Journal of Finsler Geometry and its Applications
Vol. 6, No. 2 (2025), pp 120-131
https://doi.org/10.22098/jfga.2025.17098.1155

Quater-Symmetric metric connection on f-Kenmotsu manifolds

Ashis Mondal[a*]

[a]Department of Mathematics, Jangipur College, Jangipur-742213
Murshidabad, West Bengal, India.

E-mail: ashism750@gmail.com

Abstract. In this paper, we study projectively flat and conharmonically flat three-dimensional f-Kenmotsu manifolds with respect to quater-symmetric metric connection. Also, we consider η-Ricci solitons of a three-dimensional f-Kenmotsu manifold with respect to quater-symmetric metric connection. Finally, we have cited an example which verifies one of our main Theorems.

Keywords: f-Kenmotsu manifolds, η-Ricci soliton, projectively flat, conhrarmonically flat, η-Einstein manifold.

1. Introduction

The quarter-symmetric linear connections in a differential manifold was introduced by S. Golab [14]. A linear connection $\overset{s}{\nabla}$ on an n-dimensional Riemannian manifold is called a quarter-symmetric connection(QSMC) [14] if its torsion tensor Γ of the connection $\overset{s}{\nabla}$ defined by

$$\Gamma(\mathcal{I}_1, \mathcal{I}_2) = \overset{s}{\nabla}_{\mathcal{I}_1}\mathcal{I}_2 - \overset{s}{\nabla}_{\mathcal{I}_2}\mathcal{I}_1 - [\mathcal{I}_1, \mathcal{I}_2],$$

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53C25, 53C15

satisfies

$$\Gamma(\mathcal{I}_1, \mathcal{I}_2) = \eta(\mathcal{I}_2)\omega\mathcal{I}_1 - \eta(\mathcal{I}_1)\omega\mathcal{I}_2, \tag{1.1}$$

where η is a 1 form and ω is a (1,1) tensor field. The quarter-symmetric connection is called a semi-symmetric connection [12] if $\omega\mathcal{I}_1 = \mathcal{I}_1$. A quarter-symmetric connection $\overset{s}{\nabla}$ satisfies the condition

$$(\overset{s}{\nabla}_{\mathcal{I}_1} g)(\mathcal{I}_2, \mathcal{I}_3) = 0, \tag{1.2}$$

for all $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \chi(\mathcal{M})$, where $\chi(\mathcal{M})$ is the Lie algebra of vector fields of the manifold $\mathcal{M}$, then $\overset{s}{\nabla}$ is said to be a QSMC, otherwise it is said to be a quarter-symmetric non-metric connection. In [2, 4, 19, 20, 26], the properties of Riemannian manifolds with QSMC have been studied by many authors.

Let $\mathcal{M}$ be a Riemannian manifold of dimension $(2n+1)$. If the value of projective curvature tensor is defined by [1, 13, 27]

$$P(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 = \overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 - \frac{1}{2n}\{\overset{a}{S}(\mathcal{I}_2, \mathcal{I}_3)\mathcal{I}_1 - \overset{a}{S}(\mathcal{I}_1, \mathcal{I}_3)\mathcal{I}_2\},$$

reduced to zero, then $\mathcal{M}$ is said to be locally projectively flat and the converse is also true, where $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \chi(\mathcal{M})$, $\overset{a}{R}$ is the curvature tensor and $\overset{a}{S}$ is the Ricci tensor with respect to the Levi-Civita connection, respectively.

A rank-four tensor H on Riemannian manifold $\mathcal{M}$ of dimension $(2n+1)$ is given by

$$\begin{aligned} H(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 &= \overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 - \frac{1}{2n-1}\{\overset{a}{S}(\mathcal{I}_2, \mathcal{I}_3)\mathcal{I}_1 - \overset{a}{S}(\mathcal{I}_1, \mathcal{I}_3)\mathcal{I}_2 \\ &+ g(\mathcal{I}_2, \mathcal{I}_3)Q\mathcal{I}_1 - g(\mathcal{I}_1, \mathcal{I}_3)Q\mathcal{I}_2\}, \end{aligned}$$

where $\overset{a}{R}, \overset{a}{S}$ and Q represents the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively. A manifold $\mathcal{M}$ on which H vanishes at every point is called conharmonically flat manifold.

A Ricci soliton is a natural generalization of Einstein metric. If there is a smooth vector field $\mathcal{V}$ on a Riemannian manifold $(\mathcal{M}, g)$ that satisfies the following condition [15, 28], the manifold is known as a Ricci soliton

$$\pounds_{\mathcal{V}} g + 2\overset{a}{S} + 2\lambda g = 0,$$

where $\pounds_{\mathcal{V}}$ stands for the Lie derivative operator along the vector field $\mathcal{V}$ and $\overset{a}{S}$ is a Ricci tensor of $\mathcal{M}$. According to $\lambda < 0, \lambda = 0$, or $\lambda > 0$, respectively, the Ricci soliton is considered to be shrinking, steady and expanding [8]. Several authors have studied Ricci solitons, such as [3, 7, 10, 11, 16, 23].

If there is a smooth vector field $\mathcal{V}$ such that the Ricci tensor satisfies the following equation, then the Riemannian manifold $(\mathcal{M}, g)$ is known as a η-Ricci soliton

$$\pounds_{\mathcal{V}} g + 2\overset{a}{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $\overset{a}{S}$ is the Ricci tensor associated to g, η is an one form, λ and μ are real constants. In this connection we mention the works of Blaga [5, 6]. Motivated by the above studies we study f-Kenmotsu manifolds admitting QSMC.

The present paper, after the introduction, we give some required preliminaries about f-Kenmotsu manifolds and study quater-symmetric connection on f-Kenmotsu manifolds in Section 2. In Section 3, we consider projectively flat f-Kenmotsu manifolds of dimension 3 with respect to QSMC. Section 4 is devoted to study conharmonically flat three-dimensional f-Kenmotsu manifolds with respect to QSMC. In Section 5, we study η-Ricci solitons on f-Kenmotsu manifolds of dimension 3 admitting QSMC. Finally, we give an example of 3-dimensional f-Kenmotsu manifolds admitting QSMC which admits an η-Ricci soliton.

2. **Preliminaries**

Let $\mathcal{M}$ be a 3-dimensional manifold. If the $(1,1)$ tensor field ω, the vector field ξ, the 1-form η and Riemannian matric g satisfies the conditions

$$\omega^2\mathcal{I}_1 = -\mathcal{I}_1 + \eta(\mathcal{I}_1)\xi, \quad \eta(\xi) = 1, \quad \omega\xi = 0, \quad \eta\omega = 0, \tag{2.1}$$

$$g(\omega\mathcal{I}_1, \omega\mathcal{I}_2) = g(\mathcal{I}_1, \mathcal{I}_2) - \eta(\mathcal{I}_1)\eta(\mathcal{I}_2), \tag{2.2}$$

$$g(\mathcal{I}_1, \omega\mathcal{I}_2) = -g(\omega\mathcal{I}_1, \mathcal{I}_2), \quad g(\mathcal{I}_1, \xi) = \eta(\mathcal{I}_1), \tag{2.3}$$

for all vector fields $\mathcal{I}_1, \mathcal{I}_2 \in \chi(\mathcal{M})$; then we say (ω, ξ, η, g), is a contact metric structure and $(\mathcal{M}, \omega, \xi, \eta, g)$, is known as a contact metric manifold. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(\mathcal{I}_1, \mathcal{I}_2) = g(\mathcal{I}_1, \omega\mathcal{I}_2), \tag{2.4}$$

for all vector fields $\mathcal{I}_1, \mathcal{I}_2 \in \chi(\mathcal{M})$. An almost contact metric manifold is normal if $[\omega, \omega](\mathcal{I}_1, \mathcal{I}_2) + 2d\eta(\mathcal{I}_1, \mathcal{I}_2)\xi = 0$. If the condition [21]

$$(\overset{a}{\nabla}_{\mathcal{I}_1}\omega)\mathcal{I}_2 = f\{g(\omega\mathcal{I}_1, \mathcal{I}_2)\xi - \eta(\mathcal{I}_2)\omega\mathcal{I}_1\}, \tag{2.5}$$

where $f \in C^\infty(\mathcal{M})$ such that $df \wedge \eta = 0$ and $\overset{a}{\nabla}$ is Levi-Civita connection on $\mathcal{M}$, satisfied by $\mathcal{M}$, then the almost contact metric manifold $\mathcal{M}$ is called f-Kenmotsu manifold. If $f = \alpha$=constant$\neq 0$, then the manifold is known as α-Kenmotsu manifold [17]. If $f = 1$, then it is a Kenmotsu manifold [18]. If f=0, then the manifold is called cosymplectic manifold [17]. If $f^2 + f' \neq 0$ and $f' = \xi(f)$, then f-Kenmotsu manifold is said to be regular. Above relation (2.5) gives us

$$\overset{a}{\nabla}_{\mathcal{I}_1}\xi = f\{\mathcal{I}_1 - \eta(\mathcal{I}_1)\xi\}. \tag{2.6}$$

Then using (2.6), we get

$$(\overset{a}{\nabla}_{\mathcal{I}_1}\eta)\mathcal{I}_2 = f\big(g(\mathcal{I}_1, \mathcal{I}_2) - \eta(\mathcal{I}_1)\eta(\mathcal{I}_2)\big). \tag{2.7}$$

If $dim\mathcal{M} \geq 5$, then the condition $df \wedge \eta = 0$ holds. If $dim\mathcal{M} = 3$, then this does not hold in general [22]. In a 3-dimensional f-Kenmotsu manifold $\mathcal{M}$, $(\mathcal{M}, \omega, \xi, \eta, g)$, satisfies the following relations [22]

$$\begin{aligned}\overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 &= (\frac{r}{2} + 2f^2 + 2f')\Big\{g(\mathcal{I}_2, \mathcal{I}_3)\mathcal{I}_1 - g(\mathcal{I}_1, \mathcal{I}_3)\mathcal{I}_2\Big\} \\ &- (\frac{r}{2} + 3f^2 + 3f')\Big\{g(\mathcal{I}_2, \mathcal{I}_3)\eta(\mathcal{I}_1)\xi - g(\mathcal{I}_1, \mathcal{I}_3)\eta(\mathcal{I}_2)\xi \\ &+ \eta(\mathcal{I}_2)\eta(\mathcal{I}_3)\mathcal{I}_1 - \eta(\mathcal{I}_1)\eta(\mathcal{I}_3)\mathcal{I}_2\Big\}, \end{aligned} \tag{2.8}$$

$$\overset{a}{S}(\mathcal{I}_1, \mathcal{I}_2) = (\frac{r}{2} + f^2 + f')g(\mathcal{I}_1, \mathcal{I}_2) - (\frac{r}{2} + 3f^2 + 3f')\eta(\mathcal{I}_1)\eta(\mathcal{I}_2), \tag{2.9}$$

$$Q\mathcal{I}_1 = (\frac{r}{2} + f^2 + f')\mathcal{I}_1 - (\frac{r}{2} + 3f^2 + 3f')\eta(\mathcal{I}_1)\xi, \tag{2.10}$$

$$\overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\xi = -(f^2 + f')\Big\{\eta(\mathcal{I}_2)\mathcal{I}_1 - \eta(\mathcal{I}_1)\mathcal{I}_2\Big\}, \tag{2.11}$$

$$\overset{a}{R}(\xi, \mathcal{I}_1)\mathcal{I}_2 = -(f^2 + f')\Big\{g(\mathcal{I}_1, \mathcal{I}_2)\xi - \eta(\mathcal{I}_2)\mathcal{I}_1\Big\}, \tag{2.12}$$

$$\eta(\overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3) = -(f^2 + f')\Big\{g(\mathcal{I}_2, \mathcal{I}_3)\eta(\mathcal{I}_1) - g(\mathcal{I}_1, \mathcal{I}_3)\eta(\mathcal{I}_2)\Big\}, \tag{2.13}$$

where $\overset{a}{R}$ denotes the curvature tensor, $\overset{a}{S}$ is the Ricci tensor of type $(0, 2)$ and r is the scalar curvature of the manifold $\mathcal{M}$.

The relation between QSMC $\overset{s}{\nabla}$ and the Levi-Civita connection $\overset{a}{\nabla}$ is given by [25]

$$\overset{s}{\nabla}_{\mathcal{I}_1}\mathcal{I}_2 = \overset{a}{\nabla}_{\mathcal{I}_1}\mathcal{I}_2 - \eta(\mathcal{I}_1)\omega\mathcal{I}_2, \tag{2.14}$$

for all vector fields $\mathcal{I}_1$ and $\mathcal{I}_2$ on $\mathcal{M}$. Let $\overset{s}{R}$ be the curvature tensor of an f-Kenmotsu manifold with respect to quarter-symmetric connection $\overset{s}{\nabla}$. Then $\overset{s}{R}$ is given by

$$\overset{s}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 = \overset{s}{\nabla}_{\mathcal{I}_1}\overset{s}{\nabla}_{\mathcal{I}_2}\mathcal{I}_3 - \overset{s}{\nabla}_{\mathcal{I}_2}\overset{s}{\nabla}_{\mathcal{I}_1}\mathcal{I}_3 - \overset{s}{\nabla}_{[\mathcal{I}_1, \mathcal{I}_2]}\mathcal{I}_3. \tag{2.15}$$

In view of (2.14), above equation takes the form [25]

$$\begin{aligned}\overset{s}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 = \overset{a}{R}(\mathcal{I}_1, \mathcal{I}_2)\mathcal{I}_3 &+ f\{\eta(\mathcal{I}_2)\omega\mathcal{I}_1 - \eta(\mathcal{I}_1)\omega\mathcal{I}_2\}\eta(\mathcal{I}_3) \\ &+ f\{g(\omega\mathcal{I}_2, \mathcal{I}_3)\eta(\mathcal{I}_1) - g(\omega\mathcal{I}_1, \mathcal{I}_3)\eta(\mathcal{I}_2)\}\xi, \end{aligned} \tag{2.16}$$

where $\overset{s}{R}$ and $\overset{a}{R}$ are the curvature tensor with respect to $\overset{s}{\nabla}$ and $\overset{a}{\nabla}$ respectively.

From equation (2.16) it follows that

$$\overset{s}{S}(\mathcal{I}_1, \mathcal{I}_2) = \overset{a}{S}(\mathcal{I}_1, \mathcal{I}_2) + fg(\omega\mathcal{I}_1, \mathcal{I}_2), \tag{2.17}$$

where $\overset{s}{S}$ and $\overset{a}{S}$ are the Ricci tensor of the connections $\overset{s}{\nabla}$ and $\overset{a}{\nabla}$ respectively.

Contracting (2.17), we get

$$\tilde{r} = r, \tag{2.18}$$

where $\tilde{r}$ and r are the scalar curvature of the connections $\overset{s}{\nabla}$ and $\overset{a}{\nabla}$ respectively.

We obtain from above [25]

$$\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\xi = \overset{a}{R}(\mathcal{I}_1,\mathcal{I}_2)\xi + f\{\eta(\mathcal{I}_2)\omega\mathcal{I}_1 - \eta(\mathcal{I}_1)\omega\mathcal{I}_2\}, \tag{2.19}$$

$$\overset{s}{S}(\mathcal{I}_1,\xi) = \overset{a}{S}(\mathcal{I}_1,\xi), \tag{2.20}$$

$$\tilde{Q}\mathcal{I}_1 = Q\mathcal{I}_1. \tag{2.21}$$

An f-Kenmotsu manifold is said to be a generalized η-Einstein manifold if its Ricci tensor $\overset{a}{S}$ of type (0,2) satisfies [27]

$$\overset{a}{S}(\mathcal{I}_1,\mathcal{I}_2) = l_1 g(\mathcal{I}_1,\mathcal{I}_2) + l_2\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) + l_3 g(\omega\mathcal{I}_1,\mathcal{I}_2),$$

where l_1, l_2 and l_3 are the scalar functions on $\mathcal{M}$. If $l_3 = 0$, then the manifold reduces to an η-Einstein manifold.

3. Projectively Flat f-Kenmotsu Manifolds of Dimensional 3 with QSMC

we study projectively flat f-Kenmotsu manifolds of dimension 3 with respect to QSMC. In a 3-dimensional f-Kenmotsu manifold, the projective curvature tensor with respect to QSMC is given by

$$\tilde{\mathcal{P}}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3 = \overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3 - \frac{1}{2}\Big\{\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1 - \overset{s}{S}(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2\Big\}. \tag{3.1}$$

If $\tilde{\mathcal{P}}$=0, then the manifold $\mathcal{M}$ is called projectively flat manifold with respect to QSMC.

Let $\mathcal{M}$ be a projectively flat manifold admitting quarter-symmetric connection. From (3.1), we have

$$\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3 = \frac{1}{2}\Big\{\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1 - \overset{s}{S}(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2\Big\}. \tag{3.2}$$

Taking the inner product with W in (3.2), we have

$$g(\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3,W) = \frac{1}{2}\Big\{\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)g(\mathcal{I}_1,W) - \overset{s}{S}(\mathcal{I}_1,\mathcal{I}_3)g(\mathcal{I}_2,W)\Big\}. \tag{3.3}$$

Using (2.16) and (2.17) in (3.3), we get

$$\begin{aligned} g(R(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3,W) &+ f\Big\{g(\omega\mathcal{I}_1,W)\eta(\mathcal{I}_2) - g(\omega\mathcal{I}_2,W)\eta(\mathcal{I}_1)\Big\}\eta(\mathcal{I}_3) \\ &+ f\Big\{g(\omega\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1) - g(\omega\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2)\Big\}\eta(W) \\ &= \frac{1}{2}\Big\{\overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3)g(\mathcal{I}_1,W) - \overset{a}{S}(\mathcal{I}_1,\mathcal{I}_3)g(\mathcal{I}_2,W) \\ &+ f(g(\omega\mathcal{I}_2,\mathcal{I}_3)g(\mathcal{I}_1,W) - g(\omega\mathcal{I}_1,\mathcal{I}_3)g(\mathcal{I}_2,W))\Big\}. \end{aligned} \tag{3.4}$$

Putting $W=\xi$ in (3.4), we get

$$-(f^2+f')\Big\{g(\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1)-g(\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2)\Big\}+f\Big\{g(\omega\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1) -g(\omega\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2)\Big\}=\frac{1}{2}\Big\{\overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1)-\overset{a}{S}(\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2) +f(g(\omega\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1)-g(\omega\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2))\Big\}. \tag{3.5}$$

Again putting $\mathcal{I}_1=\xi$ in (3.5), we get

$$\overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3)=-2(f^2+f')g(\mathcal{I}_2,\mathcal{I}_3)+4(f^2+f')\eta(\mathcal{I}_2)\eta(\mathcal{I}_3)+fg(\omega\mathcal{I}_2,\mathcal{I}_3). \tag{3.6}$$

Then $\mathcal{M}$ is a generalized η-Einstein manifold with respect to the Levi-Civita connection.

Now, using (3.6) in (2.17), we have

$$\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)=-2(f^2+f')g(\mathcal{I}_2,\mathcal{I}_3)+4(f^2+f')\eta(\mathcal{I}_2)\eta(\mathcal{I}_3)+2fg(\omega\mathcal{I}_2,\mathcal{I}_3). \tag{3.7}$$

Thus $\mathcal{M}$ is a generalized η-Einstein manifold with respect to quarter-symmetric connection.

Therefore we can state the following:

Theorem 3.1. *Consider $\mathcal{M}$ as a regular f-Kenmotsu manifold of dimension 3 that admits QSMC. If $\mathcal{M}$ is projectively flat in relation to QSMC, then $\mathcal{M}$ becomes an η-Einstein manifold concerning the Levi-Civita connection.*

4. Conharmonically Flat f-Kenmotsu Manifolds of Dimension 3 with QSMC

Now, we study conharmonicall flat 3-dimensional f-Kenmotsu manifolds with respect to QSMC. In a f-Kenmotsu manifold of dimension 3, the conharmonically curvature tensor admitting QSMC is given by

$$\tilde{H}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3=\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3-\Big\{\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1-\overset{s}{S}(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2 +g(\mathcal{I}_2,\mathcal{I}_3)\tilde{Q}\mathcal{I}_1-g(\mathcal{I}_1,\mathcal{I}_3)\tilde{Q}\mathcal{I}_2\Big\}. \tag{4.1}$$

If $\tilde{H}$=0, then the manifold $\mathcal{M}$ is called conharmonically flat with respect to QSMC.

Let $\mathcal{M}$ be a conharmonically flat manifold with respect to QSMC. From (4.2), we get

$$\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3=\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1-\overset{s}{S}(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2 +g(\mathcal{I}_2,\mathcal{I}_3)\tilde{Q}\mathcal{I}_1-g(\mathcal{I}_1,\mathcal{I}_3)\tilde{Q}\mathcal{I}_2. \tag{4.2}$$

Using (2.14), (2.17) and (2.21) in (4.2), we get

$$\begin{aligned}
&\overset{a}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3 + f\{\eta(\mathcal{I}_2)\omega\mathcal{I}_1 - \eta(\mathcal{I}_1)\omega\mathcal{I}_2\}\eta(\mathcal{I}_3)\\
&\qquad + f\{g(\omega\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1) - g(\omega\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2)\}\xi\\
&\qquad = \overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1 - \overset{a}{S}(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2\\
&+ f\{g(\omega\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1 - g(\omega\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2 + g(\mathcal{I}_2,\mathcal{I}_3)\omega\mathcal{I}_1 - g(\mathcal{I}_1,\mathcal{I}_3)\omega\mathcal{I}_2\}\\
&+ (\frac{r}{2} + f^2 + f')\{g(\mathcal{I}_2,\mathcal{I}_3)\mathcal{I}_1 - g(\mathcal{I}_1,\mathcal{I}_3)\mathcal{I}_2\}\\
&- (\frac{r}{2} + 3f^2 + 3f')\{g(\mathcal{I}_2,\mathcal{I}_3)\eta(\mathcal{I}_1) + g(\mathcal{I}_1,\mathcal{I}_3)\eta(\mathcal{I}_2)\}\xi. \qquad (4.3)
\end{aligned}$$

Putting $\mathcal{I}_1 = \xi$ in (4.3) and using (2.8) and (2.9), we obtain

$$\begin{aligned}
\overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3)\xi &= (f^2 + f')g(\mathcal{I}_2,\mathcal{I}_3)\xi + \frac{r}{2}\eta(\mathcal{I}_3)\mathcal{I}_2\\
&\quad - (\frac{r}{2} + 3f^2 + 3f')\eta(\mathcal{I}_2)\eta(\mathcal{I}_3)\xi. \qquad (4.4)
\end{aligned}$$

Taking the inner product with ξ in (4.3), we have

$$\overset{a}{S}(\mathcal{I}_2,\mathcal{I}_3) = (f^2 + f')g(\mathcal{I}_2,\mathcal{I}_3) - 3(f^2 + f')\eta(\mathcal{I}_2)\eta(\mathcal{I}_3). \qquad (4.5)$$

Thus $\mathcal{M}$ is an η-Einstein manifold with respect to the Levi-Civita connection. Therefore we can state the following:

Theorem 4.1. *Consider $\mathcal{M}$ as a regular f-Kenmotsu manifold of dimension 3 that admits QSMC. If $\mathcal{M}$ is conharmonically flat in relation to QSMC, then $\mathcal{M}$ becomes an η-Einstein manifold concerning the Levi-Civita connection.*

5. η-Ricci Soliton on f-Kenmotsu Manifolds of Dmension 3 with QSMC

Let (g,ξ,λ,μ) be an η-Ricci soliton on a three-dimensional f-Kenmotsu manifold with respect to QSMC. Then we have

$$(\tilde{\mathcal{L}}_\xi g)(\mathcal{I}_2,\mathcal{I}_3) + 2\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3) + 2\lambda g(\mathcal{I}_2,\mathcal{I}_3) + 2\mu\eta(\mathcal{I}_2)\eta(\mathcal{I}_3) = 0, \qquad (5.1)$$

where $\tilde{\mathcal{L}}_\xi$ is the Lie derivative along the vector field ξ on $\mathcal{M}$ and $\overset{s}{S}$ is the Ricci curvature tensor field with respect to QSMC $\overset{s}{\nabla}$, and λ and μ are real constants.

Using (2.14) and (2.17), we get

$$\begin{aligned}
2\overset{s}{S}(\mathcal{I}_2,\mathcal{I}_3) &= -g(\overset{s}{\nabla}_{\mathcal{I}_2}\xi,\mathcal{I}_3) - g(\mathcal{I}_2,\overset{s}{\nabla}_{\mathcal{I}_3}\xi) - 2\lambda g(\mathcal{I}_2,\mathcal{I}_3) - 2\mu\eta(\mathcal{I}_2)\eta(\mathcal{I}_3)\\
&= -2f\{g(\mathcal{I}_2,\mathcal{I}_3) - \eta(\mathcal{I}_2)\eta(\mathcal{I}_3)\} - 2\lambda g(\mathcal{I}_2,\mathcal{I}_3) - 2\mu\eta(\mathcal{I}_2)\eta(\mathcal{I}_3). \qquad (5.2)
\end{aligned}$$

So, from (5.2) we have

$$S(\mathcal{I}_2,\mathcal{I}_3) = -(f+\lambda)g(\mathcal{I}_2,\mathcal{I}_3) + (f-\mu)\eta(\mathcal{I}_2)\eta(\mathcal{I}_3) - 2fg(\omega\mathcal{I}_2,\mathcal{I}_3). \qquad (5.3)$$

Thus, we have:

Theorem 5.1. *Consider $\mathcal{M}$ as a regular f-Kenmotsu manifold of dimension 3 that admits QSMC. If (g, ξ, λ, μ) represents an η-Ricci soliton on a 3-dimensional f-Kenmotsu manifold with QSMC, then $\mathcal{M}$ becomes a generalized η-Einstein manifold that supports a Levi-Civita connection.*

Putting $\mathcal{I}_3 = \xi$ in (5.3) and using (2.9), we get

$$\lambda + \mu = 2(f^2 + f'). \tag{5.4}$$

Hence we can state the following:

Theorem 5.2. *Consider $\mathcal{M}$ as a regular f-Kenmotsu manifold of dimension 3 that admits QSMC. If (g, ξ, λ, μ) represents an η-Ricci soliton on a 3-dimensional f-Kenmotsu manifold with QSMC, then the η-Ricci soliton on $\mathcal{M}$ is expanding, steady or shrinking according as $\mu < 2(f^2+f')$, $\mu = 2(f^2+f')$ or $\mu > 2(f^2 + f')$.*

Let $\mathcal{V}$ be pointwise colinear with ξ i.e., $\mathcal{V} = b\xi$, where b is a function on f-Kenmotsu manifold with respect to QSMC. Then

$$\pounds_V g(\mathcal{I}_1, \mathcal{I}_2) + 2\overset{s}{S}(\mathcal{I}_1, \mathcal{I}_2) + 2\lambda g(\mathcal{I}_1, \mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) = 0,$$

implies

$$g(\overset{s}{\nabla}_{\mathcal{I}_1} b\xi, \mathcal{I}_2) + g(\overset{s}{\nabla}_{\mathcal{I}_2} b\xi, \mathcal{I}_1) + 2\overset{s}{S}(\mathcal{I}_1, \mathcal{I}_2) + 2\lambda g(\mathcal{I}_1, \mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) = 0, \tag{5.5}$$

or

$$\begin{aligned} & bg(\overset{s}{\nabla}_{\mathcal{I}_1}\xi, \mathcal{I}_2) + (\mathcal{I}_1 b)\eta(\mathcal{I}_2) + bg(\overset{s}{\nabla}_{\mathcal{I}_2}\xi, \mathcal{I}_1) + (\mathcal{I}_2 b)\eta(\mathcal{I}_1) \\ & + 2\overset{s}{S}(\mathcal{I}_1, \mathcal{I}_2) + 2\lambda g(\mathcal{I}_1, \mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) = 0. \end{aligned} \tag{5.6}$$

Using (2.14), we get

$$\begin{aligned} & 2bf[g(\mathcal{I}_1, \mathcal{I}_2) - \eta(\mathcal{I}_1)\eta(\mathcal{I}_2)] + (\mathcal{I}_1 b)\eta(\mathcal{I}_2) + (\mathcal{I}_2 b)\eta(\mathcal{I}_1) \\ & + 2\overset{a}{S}(\mathcal{I}_1, \mathcal{I}_2) + 2fg(\omega\mathcal{I}_1, \mathcal{I}_2) + 2\lambda g(\mathcal{I}_1, \mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) = 0. \end{aligned} \tag{5.7}$$

In (5.7) replacing $\mathcal{I}_2$ by ξ, it follows that

$$(\mathcal{I}_1 b) + (\xi b)\eta(\mathcal{I}_1) + 2(\lambda + \mu - 2(f^2 + f'))\eta(\mathcal{I}_1) = 0. \tag{5.8}$$

Again putting $\mathcal{I}_1 = \xi$ in (5.8), we obtain

$$(\xi b) = 2(f^2 + f') - \lambda - \mu. \tag{5.9}$$

Putting this value in (5.8), we get

$$(\mathcal{I}_1 b) = [2(f^2 + f') - \lambda - \mu]\eta(\mathcal{I}_1), \tag{5.10}$$

or

$$db = [2(f^2 + f') - \lambda - \mu]\eta. \tag{5.11}$$

Applying d on (5.11), we get

$$[2(f^2+f')-\lambda-\mu]d\eta = 0.$$

Since $d\eta \neq 0$, we have

$$2(f^2+f')-\lambda-\mu = 0. \tag{5.12}$$

Using (5.12) in (5.11) yields b is a constant. Therefore from (5.7) it follows that

$$\overset{a}{S}(\mathcal{I}_1,\mathcal{I}_2) = -(bf+\lambda)g(\mathcal{I}_1,\mathcal{I}_2)+(bf-\mu)\eta(\mathcal{I}_1)\eta(\mathcal{I}_2)-fg(\omega\mathcal{I}_1,\mathcal{I}_2),$$

which implies that $\mathcal{M}$ is a generalized η-Einstein manifold with respect to the Levi-Civita connection.

Thus, we can state the following theorem:

Theorem 5.3. *Consider $\mathcal{M}$ as a regular f-Kenmotsu manifold of dimension 3 that admits QSMC. If (g,ξ,λ,μ) represents an η-Ricci soliton on a 3-dimensional f-Kenmotsu manifold admitting QSMC and $\mathcal{V}$ is positive collinear with ξ, then $\mathcal{M}$ is a generalized η-Einstein manifold that supports a Levi-Civita connection.*

6. **Example**

We consider the three-dimensional manifold $\mathcal{M} = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in R^3 [24]. Let $\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3$ be linearly independent vector fields at each point of $\mathcal{M}$, given by

$$\mathcal{J}_1 = e^y\frac{\partial}{\partial x}, \qquad \mathcal{J}_2 = e^y\frac{\partial}{\partial z}, \qquad \mathcal{J}_3 = \frac{\partial}{\partial y}.$$

Let g be the Riemannian metric such that

$$g(\mathcal{J}_1,\mathcal{J}_3) = g(\mathcal{J}_2,\mathcal{J}_3) = g(\mathcal{J}_1,\mathcal{J}_2) = 0, \qquad g(\mathcal{J}_1,\mathcal{J}_1) = g(\mathcal{J}_2,\mathcal{J}_2) = g(\mathcal{J}_3,\mathcal{J}_3) = 1.$$

Let η be the 1-form defined by

$$\eta(\mathcal{I}_3) := g(\mathcal{I}_3,\mathcal{J}_3), \quad \forall \mathcal{I}_3 \in \chi(\mathcal{M}).$$

Let ω be the (1,1) tensor field defined by

$$\omega(\mathcal{J}_1) = -\mathcal{J}_2,\ \omega(\mathcal{J}_2) = \mathcal{J}_1,\ \omega(\mathcal{J}_3) = 0.$$

Then using the linearity of ω and g we have

$$\eta(\mathcal{J}_3) = 1, \qquad \omega^2\mathcal{I}_3 = -\mathcal{I}_3+\eta(\mathcal{I}_3)\mathcal{J}_3,$$
$$g(\omega\mathcal{I}_3,\omega W) = g(\mathcal{I}_3,W)-\eta(\mathcal{I}_3)\eta(W),$$

for any $\mathcal{I}_3, W \in \chi(M)$. Thus for $\mathcal{J}_3 = \xi$, (ω,ξ,η,g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[\mathcal{J}_1,\mathcal{J}_2] = 0, \qquad [\mathcal{J}_2,\mathcal{J}_3] = -\mathcal{J}_2, \qquad [\mathcal{J}_1,\mathcal{J}_3] = -\mathcal{J}_1.$$

In [24] the authors obtained the expression as follows:

$$\overset{a}{\nabla}_{\mathcal{J}_1}\mathcal{J}_3 = -\mathcal{J}_1, \qquad \overset{a}{\nabla}_{\mathcal{J}_1}\mathcal{J}_2 = 0, \qquad \overset{a}{\nabla}_{\mathcal{J}_1}\mathcal{J}_1 = \mathcal{J}_3,$$
$$\overset{a}{\nabla}_{\mathcal{J}_2}\mathcal{J}_3 = -\mathcal{J}_2, \qquad \overset{a}{\nabla}_{\mathcal{J}_2}\mathcal{J}_2 = \mathcal{J}_3, \qquad \overset{a}{\nabla}_{\mathcal{J}_2}\mathcal{J}_1 = 0,$$
$$\overset{a}{\nabla}_{\mathcal{J}_3}\mathcal{J}_3 = 0, \qquad \overset{a}{\nabla}_{\mathcal{J}_3}\mathcal{J}_2 = 0, \qquad \overset{a}{\nabla}_{\mathcal{J}_3}\mathcal{J}_1 = 0.$$

From above we see that the manifold M satisfies the condition

$$\overset{a}{\nabla}_{\mathcal{I}_1}\xi = f\{\mathcal{I}_1 - \eta(\mathcal{I}_1)\xi\}, \quad \text{for} \ \ \xi = \mathcal{J}_3,$$

where $f = -1$. Hence the manifold is a f-Kenmotsu manifold. Also $f^2+f' \neq 0$. Hence M is a regular f-Kenmotsu manifold.

Now using above relations in (2.14) we have

$$\overset{s}{\nabla}_{\mathcal{J}_1}\mathcal{J}_3 = -\mathcal{J}_1, \qquad \overset{s}{\nabla}_{\mathcal{J}_1}\mathcal{J}_2 = 0, \qquad \overset{s}{\nabla}_{\mathcal{J}_1}\mathcal{J}_1 = \mathcal{J}_3,$$
$$\overset{s}{\nabla}_{\mathcal{J}_2}\mathcal{J}_3 = -\mathcal{J}_2, \qquad \overset{s}{\nabla}_{\mathcal{J}_2}\mathcal{J}_2 = \mathcal{J}_3, \qquad \overset{s}{\nabla}_{\mathcal{J}_2}\mathcal{J}_1 = 0,$$
$$\overset{s}{\nabla}_{\mathcal{J}_3}\mathcal{J}_3 = 0, \qquad \overset{s}{\nabla}_{\mathcal{J}_3}\mathcal{J}_2 = -\mathcal{J}_1, \qquad \overset{s}{\nabla}_{\mathcal{J}_3}\mathcal{J}_1 = \mathcal{J}_2.$$

We known that

$$\overset{s}{R}(\mathcal{I}_1,\mathcal{I}_2)\mathcal{I}_3 = \overset{s}{\nabla}_{\mathcal{I}_1}\overset{s}{\nabla}_{\mathcal{I}_2}\mathcal{I}_3 - \overset{s}{\nabla}_{\mathcal{I}_2}\overset{s}{\nabla}_{\mathcal{I}_1}\mathcal{I}_3 - \overset{s}{\nabla}_{[\mathcal{I}_1,\mathcal{I}_2]}\mathcal{I}_3.$$

With the help of the above results, it gives us:

$$\overset{s}{R}(\mathcal{J}_1,\mathcal{J}_2)\mathcal{J}_3 = 0, \qquad \overset{s}{R}(\mathcal{J}_2,\mathcal{J}_3)\mathcal{J}_3 = -\mathcal{J}_1 - \mathcal{J}_2, \qquad \overset{s}{R}(\mathcal{J}_1,\mathcal{J}_3)\mathcal{J}_3 = \mathcal{J}_2 - \mathcal{J}_1,$$
$$\overset{s}{R}(\mathcal{J}_1,\mathcal{J}_2)\mathcal{J}_2 = -\mathcal{J}_1, \qquad \overset{s}{R}(\mathcal{J}_2,\mathcal{J}_3)\mathcal{J}_2 = \mathcal{J}_3, \qquad \overset{s}{R}(\mathcal{J}_1,\mathcal{J}_3)\mathcal{J}_2 = -\mathcal{J}_3,$$
$$\overset{s}{R}(\mathcal{J}_1,\mathcal{J}_2)\mathcal{J}_1 = -\mathcal{J}_2, \qquad \overset{s}{R}(\mathcal{J}_2,\mathcal{J}_3)\mathcal{J}_1 = \mathcal{J}_3, \qquad \overset{s}{R}(\mathcal{J}_1,\mathcal{J}_3)\mathcal{J}_1 = \mathcal{J}_3.$$

From the above expressions the components of the Ricci tensor with respect to QSMC as follows:

$$\overset{s}{S}(\mathcal{J}_1,\mathcal{J}_1) = \overset{s}{S}(\mathcal{J}_2,\mathcal{J}_2) = \overset{s}{S}(\mathcal{J}_3,\mathcal{J}_3) = -2.$$

Therefore for $\mathcal{I}_1 = a_1\mathcal{J}_1 + a_2\mathcal{J}_2 + a_3\mathcal{J}_3$ and $\mathcal{I}_2 = b_1\mathcal{J}_1 + b_2\mathcal{J}_2 + b_3\mathcal{J}_3$, we have

$$\begin{aligned}(\tilde{\mathfrak{L}}_\xi g)(\mathcal{I}_1,\mathcal{I}_2) + 2\overset{s}{S}(\mathcal{I}_1,\mathcal{I}_2) + 2\lambda g(\mathcal{I}_1,\mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) &= (-6+2\lambda)a_1b_1 \\ &+ (-6+2\lambda)a_2b_2 \\ &+ (-4+2\lambda+2\mu)a_3b_3.\end{aligned} \tag{6.1}$$

From (6.1) it is clear that for $\lambda = 3$ and $\mu = -1$

$$(\tilde{\mathfrak{L}}_\xi g)(\mathcal{I}_1,\mathcal{I}_2) + 2\overset{s}{S}(\mathcal{I}_1,\mathcal{I}_2) + 2\lambda g(\mathcal{I}_1,\mathcal{I}_2) + 2\mu\eta(\mathcal{I}_1)\eta(\mathcal{I}_2) = 0.$$

Therefore $(M, g, \xi, \lambda, \mu)$ is an η-Ricci soliton with respect to QSMC for $\lambda = 3$ and $\mu = -1$. Also $\lambda + \mu = 2 = 2(f^2 + f')$, which verifies the Theorem 5.1.

References

1. Z. Ahsan, *Tensors: Mathematics of Diffential Geometry and Relativity*, PHI Learning Pvt. Ltd., Delhi, December 2017, Second Printing.
2. S. Ali and R. Nivas, *On submanifolds immersed in a manifold with quarter symmetric connection*, Rivista di Matematica della Università di Par. **6**(3) (2000), 11-23.
3. G. Ayar and M. Yildirim, *η-Ricci solitons on nearly Kenmotsu manifolds*, Asian-European J. of Math. **12**(6) (2019), 2040002 (8 pages).
4. S. C. Biswas and U.C. De, *Quarter-symmetric metric connection in an SP-Sasakian manifold*, Communications, Faculty of Sciences. University of Ankara Series A., **46**(1997), 49-56.
5. A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20**(1) (2015), 1-13.
6. A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat., **30**(2) (2016), 489–496.
7. C. Călin and M. Crasmareanu, *η-Ricci solitons on Hopf hypersurfaces in complex space forms*, Revue Roumaine Math. pures Appl. **57**(1) (2012), 55–63.
8. B. Chow, P. Lu and I. Ni, *Hamilton's ricci flow, Graduate Studies in Mathematics*, **77**, AMS, Providence, RI, USA, 2006.
9. J.T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in complex space form*, Tohoku Math. J. **61**(2) (2009), 205–212.
10. S. Deshmukh, *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Rouma. **55**(1) (2012), 41–50.
11. S. Deshmukh, H. Alodan and H. Al-Sodais , *A note on Ricci solitons*, Balkan J. Geom. Appl. **16**(1) (2011), 48–55.
12. A. Friedman and J. A. Schouten, *Uber die Geometrie der halbsymmetrischen Ubertragung*, Math. Zeitschr. **21**(1924), 211–223.
13. U. K. Gautam, A. Haseeb and R. Prasad, *Some results on projective curvature tensor in Sasakian manifolds*, Commun. Korean Math. Soc. **34**(3) (2019), 881–896.
14. S. Golab, *On semi-ymmetric and quater-symmetric linear connections*, Tensor N. S. **29**(1975), 249–254.
15. R. S. Hamilton, *The Ricci flow on surfaces*, Math. and general relativity (Santa Cruz, CA, 1986), Contemp. Math. **71**(1988), 237-262.
16. T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3**(4) (1993), 301–307.
17. D. Janssens and L. Vanheck , *Almost contact structures and curvature tensors*, Kodai Math. J. **4**(1981), 1–27.
18. K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku math. J. **24**(2) (1972), 93–103.
19. R. S. Mishra and S. N. Pandey, *On quarter symmetric metric F-connections*, Tensor: New Seri. **34**(1) (1980), 1-7.
20. S. Mukhopadhyay, A. K. Roy and B. Barua, *Some properties of a quarter symmetric metric connection on a Riemannian manifold*, Soochow Journal of Math. **17**(1991), 205-211.
21. Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57**(1) (1989), 73–87.
22. Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectice manifolds*, Publ. Math. Debrecen., **39**(3) (1991), 315–323.
23. D. G. Prakasha and B. S. Hadimani, *Ricci solitons on Lorentzian para-Sasakian manifolds*, J. Geom. DOI:10.1007/s00022-016-0345-z.

24. R. Prasad, A. Haseeb and U. K. Gautam, *On three dimensional f-Kenmotsu manifolds with a certain connection*, Global J. Advd. Res. Class. Mod. Geom. **10**(1) (2021), 91–101.
25. A. Sarkar and N. Biswas, *On f-Kenmotsu manifolds and their submanifolds with quater-symmetric metric metric connection*, Facta Univ. (Nis) Ser. Math. Inform. **35**(4) (2020), 1017–1030.
26. K. Yano and T. Imai, *Quarter-symmetric metric connections and their curvature tensors*, Tensor: New Series. **38**(1982), 13-18.
27. K. Yano and M. Kon, Structures on Manifolds, Series in Pure Math., Vol. **3**, World Sci. Publishing Co., Singapore, 1984.
28. A. Yildiz, U. C. De and M. Turan, *On 3-dimensional f-Kenmotsu manifolds and Ricci solitons*, Ukrainian J. Math. **65**(5) (2013), 620-628.

Received: 01.04.2025
Accepted: 02.07.2025