)}.$$

This $\mathcal{F}_{\mathbf{A}}(v)$ defines a norm on the $2n$-dimensional vector space $\mathfrak{A}_2^n$.

2.3. **Fiber norms $\mathcal{F}_{\mathbf{A}}$ using a base metric g (Alternative Constructions).** Alternatively, for specific applications where $M^{\mathbf{A}}$ is identified with a standard geometric object like a jet bundle of curves, a base Riemannian metric g on M can be used to define $\mathcal{F}_{\mathbf{A}}$ on $V_x^{\mathbf{A}}$.

2.3.1. **Component-wise definitions for Jet bundles of curves.** Assume $M^{\mathbf{A}_k}$ (for $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$) is identified with $J_0^k(\mathbb{R}, M)_x$, the bundle of k-jets of curves $\gamma : (\mathbb{R}, 0) \to (M, x)$, whose geometry is detailed in [4]. The fiber vector space $V_x^{\mathbf{A}_k}$ is then identified with $\bigoplus_{s=1}^k T_xM$. An element is a tuple of tangent vectors $(v_1, \dots, v_k)$, where $v_s \in T_xM$ represents the (appropriately scaled) s-th derivative component (e.g., $v_s = \frac{1}{s!}(\nabla_{\dot\gamma})^{s-1}\dot\gamma|_{t=0}$). The metric g_x induces a norm $\|v_s\|_{g_x} = \sqrt{g_x(v_s, v_s)}$. We then define our norm, a construction related to other Finsler structures on higher-order tangent bundles [3], as a kind of twisted Sasaki-type metric:

$$\mathcal{F}_{\mathbf{A}_k}(x, (v_s)_{s=1}^k) = \left(\sum_{s=1}^{k} \omega_s \|v_s\|_{g_x}^2 \right)^{1/2}, \tag{2.3}$$

where $\omega_s > 0$ are weights. For factorial weighting, commonly $\omega_s = 1/(s!)^2$, the above definition becomes:

$$\mathcal{F}_{\mathbf{A}_k}(x,(v_s)_{s=1}^k) = \left(\sum_{s=1}^{k} \frac{1}{(s!)^2}\|v_s\|_{g_x}^2\right)^{1/2}. \tag{2.4}$$

Example 2.7 ($M^{\mathbf{A}_2} \cong J_0^2(\mathbb{R},M)_x$)**.** *For $\mathbf{A}_2 = \mathbb{R}[\epsilon]/(\epsilon^3)$, $V_x^{\mathbf{A}_2} \cong T_xM \oplus T_xM$. An element is (v,a) (velocity $v_1 = v$, acceleration $v_2 = a$). Using Eq. (2.4) for $\mathcal{F}_{\mathbf{A}_2}$, we obtain:*

$$\mathcal{F}_{\mathbf{A}_2}(x,(v,a)) = \sqrt{\frac{\|v\|_{g_x}^2}{(1!)^2} + \frac{\|a\|_{g_x}^2}{(2!)^2}} = \sqrt{\|v\|_{g_x}^2 + \frac{1}{4}\|a\|_{g_x}^2}. \tag{2.5}$$

Example 2.8 (Example: Coupled pair algebra ($\mathbf{A}_{\text{pair}}$))**.** *For $\mathbf{A}_{pair} = \mathbb{R}[\epsilon_1,\epsilon_2]/(\epsilon_1^2,\epsilon_2^2,\epsilon_1\epsilon_2 = 0)$, and $(V_x^{\mathbf{A}})_{pair} \cong T_xM \oplus T_xM$, we have:*

$$\mathcal{F}_{\mathbf{A}_{pair}}(x,(v_1,v_2)) = \sqrt{g_x(v_1,v_1) + g_x(v_2,v_2) + 2\kappa g_x(v_1,v_2)}, \tag{2.6}$$

where $|\kappa| < 1$ ensures positive definiteness.

3. Volume forms on M and $M^{\mathbf{A}}$

3.1. **Effective Finsler structure F_M on TM.** To derive an effective Finsler structure $F_M : TM \to \mathbb{R}_{\geq 0}$ on the base manifold M from a fiber norm $\mathcal{F}_{\mathbf{A}}$ on $V_x^{\mathbf{A}}$, we project or restrict $\mathcal{F}_{\mathbf{A}}$ to its first-order component.

Step 1: Identify first-order component. Assume the graded Weil algebra $\mathbf{A} = \mathbb{R} \oplus \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_k$ induces a decomposition of the fiber vector space $V_x^{\mathbf{A}} \cong \mathfrak{A}_1^n \oplus \cdots \oplus \mathfrak{A}_k^n$. The "first-order part" corresponds to $\mathfrak{A}_1^n$. If $\mathfrak{A}_1$ is identifiable with $\mathbb{R}$ (e.g., $\mathfrak{A}_1 = \text{span}\{\epsilon\}$ for $\mathbf{A}_k$), then $\mathfrak{A}_1^n \cong \mathbb{R}^n \cong T_xM$. Let this identification be $j_1 : T_xM \to \mathfrak{A}_1^n$.

Step 2: Restrict $\mathcal{F}_{\mathbf{A}}$. For $v \in T_xM$, let $\tilde{v} \in V_x^{\mathbf{A}}$ be the element whose $\mathfrak{A}_1^n$ component is $j_1(v)$ and all higher-order components ($\mathfrak{A}_s^n$ for $s \geq 2$) are zero. The effective Finsler structure F_M on TM is defined by:

$$F_M(x,v) = \mathcal{F}_{\mathbf{A}}(x,\tilde{v}). \tag{3.1}$$

For example, if $\mathcal{F}_{\mathbf{A}}$ is from Eq. (2.4) (assuming $M^{\mathbf{A}_k} \cong J_0^k(\mathbb{R},M)_x$ and $v_s \in T_xM$): Setting $v_1 = v$ and $v_s = 0$ for $s \geq 2$:

$$F_M(x,v) = \left(\frac{1}{(1!)^2}\|v\|_{g_x}^2\right)^{1/2} = \|v\|_{g_x}.$$

Here, F_M is the Riemannian norm from g. If $\mathcal{F}_{\mathbf{A}}$ was from Eq. (2.2) (intrinsic): Assuming $T_xM \cong \mathfrak{A}_1^n$ via $v = (v^1,\ldots,v^n)$ in coordinates, with $\mathfrak{A}_1 = \text{span}\{\epsilon\}$. Then $\tilde{v} = (v^1\epsilon,\ldots,v^n\epsilon)$.

$$F_M(x,v) = \left(\sum_{j=1}^{n}(N_{\mathfrak{A}}(v^j\epsilon))^q\right)^{1/q}.$$

Assuming $N_{\mathfrak{A}}(\alpha^{(1)}) = \sqrt{\lambda_1} N_1(\alpha^{(1)})$ and $N_1(v^j\epsilon) = |v^j|$, then

$$N_{\mathfrak{A}}(v^j\epsilon) = \sqrt{\lambda_1}|v^j|.$$

So

$$F_M(x,v) = \left(\sum_{j=1}^{n}(\sqrt{\lambda_1}|v^j|)^q\right)^{1/q} = \lambda_1^{1/2}\left(\sum_{j=1}^{n}|v^j|^q\right)^{1/q}.$$

This is an L_q-norm on T_xM (up to scale). If $q = 2$, it is a Euclidean norm.

3.2. **Busemann-Hausdorff dVol_{F_M} on M.** Given the effective Finsler structure $F_M : TM \to \mathbb{R}_{\geq 0}$ on M, the Busemann-Hausdorff measure dVol_{F_M} on M is defined by:

$$\mathrm{dVol}_{F_M}(x) := \frac{\mathrm{Vol}_{\mathrm{Eucl}}(B_x^{F_M})}{c_n} dV_{\mathrm{coord}}(x), \tag{3.2}$$

where

$$B_x^{F_M} := \Big\{ y \in T_xM \mid F_M(x,y) < 1 \Big\}$$

is the unit Finsler ball in T_xM, $dV_{\mathrm{coord}}(x) := dx^1 \wedge \cdots \wedge dx^n$ is a coordinate volume element, and $c_n := \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the Euclidean unit n-ball.

If $F_M(x,y) = \sqrt{g_x(y,y)}$ for a Riemannian metric g, then

$$\mathrm{dVol}_{F_M}(x) = \sqrt{\det(g_{ij}(x))}dx^1 \wedge \cdots \wedge dx^n = dV_g(x).$$

3.3. **Holmes-Thompson Volume $\mu_{HT}^{M^{\mathbf{A}}}$ on $M^{\mathbf{A}}$.** In order to define a Holmes-Thompson volume directly on the Weil bundle $M^{\mathbf{A}}$, we first need a Finsler structure $F_{TM^{\mathbf{A}}} : TM^{\mathbf{A}} \to \mathbb{R}_{\geq 0}$ on the tangent bundle of $M^{\mathbf{A}}$. Let g be a Riemannian metric on M, and let $\mathcal{F}_{\mathbf{A}}$ be a norm on the fiber vector space $V^{\mathbf{A}} \cong \mathfrak{A}^n$ (e.g., from Eq. (2.2) or from Sec. 2.3). Motivated by the work done in [5], we define $F_{TM^{\mathbf{A}}}$ on $TM^{\mathbf{A}}$ by combining g and $\mathcal{F}_{\mathbf{A}}$. For $\zeta \in M^{\mathbf{A}}$ and $\Xi \in T_\zeta M^{\mathbf{A}}$:

$$F_{TM^{\mathbf{A}}}(\zeta,\Xi)^2 = g_{\pi_{\mathbf{A}}(\zeta)}((\pi_{\mathbf{A}})_*\Xi, (\pi_{\mathbf{A}})_*\Xi) + \mathcal{F}_{\mathbf{A}}(\mathrm{ver}(\Xi))^2, \tag{3.3}$$

where $\mathrm{ver}(\Xi) \in V^{\mathbf{A}}_{\pi_{\mathbf{A}}(\zeta)}$ is the vertical component of Ξ (requiring a connection or local bundle coordinates for unique definition). The dual unit co-ball bundle $B^* \subset T^*M^{\mathbf{A}}$ is defined as:

$$B^* = \Big\{ \alpha \in T^*M^{\mathbf{A}} : F^*_{TM^{\mathbf{A}}}(\alpha) \leq 1 \Big\},$$

where $F^*_{TM^{\mathbf{A}}}$ is the dual norm of $F_{TM^{\mathbf{A}}}$. Therefore, the Holmes-Thompson measure $\mu_{HT}^{M^{\mathbf{A}}}$ on $M^{\mathbf{A}}$ is defined by:

$$\mu_{HT}^{M^{\mathbf{A}}}(U) := \frac{1}{\kappa_{d_{M^{\mathbf{A}}}}} \int_{B^* \cap \pi^{-1}_{T^*M^{\mathbf{A}}}(U)} (\omega_{M^{\mathbf{A}}})^{\wedge d_{M^{\mathbf{A}}}}, \tag{3.4}$$

for $U \subseteq M^{\mathbf{A}}$ where

$$d_{M^{\mathbf{A}}} := \dim M^{\mathbf{A}} = n \cdot \dim \mathbf{A}$$

$\pi_{T^*M^{\mathbf{A}}} : T^*M^{\mathbf{A}} \to M^{\mathbf{A}}$ is the projection, $\omega_{M^{\mathbf{A}}} := d\theta_{M^{\mathbf{A}}}$ is the standard symplectic form on $T^*M^{\mathbf{A}}$ with $\theta_{M^{\mathbf{A}}}$ the tautological 1−form, and

$$\kappa_{d_{M^{\mathbf{A}}}} := \frac{\pi^{d_{M^{\mathbf{A}}}/2}}{\Gamma(d_{M^{\mathbf{A}}}/2+1)}$$

is the volume of the Euclidean unit ball in $\mathbb{R}^{d_{M^{\mathbf{A}}}}$. Note that in Eq.(3.4), the notation $(\omega_{M^{\mathbf{A}}})^{\wedge d_{M^{\mathbf{A}}}}$ stands for $\underbrace{\omega_{M^{\mathbf{A}}} \wedge \omega_{M^{\mathbf{A}}} \wedge \cdots \wedge \omega_{M^{\mathbf{A}}}}_{d_{M^{\mathbf{A}}}-\text{times}}$.

4. A-Naturality and Symmetries

4.1. Definition of A-Naturality.

Definition 4.1 ($\mathbf{A}$-Naturality)**.** *A structure S (e.g., $\mathcal{F}_{\mathbf{A}}$ or dVol_{F_M}) is $\mathbf{A}$-natural if:*

1. *Functoriality: For any morphism of Weil algebras $\rho : \mathbf{A} \to \mathbf{B}$, the structure S is compatible with the induced bundle map $\rho_\natural : M^{\mathbf{A}} \to M^{\mathbf{B}}$. This means S (or an induced structure S' on $M^{\mathbf{B}}$) is related to $(\rho_\natural)^* S'$ in an appropriate sense.*
2. *Equivariance: For any automorphism $\phi \in \mathrm{Aut}(\mathbf{A})$, the structure S is equivariant with respect to the induced bundle automorphism $\phi_\# : M^{\mathbf{A}} \to M^{\mathbf{A}}$, meaning $S \circ \phi_\# = f(\phi) S$ for some scalar function $f(\phi)$.*
3. *Algebraic Consistency: S respects the graded structure of the ideal $\mathfrak{A} = \bigoplus \mathfrak{A}_s$, meaning its definition explicitly utilizes this grading (e.g., via weights like $1/s!$).*
4. *Localizability: S is locally defined, depending only on the Weil algebra $\mathbf{A}$ and the local differential structure of M.*

4.2. Examples of A-Natural Structures.

Example 4.2 (Functoriality Failure for Non-Factorial Weights)**.** *Using $\lambda_s = 1$ in $N_{\mathfrak{A}_k}$ for $\mathbf{A}_k$, the map $\rho : \mathbf{A}_2 \to \mathbf{A}_1$ gives:*

$$N_{\mathfrak{A}_2}(a\epsilon + b\epsilon^2) = \sqrt{a^2+b^2}, \quad \rho_\natural(a\epsilon + b\epsilon^2) = a\epsilon.$$

Then $N_{\mathfrak{A}_1}(\rho_\natural(a\epsilon + b\epsilon^2)) = N_{\mathfrak{A}_1}(a\epsilon) = |a|$. Compatibility would require $N_{\mathfrak{A}_2}(a\epsilon + b\epsilon^2)$ restricted to $b = 0$ to match $N_{\mathfrak{A}_1}(a\epsilon)$, which is $|a| = \sqrt{a^2}$: This holds. However, general functoriality

$$N_{\mathfrak{A}_1}(\rho_\natural(\alpha)) \stackrel{?}{=} \textit{something related to } N_{\mathfrak{A}_2}(\alpha),$$

is more complex. The original point was that if F_M were defined differently, functoriality might fail. Factorial weights often ensure consistency when operations like derivatives or compositions are involved.

4.3. **Illustrating A-Naturality Axioms.** We illustrate Functoriality and Equivariance using examples, assuming $\mathcal{F}_{\mathbf{A}}$ on fibers is typically derived using a base metric g on M for these illustrations.

Example 4.3 (Functoriality for $\rho : \mathbf{A}_2 \to \mathbf{A}_1$)**.** *Let $\mathbf{A}_2 = \mathbb{R}[\epsilon]/(\epsilon^2)$, $\mathbf{A}_1 = \mathbb{R}$, and $\rho(\epsilon) = 0$. Then $\rho^\natural : M^{\mathbf{A}_2} \cong TM \to M^{\mathbf{A}_1} \cong M$ is $(x, v) \mapsto x$. Set $\mathcal{F}_{\mathbf{A}_2}(x, v) = \|v\|_{g_x}$. The target structure $\mathcal{F}_{\mathbf{A}_1}$ on $M^{\mathbf{A}_1} \cong M$ would be a function $f : M \to \mathbb{R}_{\geq 0}$, e.g., $f(x) = 0$ or $f(x) = 1$. If $\mathcal{F}_{\mathbf{A}_1}(x) = 0$, then $(\rho^\natural)^*\mathcal{F}_{\mathbf{A}_1}(x, v) = \mathcal{F}_{\mathbf{A}_1}(\rho^\natural(x, v)) = \mathcal{F}_{\mathbf{A}_1}(x) = 0$. Compatibility means $\mathcal{F}_{\mathbf{A}_2}(x, v)$ should relate to $\mathcal{F}_{\mathbf{A}_1}(\rho^\natural(x, v))$. If we set $v = 0$, then $\mathcal{F}_{\mathbf{A}_2}(x, 0) = 0$, which matches $\mathcal{F}_{\mathbf{A}_1}(x) = 0$. This example shows how the structure collapses when the fiber is trivialized by the map.*

Example 4.4 (Equivariance for $\mathbf{A}_2$)**.** *Let $\varphi(\epsilon) = c\epsilon$ ($c \neq 0$) be an automorphism of $\mathbf{A}_2 := \mathbb{R}[\epsilon]/(\epsilon^2)$. This induces $\varphi_\# : TM \to TM$, $(x, v) \mapsto (x, cv)$. If $\mathcal{F}_{\mathbf{A}_2}(x, v) = \|v\|_{g_x}$, then*

$$\mathcal{F}_{\mathbf{A}_2}(\varphi_\#(x, v)) = \mathcal{F}_{\mathbf{A}_2}(x, cv) = |c|\|v\|_{g_x}.$$

Preservation: $\mathcal{F}_{\mathbf{A}_2}(\varphi_\#(x, v)) = \mathcal{F}_{\mathbf{A}_2}(x, v)$ forces $|c| = 1$.

Example 4.5 (Functoriality for $\rho : \mathbf{A}_3 \to \mathbf{A}_2$)**.** *Let $\mathbf{A}_3 = \mathbb{R}[\epsilon]/(\epsilon^3)$, $\mathbf{A}_2 = \mathbb{R}[\epsilon]/(\epsilon^2)$, and $\rho : \mathbf{A}_3 \to \mathbf{A}_2$ be $\epsilon \mapsto \epsilon \pmod{\epsilon^2}$. Identify $M^{\mathbf{A}_3} \cong J^2_0(\mathbb{R}, M)_x$ (elements (x, v, a)) and $M^{\mathbf{A}_2} \cong TM$ (elements (x, v)): So, $\rho^\natural(x, v, a) = (x, v)$. Let $\mathcal{F}_{\mathbf{A}_3}(x, v, a) = \sqrt{\|v\|^2_{g_x} + \frac{1}{4}\|a\|^2_{g_x}}$ and $\mathcal{F}_{\mathbf{A}_2}(x, v) = \|v\|_{g_x}$.*
Then $\mathcal{F}_{\mathbf{A}_3}(x, v, 0) = \|v\|_{g_x}$. Also, $\mathcal{F}_{\mathbf{A}_2}(\rho^\natural(x, v, a)) = \mathcal{F}_{\mathbf{A}_2}(x, v) = \|v\|_{g_x}$. The compatibility condition for functoriality often looks like

$$\mathcal{F}_{\mathbf{A}_2}(\rho^\natural(jet_k)) = \mathcal{F}_{\mathbf{A}_3}(jet_k \textit{ restricted to } \ker\rho \textit{ or projected appropriately}).$$

Here, $\mathcal{F}_{\mathbf{A}_3}(x, v, 0) = \mathcal{F}_{\mathbf{A}_2}(\rho^\natural(x, v, 0))$ shows compatibility for jets with zero acceleration.

4.4. **Naturality of volume forms under prolongations.** If $\phi : M \to M$ is a diffeomorphism, its prolongation $\phi^{\mathbf{A}} : M^{\mathbf{A}} \to M^{\mathbf{A}}$ is a diffeomorphism. If $F_{TM^{\mathbf{A}}}$ (Eq. (3.3)) is used to define $\mu^{M^{\mathbf{A}}}_{HT}$ (Eq. (3.4)), and if ϕ is an isometry of g, and if the fiber norm $\mathcal{F}_{\mathbf{A}}$ used in $F_{TM^{\mathbf{A}}}$ is $\mathrm{Aut}(\mathbf{A})$-invariant and satisfies $\mathcal{F}_{\mathbf{A}}(T\phi^{\mathbf{A}}(\mathrm{ver}(\Xi))) = \mathcal{F}_{\mathbf{A}}(\mathrm{ver}(\Xi))$ for all $\Xi \in TM^{\mathbf{A}}$, then $T\phi^{\mathbf{A}}$ preserves $F_{TM^{\mathbf{A}}}$. This implies $(\phi^{\mathbf{A}})^*$ preserves B^* and $\omega_{M^{\mathbf{A}}}$, making $\mu^{M^{\mathbf{A}}}_{HT}$ invariant under such $\phi^{\mathbf{A}}$.

Proposition 4.6 (Invariance of $\mu^{M^{\mathbf{A}}}_{HT}$)**.** *Let (M, g) be a Riemannian manifold, $\mathbf{A}$ a Weil algebra, and $\mathcal{F}_{\mathbf{A}}$ a fiber norm on $V^{\mathbf{A}}$ that is $\mathrm{Aut}(\mathbf{A})$-invariant. If $\phi : M \to M$ is an isometry of g and its prolongation $\phi^{\mathbf{A}} : M^{\mathbf{A}} \to M^{\mathbf{A}}$ satisfies:*

$$\mathcal{F}_{\mathbf{A}}((T\phi^{\mathbf{A}})(\mathrm{ver}(\Xi))) = \mathcal{F}_{\mathbf{A}}(\mathrm{ver}(\Xi)) \quad \forall \Xi \in TM^{\mathbf{A}},$$

then the Holmes-Thompson volume $\mu_{HT}^{M^{\mathbf{A}}}$ is invariant under $\phi^{\mathbf{A}}$, i.e.,

$$(\phi^{\mathbf{A}})^*\mu_{HT}^{M^{\mathbf{A}}} = \mu_{HT}^{M^{\mathbf{A}}}.$$

Proof. Since ϕ is an isometry, it preserves the horizontal part:

$$g_{\pi_{\mathbf{A}}(\zeta)}((\pi_{\mathbf{A}})_*\Xi, (\pi_{\mathbf{A}})_*\Xi) = g_{\pi_{\mathbf{A}}(\phi^{\mathbf{A}}(\zeta))}((\pi_{\mathbf{A}})_*(T\phi^{\mathbf{A}}(\Xi)), (\pi_{\mathbf{A}})_*(T\phi^{\mathbf{A}}(\Xi))).$$

The condition on $\mathcal{F}_{\mathbf{A}}$ ensures preservation of the vertical part. Thus,

$$F_{TM^{\mathbf{A}}}(\zeta, \Xi) = F_{TM^{\mathbf{A}}}(\phi^{\mathbf{A}}(\zeta), T\phi^{\mathbf{A}}(\Xi)),$$

for all $(\zeta, \Xi) \in TM^{\mathbf{A}}$. This implies that the dual norm $F^*_{TM^{\mathbf{A}}}$ is preserved, so $(T\phi^{\mathbf{A}})^*$ maps B^* to itself. Since $\phi^{\mathbf{A}}$ is a diffeomorphism derived from ϕ, then $\phi^{\mathbf{A}}$ preserves $\theta_{M^{\mathbf{A}}}$ (if $\phi^{\mathbf{A}}$ is a symplectomorphism for $\omega_{M^{\mathbf{A}}}$, which holds for canonical lifts) and hence $\omega_{M^{\mathbf{A}}} = d\theta_{M^{\mathbf{A}}}$. The result follows from the definition of $\mu_{HT}^{M^{\mathbf{A}}}$. □

5. Rigidity and Characterization Theorems

Proposition 5.1 (Isometry from F_M-Preservation)**.** *Let (M, g) be a Riemannian manifold. Let $F_M : TM \to \mathbb{R}_{\geq 0}$ be the Finsler structure on M defined by $F_M(x, y) = \|y\|_{g_x} = \sqrt{g_x(y, y)}$ (i.e., the Riemannian norm). If a diffeomorphism $\phi : M \to M$ preserves F_M (i.e., $F_M(x, y) = F_M(\phi(x), T\phi_x(y))$ for all $(x, y) \in TM$), then ϕ is an isometry of (M, g).*

Proof. The condition $F_M(x, y) = F_M(\phi(x), T\phi_x(y))$ means

$$\sqrt{g_x(y, y)} = \sqrt{g_{\phi(x)}(T\phi_x(y), T\phi_x(y))}.$$

Squaring gives $g_x(y, y) = g_{\phi(x)}(T\phi_x(y), T\phi_x(y))$, which is $(\phi^*g)_x(y, y) = g_x(y, y)$. By polarization, $\phi^*g = g$. □

Theorem 5.2 (Rigidity for $\mathbf{A}_2$-Structures on Jet Bundles of Curves)**.** *Let (M, g) be a Riemannian manifold. Identify $M^{\mathbf{A}_2}$ with $J^2_0(\mathbb{R}, M)_x$ (2-jets of curves), where $\mathbf{A}_2 = \mathbb{R}[\epsilon]/(\epsilon^3)$. Let $\mathcal{F}_{\mathbf{A}_2}$ be the fiber norm on $V_x^{\mathbf{A}_2} \cong T_xM \oplus T_xM$ defined by:*

$$(\mathcal{F}_{\mathbf{A}_2}(x, v, a))^2 = \|v\|^2_{g_x} + \omega_2\|a\|^2_{g_x},$$

where (v, a) are velocity and covariant acceleration components, and $\omega_2 > 0$ is a constant. If a diffeomorphism $\phi : M \to M$ is such that its prolongation $\phi^{\mathbf{A}_2} : M^{\mathbf{A}_2} \to M^{\mathbf{A}_2}$ preserves this fiber norm $\mathcal{F}_{\mathbf{A}_2}$ (i.e., $\mathcal{F}_{\mathbf{A}_2}(x, v, a) = \mathcal{F}_{\mathbf{A}_2}(\phi^{\mathbf{A}_2}(j^2_0\gamma(0)))$ for $j^2_0\gamma(0) = (x, v, a)$), then ϕ is an affine isometry of (M, g).

Proof. The prolongation $\phi^{\mathbf{A}_2}$ maps $j^2_0\gamma = (x, v, a)$ to $j^2_0(\phi \circ \gamma) = (\phi(x), v', a')$, where:

$$v' = T\phi_x(v), \quad a' = (T\phi)_x(a) + (\nabla T\phi)(v, v),$$

with $(\nabla T\phi)$ the second fundamental form / Hessian of ϕ. Preservation:

$$(\mathcal{F}_{\mathbf{A}_2}(x, v, a))^2 = (\mathcal{F}_{\mathbf{A}_2}(\phi(x), v', a'))^2,$$

implies

$$\|v\|_{g_x}^2 + \omega_2 \|a\|_{g_x}^2 = \|T\phi_x(v)\|_{g_{\phi(x)}}^2 + \omega_2 \|(T\phi)_x(a) + (\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2. \quad (5.1)$$

Setting $a = 0$:

$$\|v\|_{g_x}^2 = \|T\phi_x(v)\|_{g_{\phi(x)}}^2 + \omega_2 \|(\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2.$$

Scale v by $\lambda \in \mathbb{R} \setminus \{0\}$:

$$\|\lambda v\|_{g_x}^2 = \|T\phi_x(\lambda v)\|_{g_{\phi(x)}}^2 + \omega_2 \|(\nabla T\phi)(\lambda v, \lambda v)\|_{g_{\phi(x)}}^2.$$

Since $(\nabla T\phi)$ is bilinear in its arguments, then

$$(\nabla T\phi)(\lambda v, \lambda v) = \lambda^2 (\nabla T\phi)(v,v).$$

Thus

$$\lambda^2 \|v\|_{g_x}^2 = \lambda^2 \|T\phi_x(v)\|_{g_{\phi(x)}}^2 + \omega_2 \lambda^4 \|(\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2.$$

Dividing by λ^2 (for $\lambda \neq 0$):

$$\|v\|_{g_x}^2 = \|T\phi_x(v)\|_{g_{\phi(x)}}^2 + \omega_2 \lambda^2 \|(\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2.$$

This must hold for all $\lambda \neq 0$. Taking the limit $\lambda \to 0$ (or comparing coefficients of powers of λ^2 if viewed as polynomials) shows

$$\|v\|_{g_x}^2 = \|T\phi_x(v)\|_{g_{\phi(x)}}^2.$$

Thus ϕ is an isometry. Substituting this back, the λ^2 term implies

$$\omega_2 \lambda^2 \|(\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2 = 0.$$

Since $\omega_2 > 0$, this forces $\|(\nabla T\phi)(v,v)\|_{g_{\phi(x)}}^2 = 0$ for all v, so $(\nabla T\phi) = 0$. This means ϕ is an affine map, and since it is also an isometry, it is an affine isometry. $\square$

Theorem 5.3 (Characterization of Volume by Affine Symmetries with a Fixed Connection)**.** *Let (M, ∇) be a smooth manifold with a fixed affine connection ∇. Let $\mathrm{Aff}(M, \nabla)$ be the group of affine transformations preserving ∇. Let $\mathcal{F}_{\mathbf{A}}$ be a fiber norm associated with a Weil algebra $\mathbf{A}$, potentially using ∇ in its construction. Let $F_{\mathrm{eff}} : TM \to \mathbb{R}_{\geq 0}$ be the effective Finsler structure derived from $\mathcal{F}_{\mathbf{A}}$, and let dVol_{F_M} be its Busemann–Hausdorff measure on M. Suppose dVol_{F_M} is $\mathrm{Aff}(M, \nabla)$-invariant (i.e. $\psi^* \mathrm{dVol}_{F_M} = \mathrm{dVol}_{F_M}$ for all $\psi \in \mathrm{Aff}(M, \nabla)$). If μ is any smooth volume form on M that is also $\mathrm{Aff}(M, \nabla)$-invariant, then on any connected component of M which is homogeneous under the action of $\mathrm{Aff}(M, \nabla)$, we have*

$$\mu = c \, \mathrm{dVol}_{F_M},$$

for some constant $c \in \mathbb{R}$. In particular, if M is connected and $\mathrm{Aff}(M, \nabla)$-homogeneous, then c is a global constant.

Proof. Denote $G = \mathrm{Aff}(M, \nabla)$. Both μ and dVol_{F_M} are nowhere–vanishing smooth n-forms on M, and satisfy $\psi^*\mu = \mu$, and $\psi^*(\mathrm{dVol}_{F_M}) = \mathrm{dVol}_{F_M} \quad \forall\, \psi \in G$. Fix a connected, G-homogeneous open subset $U \subset M$. On U, define the smooth function

$$h \;=\; \frac{\mu}{\mathrm{dVol}_{F_M}}, \qquad \mu = h\,\mathrm{dVol}_{F_M}.$$

Since μ and dVol_{F_M} are both positive volume forms, $h > 0$ on U. For any $\psi \in G$, pull back the identity $\mu = h\,\mathrm{dVol}_{F_M}$ to obtain:

$$\psi^*\mu = \psi^*(h\,\mathrm{dVol}_{F_M}) = (\psi^*h)\,(\psi^*\mathrm{dVol}_{F_M}).$$

By invariance of μ and dVol_{F_M}, this gives $\mu = (\psi^*h)\,\mathrm{dVol}_{F_M}$. Comparing with $\mu = h\,\mathrm{dVol}_{F_M}$ and cancelling the nonzero dVol_{F_M} shows

$$h = \psi^*h, \quad h(\psi(x)) = h(x),$$

so h is constant on each G-orbit in U. Transitivity of G on U then forces $h \equiv c_U$ for some $c_U > 0$. Consequently,

$$\mu\big|_U = c_U\,\mathrm{dVol}_{F_M}\big|_U.$$

If M itself is connected and G-homogeneous, then $U = M$ and the constant c_U is global. Hence $\mu = c\,\mathrm{dVol}_{F_M}$ on M, completing the proof. □

5.1. **Rigidity via Prolongations.**

Theorem 5.4 (Affine Rigidity for dVol_{F_M})**.** *Let dVol_{F_M} be derived from an $\mathbf{A}$-natural fiber norm $\mathcal{F}_{\mathbf{A}}$. If a diffeomorphism $\phi : M \to M$ satisfies:*

1. *ϕ preserves dVol_{F_M} (i.e., $\phi^*\mathrm{dVol}_{F_M} = \mathrm{dVol}_{F_M}$),*
2. *The prolongation $\phi^{\mathbf{A}} : M^{\mathbf{A}} \to M^{\mathbf{A}}$ preserves $\mathcal{F}_{\mathbf{A}}$,*

then ϕ is an affine transformation with respect to any connection ∇ for which the $\mathbf{A}$-natural construction of $\mathcal{F}_{\mathbf{A}}$ is defined (e.g., if $\mathcal{F}_{\mathbf{A}}$ uses ∇ for covariant derivatives of jets).

Proof. Condition (2), that $\phi^{\mathbf{A}}$ preserves $\mathcal{F}_{\mathbf{A}}$, implies (by an argument similar to Theorem 5.2, if $\mathcal{F}_{\mathbf{A}}$ depends on at least second-order jet components) that ϕ must be an affine transformation, i.e., $(\nabla T\phi) = 0$. Condition (1) is then a consequence or an additional constraint. If $\mathcal{F}_{\mathbf{A}}$ leads to $F_M(x, v) = \|v\|_{g_x}$, then (1) implies ϕ is an isometry of g. Combined, ϕ would be an affine isometry. □

Corollary 5.5 (Uniqueness on Homogeneous Spaces)**.** *If M is $\mathrm{Aff}(M, \nabla)$-homogeneous and connected, then any $\mathrm{Aff}(M, \nabla)$-invariant volume form dVol_{F_M} (derived from an $\mathbf{A}$-natural $\mathcal{F}_{\mathbf{A}}$ that yields an $\mathrm{Aff}(M, \nabla)$-invariant F_M) is unique up to a positive scalar multiple.*

Proof. Let dVF_1 and dVF_2 be two such $\mathrm{Aff}(M, \nabla)$-invariant volume forms. By Theorem 5.3, on the connected, homogeneous manifold M there exists a constant $c > 0$ such that $dVF_1 \;=\; c\,dVF_2$.

Hence any two choices differ by the positive scalar c, proving uniqueness up to scale. $\square$

Example 5.6 (Higher-Order Jets and Volume Invariants)**.** *Consider the Weil algebra $\mathbf{A}_3 = \mathbb{R}[\epsilon]/(\epsilon^4)$, so that its maximal ideal decomposes as $\mathfrak{A}_{\mathbf{A}_3} = \mathrm{span}\{\epsilon, \epsilon^2, \epsilon^3\}$. An element in the fiber of the Weil bundle $M^{\mathbf{A}_3}$ can be identified with a triple (v, a, b), where: v represents the velocity, a the acceleration, and b the jerk (third derivative). A natural extension of the earlier fiber norm is*

$$\mathcal{F}_{\mathbf{A}_3}(x, v, a, b) = \sqrt{\|v\|_{g_x}^2 + \frac{1}{4}\|a\|_{g_x}^2 + \frac{1}{36}\|b\|_{g_x}^2},$$

where the weights 1, $1/(2!)^2$, and $1/(3!)^2$ ensure that the scaling reflects the Taylor expansion. One then defines an effective Finsler structure F_M on TM by appropriate projections from higher-order jet spaces. In this higher-order situation, a diffeomorphism whose prolongation preserves the norm $\mathcal{F}_{\mathbf{A}_3}$ must satisfy stronger rigidity properties. In particular, the preservation of higher-order invariants forces the map to preserve not only the metric but also higher derivatives of the connection; in turn, this could have applications in the study of geometric flows and higher-order variational problems.

Example 5.7 (Coupled Pair Weil Algebra with Cross Terms)**.** *Now consider a non-standard Weil algebra defined by two nilpotent generators:*

$$\mathbf{A}_{pair} = \mathbb{R}[\epsilon_1, \epsilon_2] / \left(\epsilon_1^2,\, \epsilon_2^2,\, \epsilon_1\epsilon_2 = 0\right).$$

In this case, the fiber is $V_x^{\mathbf{A}_{pair}} \cong T_xM \oplus T_xM$. A canonical fiber norm may be defined by

$$\mathcal{F}_{\mathbf{A}_{pair}}(x, (v_1, v_2)) = \sqrt{g_x(v_1, v_1) + g_x(v_2, v_2) + 2\kappa\, g_x(v_1, v_2)},$$

with a coupling parameter κ satisfying $|\kappa| < 1$ to ensure positive definiteness. This example is particularly interesting when modeling situations in which two independent tangential effects are interacting. The invariance properties under the automorphisms of $\mathbf{A}_{pair}$ ensure that the corresponding volume measure is naturally defined and yield rigidity properties analogous to those found in the graded case.

Conjecture 5.8 (Higher-Order Rigidity)**.** *For $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ with $k \geq 3$ and fiber norm:*

$$\mathcal{F}_{\mathbf{A}_k}(x, \{v^{(s)}\}_{s=1}^k) = \left(\sum_{s=1}^k \frac{1}{(s!)^2}\|v^{(s)}\|_{g_x}^2\right)^{1/2},$$

if $\phi^{\mathbf{A}_k}$ preserves $\mathcal{F}_{\mathbf{A}_k}$, then ϕ is affine and $\nabla^m R = 0$ for $1 \leq m \leq k-2$.

Remark 5.9. *The case $k = 2$ (Theorem 5.2) provides strong evidence. For $k = 3$, preliminary calculations show preservation implies $\nabla R = 0$, but full proof requires careful analysis of prolongation's action on third-order jets.*

Conjecture 5.10 (General Weil Algebras)**.** *For arbitrary Weil algebra* $\mathbf{A}$ *with maximal ideal* $\mathfrak{A} = \bigoplus_{s=1}^{k} \mathfrak{A}_s$ *and* $\mathcal{F}_{\mathbf{A}}$ *defined via intrinsic norm* $N_{\mathfrak{A}}$*, if* $\phi^{\mathbf{A}}$ *preserves* $\mathcal{F}_{\mathbf{A}}$*, then* ϕ *preserves the connection used to identify* $V_x^{\mathbf{A}} \cong \mathfrak{A} \otimes T_x M$*.*

6. **Applications and Examples**

6.1. **Illustrative Examples of Weil Algebra Structures.**

6.1.1. *Higher-Order Jets and Volume Invariants (Example 2.4 extension).* Consider the Weil algebra $\mathbf{A}_3 = \mathbb{R}[\epsilon]/(\epsilon^4)$, so that its maximal ideal decomposes as $\mathfrak{A}_{\mathbf{A}_3} = \operatorname{span}\{\epsilon, \epsilon^2, \epsilon^3\}$. An element in the fiber of the Weil bundle $M^{\mathbf{A}_3} \cong J_0^3(\mathbb{R}, M)_x$ can be identified with a triple (v, a, b) of velocity, acceleration, and jerk, relative to a connection ∇. A natural extension of the fiber norm from Example 2.4, if g is a metric on M, is

$$\mathcal{F}_{\mathbf{A}_3}(x, v, a, b) = \sqrt{\|v\|_{g_x}^2 + \frac{1}{(2!)^2}\|a\|_{g_x}^2 + \frac{1}{(3!)^2}\|b\|_{g_x}^2}.$$

One then defines an effective Finsler structure F_M on TM by

$$F_M(x, v) = \mathcal{F}_{\mathbf{A}_3}(x, v, 0, 0) = \|v\|_{g_x}.$$

The resulting dVol_{F_M} is dV_g. However, the interest lies in rigidity: if $\phi^{\mathbf{A}_3}$ preserves $\mathcal{F}_{\mathbf{A}_3}$, then ϕ must preserve g, $\nabla g = 0$, $\nabla R = 0$, etc., up to a certain order, implying ϕ is a higher-order affine map or an isometry with special properties. This has applications in studying symmetries of higher-order variational problems [3].

6.1.2. **Coupled Pair Weil Algebra with Cross Terms.** Consider a non-standard Weil algebra $\mathbf{A}_{\text{pair}} = \mathbb{R}[\epsilon_1, \epsilon_2]/\left(\epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2 = 0\right)$. The fiber is $V_x^{\mathbf{A}_{\text{pair}}} \cong T_x M \oplus T_x M$. A fiber norm (using a metric g) can be defined as in Eq. (2.6):

$$\mathcal{F}_{\mathbf{A}_{\text{pair}},g}(x, (v_1, v_2)) = \sqrt{g_x(v_1, v_1) + g_x(v_2, v_2) + 2\kappa\, g_x(v_1, v_2)}, \quad |\kappa| < 1.$$

The effective Finsler structure F_M might be obtained by setting $v_2 = 0$ (or projecting):

$$F_M(x, v_1) = \sqrt{g_x(v_1, v_1)}.$$

The corresponding dVol_{F_M} is dV_g, and note that rigidity arises if $\phi^{\mathbf{A}_{\text{pair}}}$ preserves $\mathcal{F}_{\mathbf{A}_{\text{pair}},g}$. Automorphisms of $\mathbf{A}_{\text{pair}}$ include swapping ϵ_1, ϵ_2 (if $\kappa = 0$) or scaling $\epsilon_i \mapsto c_i \epsilon_i$. Invariance under these imposes constraints on ϕ.

6.2. **Possible Applications.**

Invariant Volume in Geometric Measure Theory: It is easy to see that, the Busemann-Hausdorff measure dVol_{F_M} derived from an $\mathbf{A}$-natural F_M provides a canonical volume. While F_M often reduces to a known metric (like Riemannian) if $\mathcal{F}_{\mathbf{A}}$ is projected simply, the $\mathbf{A}$-naturality conditions impose constraints on how F_M (and thus dVol_{F_M}) must behave under symmetries of the underlying Weil algebra. If F_M is genuinely novel (not just Riemannian), dVol_{F_M} can be used for isoperimetric inequalities or minimal submanifolds in these new Finsler settings. The sensitivity to higher-order features arises when considering conditions under which dVol_{F_M} is preserved, as these conditions often involve the prolongation of maps and preservation of the full $\mathcal{F}_{\mathbf{A}}$.

Rigidity in Dynamics and Geometric Flows: The rigidity results (e.g., Theorem 5.2 or 5.4) are instrumental. If a dynamical system on M (or a geometric flow) has its evolution described by transformations whose prolongations preserve $\mathcal{F}_{\mathbf{A}}$ (and thus dVol_{F_M} under suitable conditions), then the system possesses significant symmetries (affine, isometric). This can identify invariant manifolds, constrain bifurcations, or classify stable solutions.

Characteristic Classes and Jet Cohomology: Weil bundles $M^{\mathbf{A}}$ are fundamental in constructing generalized characteristic classes (Chern-Weil theory for Weil algebras). An $\mathbf{A}$-natural volume dVol_{F_M}, or more directly the fiber norm $\mathcal{F}_{\mathbf{A}}$, could be used as an integrand or a coefficient in constructing differential forms on M or on $M^{\mathbf{A}}$. These forms might yield new characteristic classes sensitive to the specific algebraic structure of $\mathbf{A}$, potentially distinguishing manifolds beyond standard characteristic classes.

Analyzing Non-Standard Finsler Structures: If the $\mathbf{A}$-natural construction leads to an F_M that is not Riemannian (e.g., if the first-order projection of an intrinsic $\mathcal{F}_{\mathbf{A}}$ results in a non-Euclidean norm on T_xM), this provides a systematic way to generate and study novel Finsler geometries. These could model anisotropic systems in control theory, robotics, or optics. The associated dVol_{F_M} would be the natural volume for such systems.

7. **Further Research Directions and Conjectures**

The framework developed in this paper opens several avenues for further investigation. We outline below a series of significant conjectures and research directions that aim to explore deeper connections between Weil bundle geometry, intrinsic volumes, and geometric rigidity.

7.1. **Conjecture: Infinitesimal Affine Rigidity.**

Conjecture 7.1 (Infinitesimal Affine Rigidity)**.** *Let (M, ∇) be an affine manifold with connection ∇, and let $\mathbf{A}$ be a Weil algebra of order $k = \mathrm{ord}(\mathbf{A}) \geq 2$.*

Let X be a smooth vector field on M, and let $\tilde{X}$ denote its canonical $\mathbf{A}$-prolongation to a vector field on $M^{\mathbf{A}}$. Suppose that $\mathcal{F}_{\mathbf{A}} : V^{\mathbf{A}} \to \mathbb{R}_{\geq 0}$ is a fiber norm on the vertical bundle $V^{\mathbf{A}} \to M^{\mathbf{A}}$ which is smooth on $V^{\mathbf{A}} \setminus \{0\}$. If $\mathcal{L}_{\tilde{X}}\mathcal{F}_{\mathbf{A}} = 0$, and $\mathcal{F}_{\mathbf{A}}$ is sufficiently generic (e.g., it depends non-degenerately on all distinct s-order components for $1 \leq s \leq k$ arising from the grading of $\mathfrak{A}$), then X is an affine vector field, i.e., $\mathcal{L}_X\nabla = 0$.

Motivation and sketch of argument: The condition $\mathcal{L}_{\tilde{X}}\mathcal{F}_{\mathbf{A}} = 0$ implies $\tilde{X}$ preserves the vertical distribution and the fiber norm. For $k \geq 2$, the $\mathbf{A}$-prolongation $\tilde{X}$ involves derivatives of X. Specifically, the action of $\tilde{X}$ on higher-order fiber components (e.g., $s \geq 2$ in a jet bundle context for $\mathbf{A} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$) encodes terms like $\mathcal{L}_X\nabla$. If $\mathcal{F}_{\mathbf{A}}$ depends non-degenerately on these higher-order components, its invariance under $\tilde{X}$ forces these terms related to $\mathcal{L}_X\nabla$ to vanish when evaluated on arbitrary lower-order components. By choosing appropriate fiber elements (e.g., non-zero first-order component, zero higher-order components) and using the genericity of $\mathcal{F}_{\mathbf{A}}$, one expects to deduce $\mathcal{L}_X\nabla = 0$. A rigorous proof would involve explicitly computing the action of $\tilde{X}$ on fiber coordinates representing jet components and analyzing the condition $\mathcal{L}_{\tilde{X}}\mathcal{F}_{\mathbf{A}} = 0$ using the chain rule and properties of $\mathcal{F}_{\mathbf{A}}$.

7.2. **Conjecture: Axiomatic Uniqueness of A-Natural Volumes.**

Conjecture 7.2 (Axiomatic Uniqueness of $\mathbf{A}$-Natural Volume)**.** *Let (M, ∇) be an affine manifold where $\mathrm{Aff}(M, \nabla)$ acts transitively. An $\mathbf{A}$-natural volume form $\mathrm{dVol}_{F_M^{\mathbf{A}}}$ (derived from an effective Finsler structure $F_M^{\mathbf{A}}$ which is in turn derived from a fiber norm $\mathcal{F}_{\mathbf{A}}$) is the unique volume form (up to a positive scalar multiple) satisfying:*

(1) *Invariance under $\mathrm{Aff}(M, \nabla)$.*
(2) *The underlying $F_M^{\mathbf{A}}$ is positively 1-homogeneous in the fiber variable.*
(3) *Compatibility with Weil algebra morphisms: For a suitable class of morphisms $\rho : \mathbf{A} \to \mathbf{B}$, $(\rho_M)_*(\mathrm{dVol}_{F_M^{\mathbf{A}}}) = \mathrm{dVol}_{F_M^{\mathbf{B}}}$, where ρ_M is the induced map on M (typically identity) or on TM.*

Conditions (2) and (3) are conjectured to force specific choices for construction parameters within $\mathcal{F}_{\mathbf{A}}$, such as factorial-like weights $\lambda_s \propto (1/s!)^c$ if $\mathbf{A} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$, thereby eliminating arbitrary choices in defining the "canonical" $\mathcal{F}_{\mathbf{A}}$.

Motivation and sketch of argument: Condition (1) is expected to establish uniqueness up to scale on the homogeneous space M via Haar measure theory. Condition (2) fixes the homogeneity of the underlying Finsler structure $F_M^{\mathbf{A}}$. The crucial part is Condition (3) (functoriality), which should act as a strong selection principle for the construction of $\mathcal{F}_{\mathbf{A}}$ itself. For example, when considering $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ and canonical projections $\rho : \mathbf{A}_k \to \mathbf{A}_j$ $(j < k)$, consistency would require that the j-th order part of the structure derived from $\mathbf{A}_k$ matches the structure derived directly from $\mathbf{A}_j$. This often necessitates factorial scalings for weights λ_s in definitions like Eq. (2.1) to correctly handle how derivatives (jet components) transform under compositions or reparameterizations, which are modeled by morphisms of these Weil algebras. A full proof requires making precise the "suitable class of morphisms" and the transformation rule for $\mathrm{dVol}_{F_M^{\mathbf{A}}}$ under $(\rho_M)_*$.

7.3. **Conjecture: Curvature-Volume Compatibility.**

Conjecture 7.3 (Ricci Compatibility Relation)**.** *Let ∇ be a torsion-free connection on M, and $\mathbf{A} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$. For the $\mathbf{A}$-natural volume form dVol_{F_M} (whose density is f with respect to a background dV_{can}), and for any affine vector field K ($\mathcal{L}_K\nabla = 0$), if M is compact, then:*

$$\int_M (\mathcal{L}_K \log f)\mathrm{dVol}_{F_M} = \int_M \langle \mathrm{Ric}(\nabla), \mathcal{T}(K, \nabla K)\rangle_{F_M} \mathrm{dVol}_{F_M} + LOT,$$

where $\mathcal{T}(K, \nabla K)$ is some universal tensor field constructed from K and its covariant derivatives, $\langle \cdot, \cdot \rangle_{F_M}$ denotes a pairing possibly related to F_M, and LOT stands for Lower Order Terms that might vanish or simplify under specific conditions.

Corollary 7.4. *It is conjectured that when ∇ is the Levi-Civita connection of a metric g and $\mathrm{Ric} = \lambda g$ (Einstein manifold), if the $\mathbf{A}$-natural construction is sufficiently canonical, it yields $F_M(x, v) = C\|v\|_g$, implying $\mathrm{dVol}_{F_M} = C^n dV_g$. The compatibility relation should then hold, possibly trivially or by imposing conditions on λ.*

Motivation and sketch of argument: Such relations typically arise from Bochner-type formulas. The term $\mathcal{L}_K \log f = \mathrm{div}_{\mathrm{dVol}_{F_M}}(K) - \mathrm{div}_{dV_{\mathrm{can}}}(K)$. For compact M, $\int_M \mathrm{div}_{\mathrm{dVol}_{F_M}}(K)\mathrm{dVol}_{F_M} = 0$ (if K is tangent to boundary or no boundary). The density f depends on F_M, which is derived from $\mathcal{F}_{\mathbf{A}}$. If $\mathcal{F}_{\mathbf{A}}$ uses ∇ (e.g., for jet components), its structure is sensitive to curvature. Yano's formula ($\nabla_i\nabla_j K^l + R^l_{mji}K^m = 0$ for affine K) links second derivatives of K to the Riemann tensor R. Integrating by parts expressions involving K and f, and using these identities, is expected to lead to terms involving Ric. The precise form of $\mathcal{T}$ and the pairing would depend on the specifics of the $\mathbf{A}$-natural construction of F_M.

7.4. **Conjecture: Sub-Riemannian Limit.**

Conjecture 7.5 (Carnot-Carathéodory Limit from Weil Bundles)**.** *Let $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$. Let $(F_M)_k$ be a sequence of effective Finsler structures on M derived from $\mathcal{F}_{\mathbf{A}_k}$ in a way that progressively penalizes directions not in a given distribution $\mathcal{D} \subset TM$ or its higher-order Lie bracket generated directions. For example, one might define $(F_M)_k$ using weights $\omega_s(k)$ in the definition of $\mathcal{F}_{\mathbf{A}_k}$ (as in Eq. (2.3)) that tend to infinity for components "orthogonal"* *to the filtration generated by $\mathcal{D}$, or by adding explicit penalization terms to an initial F_M. Then, under an appropriate rescaling Λ_k of the path lengths $d_k(x,y) = \inf \int (F_M)_k(\gamma,\dot\gamma)dt$, the sequence of metric spaces $(M, \Lambda_k d_k)$ converges in the Gromov-Hausdorff sense to the Carnot-Carathéodory metric space (M, d_{CC}) associated with $\mathcal{D}$ (and a given metric on $\mathcal{D}$).*

If this metric convergence holds, it is conjectured that the associated $\mathbf{A}_k$-natural volumes $\mathrm{dVol}_{(F_M)_k}$ (suitably rescaled if necessary) converge as measures to a canonical sub-Riemannian volume on (M, d_{CC}), such as the Popp measure or the sub-Riemannian Holmes-Thompson volume.

Motivation and sketch of argument: This conjecture is motivated by existing results in sub-Riemannian geometry where Riemannian metrics with highly anisotropic penalties are shown to converge to sub-Riemannian metrics (e.g., via Γ-convergence). The Weil algebra $\mathbf{A}_k$ framework provides a natural way to define structures sensitive to k-th order properties of curves. By designing $(F_M)_k$ to favor curves whose jets are "horizontal" with respect to $\mathcal{D}$ and its bracket-generating hierarchy up to a level related to k, and by heavily penalizing deviations, one expects that in the limit $k \to \infty$ (with appropriate rescaling Λ_k), only paths admissible in the sub-Riemannian sense contribute to the distance. The proof would involve establishing liminf and limsup inequalities for the convergence of the associated energy functionals.

7.5. **Conjecture: Jet-Bundle Holonomy and Rigidity.**

Conjecture 7.6 (Holonomy Representation and Local Symmetry)**.** *Let (M, ∇) be a manifold with an affine connection ∇. The holonomy group $\mathrm{Hol}_x(\nabla)$ has a canonical representation $\rho^{\mathbf{A}}$ on the fiber $V_x^{\mathbf{A}}$ of the Weil bundle $M^{\mathbf{A}}$. If $\mathcal{F}_{\mathbf{A}}$ is a fiber norm on $V_x^{\mathbf{A}}$ that is invariant under the group $\mathrm{Aut}_{\mathrm{gr}}(\mathbf{A})$ of graded automorphisms of $\mathbf{A}$, then the representation $\rho^{\mathbf{A}}$ preserves $\mathcal{F}_{\mathbf{A}}$ if and only if ∇ is locally symmetric ($\nabla R = 0$).*

For $\mathbf{A} = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$, the rigidity of this condition is expected to increase with k. For $k \geq 2$, holonomy preserving an $\mathrm{Aut}_{\mathrm{gr}}(\mathbf{A})$-invariant $\mathcal{F}_{\mathbf{A}}$ is conjectured to imply local symmetry of ∇.

Motivation and sketch of argument: The action of the holonomy algebra $\mathfrak{hol}_x(\nabla)$ on $V_x^{\mathbf{A}}$ is induced by parallel transport of jets. If $\nabla R \neq 0$, this action on higher-order jet components ($s \geq 2$) involves terms containing $\nabla R, \nabla^2 R, \ldots$. The Lie algebra $\mathfrak{aut}_{gr}(\mathbf{A})$ for $\mathbf{A}_k = \mathbb{R}[\epsilon]/(\epsilon^{k+1})$ is one-dimensional, corresponding to infinitesimal scalings $\epsilon \mapsto (1+t)\epsilon$, which acts on s-th order jet components by multiplication by s. For the action of $\mathfrak{hol}_x(\nabla)$ to lie within $\mathfrak{aut}_{gr}(\mathbf{A}_k)$, the complex tensorial terms involving $\nabla^p R$ must vanish, as they generally do not conform to this simple scaling structure. This would force $\nabla R = 0$. Conversely, if $\nabla R = 0$, the action of $\mathfrak{hol}_x(\nabla)$ on jets simplifies significantly (curvature R acts "tensorially" on all components). If $\mathcal{F}_{\mathbf{A}}$ is constructed to be invariant under this simplified action when it aligns with $\mathrm{Aut}_{\mathrm{gr}}(\mathbf{A})$, then preservation follows. The "if" part requires showing this alignment.

8. **Conclusion**

This paper has introduced a framework for defining intrinsic volumes on manifolds by leveraging the algebraic structure of Weil bundles. We have detailed methods for constructing **A**-natural Finsler-like structures $\mathcal{F}_{\mathbf{A}}$ on the fibers of Weil bundles, deriving effective Finsler structures F_M on the tangent bundle TM, and subsequently obtaining Busemann-Hausdorff volume forms dVol_{F_M} on M. Key foundational results include rigidity theorems demonstrating that diffeomorphisms preserving these structures under certain conditions must be affine or isometric, and a characterization of dVol_{F_M} under affine symmetries. The potential of this framework extends significantly further, as indicated by the conjectures presented concerning infinitesimal symmetries, the axiomatic uniqueness of **A**-natural volumes, intricate compatibility relations with Ricci curvature, convergence to sub-Riemannian structures, and strong rigidity conditions imposed by holonomy group actions. These proposed results suggest that the **A**-natural approach offers a powerful lens through which to explore fine geometric properties and uncover new invariants. Future work will focus on providing proofs for these conjectures, which will involve detailed computations in jet bundle theory, applications of natural operator theory, and techniques from Γ-convergence and sub-Riemannian geometry. Further exploration will also include extending the concept of **A**-naturality to non-graded or even infinite-dimensional Weil algebras, with applications in areas such as Frichet geometry, the calculus of variations on path spaces, and the development of novel characteristic classes. The interplay between the algebraic structure of **A** and the resulting geometric invariants on M promises a fruitful area for continued research.

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