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Research Paper

SOME CHARACTERIZATIONS OF THE MAXIMAL $\mathbb{Z}G$-REGULAR IDEAL IN A RING

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ABSTRACT

Let R be an associative ring with identity. The ring R is called $\mathbb{Z}G$-regular (resp. strongly $\mathbb{Z}G$-regular) if, for every $a \in R$, there exist positive integer n and $g \in G$, such that $a^{ng} \in a^{ng}Ra^{ng}$ (resp. $a^{ng} \in a^{(n+1)g}R$). In this paper, we shall show that the join of all $\mathbb{Z}G$-regular ideals in an arbitrary ring R is a $\mathbb{Z}G$-regular ideal, and so there exists a unique maximal $\mathbb{Z}G$-regular ideal $M = M(R)$ in R, whose structure we investigate. Furthermore, we establish the necessary and sufficient condition for a ring to be a direct sum of its ideals.

1. Introduction

Let R be an associative ring with identity. A group action (or just action) of G on X is a binary operation:

$$\mu : X \times G \longmapsto X$$

(If there is no fear of confusion, we write $\mu(x, g)$ simply as by x^g) such that
(I) $(x^g)^h = x^{gh}$ for all $x \in X$ and $g, h \in G$,
(II) $x^1 = x$ for all $x \in X$.
Following [12], we say that R is $\mathbb{Z}G$-regular if, for every $a \in R$, there exist positive integer

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n and $g \in G$, such that $a^{ng} \in a^{ng}Ra^{ng}$. A typical example is the class of classical π-regular rings. In a similar way, a ring R is said to be strongly $\mathbb{Z}G$-regular if, for every $a \in R$, there exist positive integer n and $g \in G$, such that $a^{ng} \in a^{(n+1)g}R$. For examples locally finite ring and the ring of all $n \times n$ matrices and the $n \times n$ lower (upper) triangular matrices over locally finite ring R are strongly $\mathbb{Z}G$-regular rings.

For a more detailed information about $\mathbb{Z}G$-regular rings and strongly $\mathbb{Z}G$-regular rings, we refer interested reader to [12].

The standard notations $J(R)$, $M(R)$, R_n will stand for the Jacobson radical, a unique maximal $\mathbb{Z}G$-regular ideal, the complete matrix ring of order n over R, respectively. We also denote by M^* the ideal consisting of all elements a of R such that $aM = Ma = 0$.

Recall an element x of R is called regular (unit regular) if there exists $y \in R$ (a unit $u \in R$) such that $xyx = x$ ($xux = x$). Some properties of regular rings and strongly regular has been studied in [6, 9].

An element $x \in R$ is said to be π-regular if there exist $y \in R$ and a positive integer n such that $x^n = x^n y x^n$. An element $x \in R$ is said to be strongly π-regular if $x^n = x^{2n}y$. The ring R is π-regular if every element of R is π-regular and is strongly π-regular if every element of R strongly π-regular. By a result of Azumaya [2] and Dischinger [8], the element x can be chosen to commute with y. In particular this definition is left-right symmetric. π-regular and strongly π-regular rings, are studied in particular in [1-5, 7].

Also in a $\mathbb{Z}G$-regular ring, we define:

$(a^g)^h = a^{gh}$ for all $a \in R$ and $g, h \in G$,

$a^1 = a$ for all $a \in R$,

$[a_{ij}]^{ng} = \left[a_{ij}^{ng}\right]$,

$a^{g_1+g_2} = a^{g_1}a^{g_2}$,

$(x_i)_{i\in I}^g = (x_i{}^g)_{i\in I}$

In this note, we first show that the join of all $\mathbb{Z}G$-regular ideals in an arbitrary ring R is $\mathbb{Z}G$-regular ideal, and that there exists a unique maximal $\mathbb{Z}G$-regular ideal $M = M(R)$ in R. Also we prove a few fundamental properties of $M = M(R)$ in R. For example these are the following properties:

Theorem 3.1: $M(R/M(R)) = 0$, Theorem 3.2: if B an ideal in R, $M(B) = B \cap M(B)$. Theorem 3.3: if R_n is complete matrix ring of order n over R, then $M(R_n) = (M(R))_n$. Also as final consequence, we prove that, under the decending chain condition for right ideals, R is experssible as a direct sum $R = M + M^*$, where M^* is the ideal consisting of all elements $a \in R$ such that $aM = Ma = 0$.

2. Preliminaries

In this section, we present several lemmas and propositions that will be used in the subsequent results.

Lemma 2.1. *Let R be a ring with Jacobson radical $J = J(R)$. Suppose that for all $x \in R$ and $y \in J(R)$ we have $xy = x (yx = x)$. Then $x = 0$.*

Proof. Clearly from[15, Lemma 1]. □

Proposition 2.2. *Every $\mathbb{Z}G$-regular ring has zero Jacobson radical.*

Proof. Let $a \in J$, if there exist $y \in R$, $n \in \mathbb{Z}$, $g \in G$ such that $a^{ng} = a^{ng}ya^{ng}$. Since $a^{ng} \in J$ by Lemma 2.1 it follow that $a^{ng} = 0$ and thus $a = 0$. □

Proposition 2.3. *A $\mathbb{Z}G$-regular ideal $I \in R$ is itself a $\mathbb{Z}G$-regular ring.*

Proof. For if $a \in I$, there exist an element $y \in R$, $n \in \mathbb{Z}$, $g \in G$ such that $a^{ng} = a^{ng}ya^{ng}$, $a^{ng} \in I$. It follows that $a^{ng}ya^{ng}ya^{ng} = a^{ng}$ and $ya^{ng}y \in I$, so a^{ng} is regular in the ring I, therefore, by [16, Theorem 2.4], a is $\mathbb{Z}G$-regular in the ring I. □

Lemma 2.4. *Let $y \in R$ such that $a^{ng} - a^{ng}ya^{ng} = a'$ and suppose that a' is $\mathbb{Z}G$-regular and the group action satisfies $(ay)^g = a^g y^g$ for all $a, y \in R$. Then a is $\mathbb{Z}G$-regular.*

Proof. Since $a^{ng} - a^{ng}ya^{ng} = a'$ then we have:

$$\begin{aligned}
a^{ng} &= a' + a^{ng}ya^{ng} \\
&= (a'^{n'h}b^{n'}a'^{n'h})^{n'^{-1}h^{-1}} + a^{ng}ya^{ng} \\
&= a'b^{h^{-1}}a' + a^{ng}ya^{ng} \\
&= (a^{ng} - a^{ng}ya^{ng})b^{h^{-1}}(a^{ng} - a^{ng}ya^{ng}) \\
&\quad + a^{ng}ya^{ng} \\
&= (a^{ng}b^{h^{-1}} - a^{ng}ya^{ng}b^{h^{-1}})(a^{ng} - a^{ng}ya^{ng}) \\
&\quad + a^{ng}ya^{ng} \\
&= a^{ng}b^{h^{-1}}a^{ng} - a^{ng}ya^{ng}b^{h^{-1}}a^{ng} \\
&\quad - a^{ng}b^{h^{-1}}a^{ng}ya^{ng} + a^{ng}ya^{ng}b^{h^{-1}}a^{ng}ya^{ng} \\
&\quad + a^{ng}ya^{ng} \\
&= a^{ng}(b^{h^{-1}} - ya^{ng}b^{h^{-1}} - b^{h^{-1}}a^{ng}y \\
&\quad + ya^{ng}b^{h^{-1}}a^{ng}y + y)a^{ng}
\end{aligned}$$

Now by taking $b = b^{h^{-1}} - ya^{ng}b^{h^{-1}} - b^{h^{-1}}a^{ng}y + ya^{ng}b^{h^{-1}}a^{ng}y + y$, we have $a^{ng} = a^{ng}ba^{ng}$, then a is $\mathbb{Z}G$-regular. □

We shall indicate by (a) the principal ideal in R generated by a.

Proposition 2.5. *If M is the set of all elements a of R such that the principal ideal (a) is $\mathbb{Z}G$-regular and satisfies the property $(a + b)^{ng} = a^{ng} + b^{ng}$ for all $a, b \in R$, $n \in \mathbb{Z}$, $g \in G$, then M is an ideal in R.*

Proof. Let $z \in M$ and $t \in R$. Then $zt \in M$, since $(zt) \subseteq (z)$. Similarly, $tz \in M$. If $z, w \in (z - w)$, then $a = u - v$ for some $u \in (z)$ and $v \in (w)$. Since(z) is $\mathbb{Z}G$-regular, then there exist $r \in R$ and $n \in \mathbb{Z}$, $g \in G$ such that $u^{ng} = u^{ng}ru^{ng}$, then

$$\begin{aligned}
&a^{ng} - a^{ng}ra^{ng} = (u - v)^{ng} - (u - v)^{ng}r(u - v)^{ng} \\
&= u^{ng} - v^{ng} - u^{ng}ru^{ng} + u^{ng}rv^{ng} + v^{ng}ru^{ng} - v^{ng}rv^{ng} \\
&= -v^{ng} + u^{ng}rv^{ng} + v^{ng}ru^{ng} - v^{ng}rv^{ng}
\end{aligned}$$

Since $v \in (w)$, this shows that $a^{ng} - a^{ng}ra^{ng} \in (w)$ and is therefore $\mathbb{Z}G$-regular. By applying Lemma 2.4, we conclude that a is $\mathbb{Z}G$-regular, and $z - w \in M$. This completes the proof of the theorem. □

Lemma 2.6. *Let R be a $\mathbb{Z}G$-regular ring. Then R_n, complete matrix ring of order n over R, is a $\mathbb{Z}G$-regular ring.*

Proof. The first being the proof for $n = 2$, and then the extension to arbitrary n. If $r \in R$, let us denote by r' an element of R and $n \in \mathbb{Z}$, $g \in G$ such that $r^{ng}r'r^{ng} = r^{ng}$. Now let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an arbitrary element of R_n. If we set

$$X = \begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix}$$

and denote $A^{ng} - A^{ng}XA^{ng}$ by B^{ng}, we have:

$$\begin{aligned} &\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{ng} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{ng} \begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{ng} \\ &= \begin{bmatrix} a^{ng} & b^{ng} \\ c^{ng} & d^{ng} \end{bmatrix} - \begin{bmatrix} a^{ng} & b^{ng} \\ c^{ng} & d^{ng} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix} \begin{bmatrix} a^{ng} & b^{ng} \\ c^{ng} & d^{ng} \end{bmatrix} \\ &= \begin{bmatrix} a^{ng} & b^{ng} \\ c^{ng} & d^{ng} \end{bmatrix} - \begin{bmatrix} b^{ng}b'a^{ng} & b^{ng}b'b^{ng} \\ 0 & d^{ng}b'b^{ng} \end{bmatrix} \\ &= \begin{bmatrix} a^{ng} - b^{ng}b'a^{ng} & b^{ng} - b^{ng}b'b^{ng} \\ c^{ng} & d^{ng}d^{ng}b'b^{ng} \end{bmatrix} \\ &= \begin{bmatrix} a^{ng} - b^{ng}b'a^{ng} & 0 \\ c^{ng} & d^{ng}d^{ng}b'b^{ng} \end{bmatrix} \end{aligned}$$

Upper simple calculation shows that $B^{ng} = \begin{bmatrix} j & 0 \\ h & i \end{bmatrix}$ for suitable choice of element j, h, i of R. If $Y = \begin{bmatrix} j' & 0 \\ 0 & i' \end{bmatrix}$ then $C^{ng} = B^{ng} - B^{ng}YB^{ng} = \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix}$, for $k = c^{ng}j'(a^{ng} - b^{ng}b'a^{ng}) + (d^{ng} - d^{ng}b'b^{ng})i'c^{ng}$ of R. Finally, if $Z = \begin{bmatrix} 0 & k' \\ 0 & 0 \end{bmatrix}$, we see that $C^{ng} - C^{ng}ZC^{ng} = 0$, since $\begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} \begin{bmatrix} 0 & k' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k - k'kk & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
This means that C is $\mathbb{Z}G$-regular and then by Lemma 2.4, B is $\mathbb{Z}G$-regular. Again applying Lemma 2.4, we see that A is $\mathbb{Z}G$-regular, and this completes the proof for $n = 2$. Since $(R_2)_2 \cong R_4$, it follows from the case just proved that R_4 is $\mathbb{Z}G$-regular, and similarly R_{2^k} is $\mathbb{Z}G$-regular for any positive integer k. Let n be an arbitrary positive integer. Choose an integer k such $2^k \geqslant n$. If $A \in R_n$, let A_1 be the matrix of R_{2^k} with A is the upper left-hand corner and zeros else where. Now, as an element of R_{2^k}, A_1 is $\mathbb{Z}G$-regular, because there exist an element $X = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \in R_{2^k}$ that $(B \in R_n)$, $g \in G$ and $n \in \mathbb{Z}$ such that $A_1^{ng} = A_1^{ng}XA_1^{ng}$. However, this implies that $A^{ng} = A^{ng}BA^{ng}$ and hence A is $\mathbb{Z}G$-regular. The proof of the lemma is therefore complete. $\square$

Lemma 2.7. *The only idempotent element in the Jacobson radical R is zero.*

Proof. Let $\alpha \in J(R)$ be an idempotent, i.e., $\alpha^2 = \alpha$. Since $1 - \alpha$ is a unit in R, then $\alpha = 0$. $\square$

Lemma 2.8. *Let I be an ideal of R. Then $J(I) = I \cap J(R)$.*

Proof. The result follows from [13, Exercise 7, p. 68] $\square$

Lemma 2.9. *For any ring R, the following are equivalent.*

1. *For any $a \in R$, there exist $x \in R$, $n \in \mathbb{Z}$, $g \in G$ such that $a^{ng} = a^{ng}xa^{ng}$.*
2. *Every principal left(right) ideal is generated by an idempotent.*
3. *Every principal left(right) ideal is a direct summand of ${}_RR$.*
4. *Every finitely generated left(right) ideal is generated by an idempotent.*
5. *Every finitely generated left(right) ideal is a direct summand of ${}_RR$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) follows from standard arguments (See 13, Exercise 7, P. 68).
Let us prove (1) $\Leftrightarrow$ (2). Assume (1), let $a \in R$ then there exist $n \in \mathbb{Z}$, $g \in G$ such that $a^{ng} \in R$ and consider a principal left ideal $R.a^{ng}$. Choose $x \in R$ such that $a^{ng}xa^{ng} = a^{ng}$. Then $e := xa^{ng} = xa^{ng}xa^{ng} = e^2$ and $e \in R.a^{ng}$ while $a^{ng} = a^{ng}xa^{ng} = a^{ng}e \in R.e$, so $R.a^{ng} = R.e$. Conversely, assume (2) and let $a \in R$. Writing $R.a^{ng} = R.e$,where $e = e^2$, we have $e = xa^{ng}$ and $a^{ng} = y.e$ for some $x, y \in R$. Then $a^{ng}xa^{ng} = ye.e = ye = a^{ng}$.
(4) $\Leftrightarrow$ (2): since (4) obviously implies (2), it only remains to show that (2) $\Rightarrow$ (4). By induction, it suffices to show that, for any two idempotents e, f, $I = Re+Rf$ is generated by an idempotent. Now $I = Re+Rf(1-e)$ and $Rf(1-e) = Re'$ for some idempotent e', for which $e'e \in Rf(1-e)e = 0$. Thus $e'(e'+e) = e'$ which leads easily to $I = Re+Re' = R(e'+e)$. $\square$

Lemma 2.10. *A ring is semisimple if and only if it is Artinian and $\mathbb{Z}G$-regular.*

Proof. We have already see that semi simple rings are exactly the artinian and $\mathbb{Z}G$-regular by (13, Corollary 2.6) and Lemma 2.9.
Conversely, if a ring R is artinian and $\mathbb{Z}G$-regular, then every ideal of R is finitely generated and hence a direct summand or ${}_RR$, by using the characterization (5) of Lemma 2.9. Therefore R is semi simple. $\square$

3. Main Results

It is clear that $M(R)$ in Proposition 2.5, being of the join of all $\mathbb{Z}G$-regular ideals in R, and being itself $\mathbb{Z}G$-regular, is the unique maximal $\mathbb{Z}G$-regular ideal in R.

Theorem 3.1. *Let R be a ring. Then $M(R/M(R)) = 0$*

Proof. Let $\overline{a}$ denote the residue class modulo $M(R)$ which contains the element a of R. If $\overline{b} \in M(R/M(R))$ and $a \in (b)$, then $\overline{a} \in (\overline{b})$. Since $(\overline{b})$ is $\mathbb{Z}G$-regular ideal in $R/M(R)$, then $\overline{a}$ is $\mathbb{Z}G$-regular. If $\overline{a}^{ng} = \overline{a}^{ng}\overline{xa}^{ng}$, $a^{ng} - a^{ng}xa^{ng} \in M(R)$ therefore $a^{ng} - a^{ng}xa^{ng}$ is $\mathbb{Z}G$-regular and Lemma 2.4 implies that a is $\mathbb{Z}G$-regular. This shows that every element of (b) is $\mathbb{Z}G$-regular, and hence $b \in M(R)$. Thus $\overline{b} = 0$, completing the proof. $\square$

Theorem 3.2. *Let B be an ideal in R. Then $M(B) = B \cap M(R)$.*

Proof. Suppose that B is an ideal in R, and let b be an element of B which generates a $\mathbb{Z}G$-regular ideal $(b)'$ in the ring B. Let (b) be the ideal in R generated by the element b, and let $c = mb^{ng} + rb^{ng} + b^{ng}s + \sum r_i b^{ng} s_i$, ($m$ is integer; $r, sr_i, s_i \in R$, $n \in \mathbb{Z}$, $g \in G$) be any element of (b). Since b is $\mathbb{Z}G$-regular in B, we have $b^{ng} = b^{ng} b_1 b^{ng}$ for some b_1 in B. Hence $c = nb^{ng} + (rb^{ng}b_1)b^{ng} + b^{ng}(b_1 b^{ng} s) + \sum (r_i b^{ng} b_1) b^{ng} (b_1 b^{ng} s_i)$, and thus $c \in (b)'$, therefore (b) is $\mathbb{Z}G$-regular, since it coincides with $(b)'$. This shows that if $b \in M(R)$, then $b \in B \cap M(R)$. Conversely, if $b \in B \cap M(R)$, then b is an element of B which is $\mathbb{Z}G$-regular in R, and it is easy to see that b is therefore $\mathbb{Z}G$-regular in the ring B. Since $B \cap M(R)$ is $\mathbb{Z}G$-regular ideal in the ring R, it follows that $B \cap M(R) \subseteq M(B)$. We have therefore proved theorem. □

Theorem 3.3. *If R_n is complete matrix ring of order n over R, then $M(R_n) = (M(R))_n$.*

Proof. By Lemma 2.6, we proved $(M(R))_n$ is a $\mathbb{Z}G$-regular ideal in R_n and hence $(M(R))_n \subseteq M(R_n)$. Conversely, let A be a matrix in $M(R_n)$, and let a_{ij} be a fixed element of A. Since (A) is a $\mathbb{Z}G$-regular ideal, there exist an element X of R_n and $n \in \mathbb{Z}$, $g \in G$ such that $A^{ng} = A^{ng} X A^{ng} = A^{ng} X A^{ng} X A^{ng}$, and therefore $a_{ij}^{ng} = \sum t_{pq}^{ng} a_{pq} s_{pq}^{ng}$, for suitable elements t_{pq}, s_{pq} of R. But it is easy to see that there exists a matrix of (A) with $t_{pq} a_{pq} s_{pq}$ in $(1,1)$ position and zeros elsewhere, and hence an element of (A) with a_{ij} in $(1,1)$ position and zeros elsewhere. Now if b is any element of the principal ideal in R generated by a_{ij}, it is clear that there exists an element B of (A) with b in the $(1,1)$ position and zeros elsewhere. Furthermore for $n \in \mathbb{Z}$, $g \in G$, we have $B^{ng} = B^{ng} Y B^{ng}$ for suitable choice of Y in R_n, since (A) is $\mathbb{Z}G$-regular. But this implies that $b^{ng} = b^{ng} y_{11} b^{ng}$ and hence b is $\mathbb{Z}G$-regular. This shows $a_{ij} \in M(R)$, and so that $M(R_n) \subseteq (M(R))_n$, completing the proof of the theorem. □

Definition 3.4. For an ideal B of a ring R, annihilator B^* is meant the ideal consisting of all $a \in R$ such that $aB = Ba = 0$.

Theorem 3.5. *If M is the maximal $\mathbb{Z}G$-regular ideal of a ring R and J is the Jacobson radical of R. Then:*

1. $M \cap J = 0$.

2. $J \subseteq M^*$, $M \subseteq J^*$.

3. $M \cap M^* = 0$.

4. J is the radical of the ring M^* and M is the maximal $\mathbb{Z}G$-regular ideal of the ring J^*

Proof. Since J contains no nonzero idempotent element by Lemma 2.7, $M \cap J = 0$.
From 1, it follows that $MJ = JM = 0$, so $J \subseteq M^*$, $M \subseteq J^*$.
If $a \in M \cap M^*$, then $a = axa$ for some x and $1 \in \mathbb{Z}$, $1 \in G$. But $a \in M$ and $xa \in M^*$, hence $M \cap M^* = 0$.
By Lemma 2.8, if B is any ideal in R, the radical of the ring B is just $B \cap J$. Since $J \subseteq M^*$, then J is the radical of the ring M^*. Also M is the maximal $\mathbb{Z}G$-regular ideal of the ring J^* follow from the analogous of Theorem 3.2. □

Definition 3.6. A ring is R is bound to its radical J if and only if $J^* \subseteq J$.

Theorem 3.7. *If R is a ring such that R/J is $\mathbb{Z}G$-regular, then $M = 0$ if and only if R is bound to J.*

Proof. If R is bound to J, it follows that $M = 0$, even without the condition that R/J be $\mathbb{Z}G$-regular. For $M \cap J = 0$, and this implies, as in Theorem 3.5, that $M \subseteq J^* \subseteq J$. Hence $M = 0$.
Conversely, let R/J be $\mathbb{Z}G$-regular and $M = 0$. We show first by induction that $J \cap J^{*^2} = 0$.

Suppose that $j \in J$ and that $j = \sum_{i=0}^{m} a_i b_i$ where $a_i, b_i \in J^*$. It must be proved that $j = 0$. In the $\mathbb{Z}G$-regular ring R/J, $\overline{a_i}$ is $\mathbb{Z}G$-regular, so there exist elements $x_i \in R$ and $1 \in \mathbb{Z}$, $1 \in G$ such that $\overline{a_i} - \overline{a_i} x_i \overline{a_i} = j_i \in J$. Since $b_i \in J^*$; we have:

$$j = \sum_{i=0}^{m} a_i b_i = \sum_{i=0}^{m} (j_i + a_i x_i a_i) b_i \sum_{i=0}^{m} a_i x_i a_i b_i \tag{1}$$

If $m = 1$, this implies that $j = a_1 x_1 a_1 b_1 = a_1 x_1 j = 0$ since $a_1 \in J^*$. If $m \neq 1$, then $a_m b_m = j - \sum_{i=0}^{m-1} a_i b_i$. Thus by (1)

$$j = \sum_{i=0}^{m-1} a_i x_i a_i) b_i + a_m x_m (j - \sum_{i=0}^{m-1} a_i b_i) = \sum_{i=0}^{m-1} (a_i) x_i - a_m b_m) a_i) b_i$$

But the induction hypothesis asserts that if $j = \sum_{i=0}^{m-1} c_i d_i$ that $c_i, d_i \in J^*$, then $j = 0$. because$((a_i) x_i - a_m b_m) a_i) b_i) a_i, a_i \in J^*$, it follows that $j = 0$ and we proved that $J \cap {J^*}^2 = 0$. This implies, however, that ${J^*}^2$ is a $\mathbb{Z}G$-regular ideal. For if $a \in {J^*}^2$, then in the $\mathbb{Z}G$-regular ring R/J, the element $\overline{a_i}$ is $\mathbb{Z}G$-regular, that is, for some x and $1 \in \mathbb{Z}$, $1 \in G$, $a - axa \in J \cap {J^*}^2 = 0$, so a is $\mathbb{Z}G$-regular. Hence ${J^*}^2 \subseteq M = 0$, from which it follows that $J^* \subseteq J$ Since the radical contains all nill ideal [13, Lemma 4.11, P. 56]. Thus R is bound to J. □

By applying before results, we obtain our final consequence.

Theorem 3.8. *If a ring R satisfies the descending chain condition for right ideals, then $R = M + M^*$, where the ring M is semisimple and the ring M^* is bound to its radical.*

Proof. If an ideal I in a ring R has a unit element e, then $R = I + I^*$ (1), since the existence of a unit element in I implies that $I \cap I^* = 0$. If $x \in R$, then $ex + xe \in I$ and hence $(ex + xe)e = e(ex + xe)$, from which it follows that $xe = ex$ and e is in the center of R. Thus the peirce decomposition $x = ex + (x - ex)$ expresses each element $x \in R$ as a sum of elements $ex \in I$ and $(x - ex) \in I^*$.
On the other hand, a right ideal I in the ring M is right ideal in R. For if $a \in I$, $r \in R$ then $ar \in M$, hence for some element $y \in R$ and $1 \in \mathbb{Z}$, $1 \in G$, $aryar = ar$. But $ryar \in M$, so $ar \in I$. Thus I is a right ideal in R. Then, it follows that if the descending chain condition for right ideals hold in R, it holds also in M. In the presence of this chain condition, $\mathbb{Z}G$-regularity is equivalent to semisimplicity by Lemma 2.10. Hence M has a unit element, and so $R = M + M^*$ by (1).
The semi simplicity of M is implied by the $\mathbb{Z}G$-regularity of M. Since the maximal $\mathbb{Z}G$-regular ideal of M^* is zero by Theorem 3.2, and the chain condition holds in M^*, it follows from Theorem 3.7 that M^* is bound to its radical. □

Example 3.9. For the ring $R = \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. The ring M can be taken as $\mathbb{R} \times 0$, which is semisimple, since it is isomorphic to $\mathbb{R}$ (a simple ring). Also the ring M^* can be taken as $0 \times \mathbb{R}$. This is a simple ring but it is also bounded to its radical as its Jacobson radical is zero. The direct sum $M + M^*$ gives:

$$M + M^* = (\mathbb{R} \times 0) + (0 \times \mathbb{R}) = \mathbb{R} \times \mathbb{R} = R$$

The descending chain condition for right ideals holds, because in $\mathbb{R} \times \mathbb{R}$, any descending sequence of right ideals will eventually stabilize.

4. Conclusions

In this paper, various characterizing properties of maximal $\mathbb{Z}G$-regular ideal $M(R)$ have been investigated. Results have contained a description of $M(R)$. Also final theorem in this research shows that the semisimple component is precisely the maximal $\mathbb{Z}G$-regular ideal M of R, and the bound component is the annihilator of M.

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