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On Berwald-Matsumoto type Finsler metrics

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Abstract. In this paper, we looked at the basic properties of the Berwald and Douglas spaces of a Finsler space with a deformed Berwald-Matsumoto metric. We also examined the conditions that make the Finsler space, with the deformed Berwald-Matsumoto metric, a Berwald and Douglas space.

Keywords: Finsler space, Berwald space, Douglas space, (α, β)-metric, Berwald-Matsumoto metric.

1. Introduction

P. Finsler was the first to present the slope of a mountain for measuring time, although he considered it was a typical model of the Finsler metric (as noted in his letter [13] to Matsumoto). Matsumoto [13] went on to work on the problem in 1989 and introduced the idea of the slope metric, which is defined as

$$L = \frac{\alpha^2}{v\alpha - w\beta},$$

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where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on n-dimensional manifold M^n, and w and v are the non-zero constants. In 1990, Aikou and coauthors [1] examined the aforementioned measure in depth and dubbed it the Matsumoto metric. They achieved interesting findings for this metric in comparison to the Finsler metric. For more progress, see [7] and [15].

Berwald [5] established the Finsler metric in 1929. It is defined on the unit ball $B^n(1)$ and includes all straight line segments. Its geodesics have constant flag curvature $K = 0$ and take the form of

$$L = \frac{\{\sqrt{1 - |x|^2|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\}^2}{\{1 - |x|^2\}^2\sqrt{1 - |x|^2|y|^2 + \langle x, y\rangle^2}}. \tag{1.1}$$

From a modern perspective, Berwald's metric corresponds to a unique sort of Finsler metric called Berwald type metric, which is defined as

$$L = \frac{(\alpha + \beta)^2}{\alpha}.$$

See [16]. The authors of the papers provided highly important results in the field of Finsler geometry.

In 2018, Chaubey and Tripathi [6] merged the Berwald and Matsumoto metrics, naming it the Berwald-Matsumoto metric. They investigated the fundamental characteristics of Finsler space and numerous hypersurfaces using this essential metric. In this study, we combine the Berwald and Matsumoto metrics to produce a new metric known as the deformed Berwald-Matsumoto metric. We also investigate the conditions under which the Finsler space F^n with Berwald-Matsumoto metric is both a Berwald space and a Douglas space.

2. **Preliminaries**

Here we investigate an n-dimensional Finsler space $F^n = (M^n, L(\alpha, \beta))$, that is, a pair consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L as a particular Finsler space with the metric

$$L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha} + \frac{\alpha^2}{\alpha - \beta}, \tag{2.1}$$

i.e., the deformed Berwald-Matsumoto metric [6] is the combination of Berwald and Matsumoto metrics, and the Finsler space F^n with this metric is called the Berwald-Matsumoto Finsler space.

The geodesics of a Finsler space $F^n = (M^n, L)$ are provided by the system of differential equations that include the function

$$G^i(x, y) = \frac{1}{4}g^{ij}(y^r\dot{\partial}_j\partial_r L^2 - \partial_j L^2).$$

The corresponding Riemannian space for an (α,β)-metric $L(\alpha,\beta)$ is $R^n = (M^n,\alpha)$ with $F^n = \{M^n, L(\alpha,\beta)\}$ [10, 2]. (;) represents the covariant differentiation with regard to the Levi-Civita connection $\gamma^i_{jk}(x)$ of R^n. We write $(a^{ij}) = (a_{ij})^{-1}$ and use the following symbols:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}),\ s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),\ r^i_j = a^{ir} r_{rj},\ s^i_j = a^{ir} s_{rj},\ r_j = b_r r^r_j,$$
$$s_j = b_r s^r_j,\ b^i = a^{ir} b_r,\ b^2 = a^{rs} b_r b_s.$$

On the basis of [11], if $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$, therefore the function $G^i(x,y)$ of F^n with an (α,β)-metric is expressed in the form

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.2}$$
$$B^i = \alpha\frac{L_\beta}{L_\alpha} s^i_0 + \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \alpha\frac{L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right)\right\} C^*,$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_\beta = \frac{\partial L}{\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha}.$$

The subscript '0' indicates a contraction by y^i, and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2(\alpha\gamma^2 L_{\alpha\alpha} + \beta^2 L_\alpha)}. \tag{2.3}$$

For clarity, we will represent the homogeneous polynomials in (y^i) of degree r as $hp(r)$. For example, γ^i_{00} is $hp(2)$.
Based on (2.2), the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with an (α,β)-metric is defined as

$$G^i_j = \dot{\partial}_j G^i = \gamma^i_{0j} + B^i_j,$$
$$G^i_{jk} = \dot{\partial}_k G^i_j = \gamma^i_{jk} + B^i_{jk},$$

where we placed

$$B^i_j = \dot{\partial}_j B^i, \quad B^i_{jk} = \dot{\partial}_k B^i_j.$$

$B^i(x,y)$ is known as the *difference vector* [11]. According to [11], B^i_{jk} is defined as

$$L_\alpha B^t_{ji} y^j y_t + \alpha L_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0, \tag{2.4}$$

where $y_k = a_{ik} y^i$.

A Finsler space F^n with an (α,β)-metric is a Douglas space, if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$. Several authors [14, 3] examined the features of this space in depth. Based on (2.2), B^{ij} is written as follows:

$$B^{ij} = \alpha\frac{L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}(b^i y^j - b^j y^i) C^*. \tag{2.5}$$

Lemma 2.1. [4]. *If $\alpha^2 \equiv 0(mod\beta)$, that is, α^2 contains β as a factor, then the dimension equals to two and b^2 vanishes. In this scenario, $\delta = d_i(x)y^i$, where $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

3. The condition for F^n to be a Berwald space

In this section, we discover the conditions under which a Finsler space F^n with a deformed Berwald-Matsumoto metric becomes a Berwald space. In a n-dimensional Finsler space F^n with deformed Berwald-Matsumoto metric, we obtain the following results:

$$\begin{aligned} L_\alpha &= \frac{(\alpha-2\beta)\alpha^3+(\alpha+\beta)(\alpha-\beta)^3}{(\alpha-\beta)^2\alpha^2}, \\ L_\beta &= \frac{\alpha^3+2(\alpha+\beta)(\alpha-\beta)^2}{(\alpha-\beta)^2\alpha}, \\ L_{\alpha\alpha} &= \frac{2\beta^2\left\{\alpha^3+(\alpha-\beta)^3\right\}}{(\alpha-\beta)^3\alpha^3}, \\ L_{\beta\beta} &= \frac{2\left\{\alpha^3+(\alpha-\beta)^3\right\}}{(\alpha-\beta)^3\alpha}. \end{aligned} \tag{3.1}$$

By substituting (3.1) into (2.4), we obtain

$$\begin{aligned} &\Big\{(2\alpha^4-\beta^4)B^t_{ji}y^jy_t+2\alpha^2\beta(\beta^2-\alpha^2)(B^t_{ji}b_t-b_{j;i})y^j\Big\} \\ &+\alpha\Big\{2\beta(\beta^2-2\alpha^2)B^t_{ji}y^jy_t+\alpha^2(3\alpha^2-2\beta^2)(B^t_{ji}b_t-b_{j;i})y^j\Big\}=0. \end{aligned} \tag{3.2}$$

Assume F^n is a Berwald space, where $G^i_{jk} = G^i_{jk}(x)$. Then, $B^t_{ji} = B^t_{ji}(x)$. α is irrational in (y^i), therefore from (3.2) we obtain

$$(2\alpha^4-\beta^4)B^t_{ji}y^jy_t+2\alpha^2\beta(\beta^2-\alpha^2)(B^t_{ji}b_t-b_{j;i})y^j=0$$

and

$$2\beta(\beta^2-2\alpha^2)B^t_{ji}y^jy_t+\alpha^2(3\alpha^2-2\beta^2)(B^t_{ji}b_t-b_{j;i})y^j=0,$$

which may be expressed as a matrix of homogeneous linear equations,

$$AX = 0$$

$$\begin{bmatrix} (2\alpha^4-\beta^4) & 2\alpha^2\beta(\beta^2-\alpha^2) \\ 2\beta(\beta^2-2\alpha^2) & \alpha^2(3\alpha^2-2\beta^2) \end{bmatrix} \begin{bmatrix} B^t_{ji}y^jy_t \\ (B^t_{ji}b_t-b_{j;i})y^j \end{bmatrix} = 0.$$

Let,

$$A = \begin{bmatrix} (2\alpha^4-\beta^4) & 2\alpha^2\beta(\beta^2-\alpha^2) \\ 2\beta(\beta^2-2\alpha^2) & \alpha^2(3\alpha^2-2\beta^2) \end{bmatrix},$$

where

$$|A| = 6\alpha^8 + 9\alpha^4\beta^4 - 12\alpha^6\beta^2 - 2\alpha^2\beta^6 \neq 0,$$

This suggests

$$B^t_{ji} y^j y_t = 0 \quad \text{and} \quad (B^t_{ji} b_t - b_{j;i}) y^j = 0,$$

which show

$$B^t_{ji} a_{th} + B^t_{hi} a_{tj} = 0 \quad \text{and} \quad (B^t_{ji} b_t - b_{j;i}) y^j = 0.$$

The former gives $B^t_{ji} = 0$ by the renowned Christoffel procedure yields $b_{j;i} = 0$. Therefore, $r_{ij} = 0$ and $s_{ij} = 0$. However, if $b_{j;i} = 0$, then $B^t_{ji} = 0$ are uniquely determined from (3.2). So, we have

Theorem 3.1. *The Finsler space F^n with deformed Berwald-Matsumoto metric is a Berwald space if and only if $b_{j;i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma^i_{jk}, \gamma^i_{0j}, 0)$.*

Theorem 3.2. *The Finsler space F^n with deformed Berwald-Matsumoto metric is a Berwald space if and only if $r_{ij} = 0$ and $s_{ij} = 0$.*

4. The condition for F^n to be a Douglas space

In this section, we will investigate the condition that a Finsler space F^n with deformed Berwald-Matsumoto metric is a Douglas space. Substituting (3.1) into (2.5), we get

$$\begin{aligned}
&\{4\alpha^9 + 8b^2\alpha^9 - 20\alpha^8\beta - 28b^2\alpha^8\beta + 24\alpha^7\beta^2 + 36b^2\alpha^7\beta^2 + 20\alpha^6\beta^3 \\
&-20b^2\alpha^6\beta^3 - 62\alpha^5\beta^4 - 8b^2\alpha^5\beta^4 + 48\alpha^4\beta^5 + 18b^2\alpha^4\beta^5 + 4\alpha^3\beta^6 \\
&-10b^2\alpha^3\beta^6 - 26\alpha^2\beta^7 + 2b^2\alpha^2\beta^7 + 15\alpha\beta^8 - 3\beta^9\}B^{ij} \\
&-\alpha^2\{6\alpha^8 + 12b^2\alpha^8 - 22\alpha^7\beta - 26b^2\alpha^7\beta + 8\alpha^6\beta^2 + 22b^2\alpha^6\beta^2 \\
&+40\alpha^5\beta^3 + 2b^2\alpha^5\beta^3 - 55\alpha^4\beta^4 - 20b^2\alpha^4\beta^4 + 11\alpha^3\beta^5 + 16b^2\alpha^3\beta^5 \\
&+28\alpha^2\beta^6 - 24\alpha\beta^7 - 4b^2\alpha^2\beta^6 + 6\beta^8\}(s^i_0 y^j - s^j_0 y^i) \\
&-\alpha^2\{r_{00}(4\alpha^7 - 14\alpha^6\beta + 18\alpha^5\beta^2 - 10\alpha^4\beta^3 - 4\alpha^3\beta^4 \\
&+9\alpha^2\beta^5 - 5\alpha\beta^6 + \beta^7) - 2s_0\alpha^2(6\alpha^6 - 13\alpha^5\beta + 11\alpha^4\beta^2 \\
&+\alpha^3\beta^3 - 10\alpha^2\beta^4 + 8\alpha\beta^5 - 2\beta^6)\}(b^i y^j - b^j y^i) = 0.
\end{aligned} \tag{4.1}$$

It is noteworthy given that $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$.

Assume that F^n is a Douglas space, where B^{ij} are $hp(3)$. We can separate (4.1) into rational and irrational terms of y^i since α is irrational in (y^i); we have

$$\begin{aligned}
&\{-20\alpha^8\beta - 28b^2\alpha^8\beta + 20\alpha^6\beta^3 - 20b^2\alpha^6\beta^3 + 48\alpha^4\beta^5 + 18b^2\alpha^4\beta^5 \\
&-26\alpha^2\beta^7 + 2b^2\alpha^2\beta^7 - 3\beta^9\}B^{ij} - \alpha^2\{6\alpha^8 + 12b^2\alpha^8 + 8\alpha^6\beta^2
\end{aligned} \tag{4.2}$$

$$+22b^2\alpha^6\beta^2 - 55\alpha^4\beta^4 - 20b^2\alpha^4\beta^4 + 28\alpha^2\beta^6 - 4b^2\alpha^2\beta^6 + 6\beta^8\}$$
$$(s^i_0 y^j - s^j_0 y^i) - \alpha^2\{r_{00}(-14\alpha^6\beta - 10\alpha^4\beta^3 + 9\alpha^2\beta^5 + \beta^7)$$
$$-2s_0\alpha^2(6\alpha^6 + 11\alpha^4\beta^2 - 10\alpha^2\beta^4 - 2\beta^6)\}(b^i y^j - b^j y^i)$$
$$+\alpha[\{4\alpha^8 + 8b^2\alpha^8 + 24\alpha^6\beta^2 + 36b^2\alpha^6\beta^2 - 62\alpha^4\beta^4 - 8b^2\alpha^4\beta^4$$
$$+4\alpha^2\beta^6 - 10b^2\alpha^2\beta^6 + 15\beta^8\}B^{ij} - \alpha^2\{-22\alpha^6\beta - 26b^2\alpha^6\beta$$
$$+40\alpha^4\beta^3 + 2b^2\alpha^4\beta^3 + 11\alpha^2\beta^5 + 16b^2\alpha^2\beta^5 - 24\beta^7\}(s^i_0 y^j - s^j_0 y^i)$$
$$-\alpha^2\{r_{00}(4\alpha^6 + 18\alpha^4\beta^2 - 4\alpha^2\beta^4 - 5\beta^6) - 2s_0\alpha^2(-13\alpha^4\beta$$
$$+\alpha^2\beta^3 + 8\beta^5)\}(b^i y^j - b^j y^i) = 0.$$

Thus, equation (4.2) is split into two equations as follows:

$$\begin{aligned}\beta\{&-20\alpha^8 - 28b^2\alpha^8 + 20\alpha^6\beta^2 - 20b^2\alpha^6\beta^2 + 48\alpha^4\beta^4 + 18b^2\alpha^4\beta^4 \\ &-26\alpha^2\beta^6 + 2b^2\alpha^2\beta^6 - 3\beta^8\}B^{ij} - \alpha^2\{6\alpha^8 + 12b^2\alpha^8 + 8\alpha^6\beta^2 \\ &+22b^2\alpha^6\beta^2 - 55\alpha^4\beta^4 - 20b^2\alpha^4\beta^4 + 28\alpha^2\beta^6 - 4b^2\alpha^2\beta^6 + 6\beta^8\} \\ &(s^i_0 y^j - s^j_0 y^i) - \alpha^2\{r_{00}\beta(-14\alpha^6 - 10\alpha^4\beta^2 + 9\alpha^2\beta^4 + \beta^6) \\ &-2s_0\alpha^2(6\alpha^6 + 11\alpha^4\beta^2 - 10\alpha^2\beta^4 - 2\beta^6)\}(b^i y^j - b^j y^i) = 0\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\{4\alpha^8 + 8b^2\alpha^8 + 24\alpha^6\beta^2 + 36b^2\alpha^6\beta^2 - 62\alpha^4\beta^4 - 8b^2\alpha^4\beta^4 \\ &+4\alpha^2\beta^6 - 10b^2\alpha^2\beta^6 + 15\beta^8\}B^{ij} - \alpha^2\beta\{-22\alpha^6 - 26b^2\alpha^6 \\ &+40\alpha^4\beta^2 + 2b^2\alpha^4\beta^2 + 11\alpha^2\beta^4 + 16b^2\alpha^2\beta^4 - 24\beta^6\}(s^i_0 y^j - s^j_0 y^i) \\ &-\alpha^2\{r_{00}(4\alpha^6 + 18\alpha^4\beta^2 - 4\alpha^2\beta^4 - 5\beta^6) - 2s_0\alpha^2\beta(-13\alpha^4 \\ &+\alpha^2\beta^2 + 8\beta^4)\}(b^i y^j - b^j y^i) = 0.\end{aligned} \tag{4.4}$$

Eliminating B^{ij} from (4.3) and (4.4), we get

$$A(s^i_0 y^j - s^j_0 y^i) + B(b^i y^j - b^j y^i) = 0, \tag{4.5}$$

where

$$\begin{aligned}A = &-24\alpha^{16} - 96b^2\alpha^{16} - 96b^4\alpha^{16} + 264\alpha^{14}\beta^2 + 480b^2\alpha^{14}\beta^2 \\ &+120b^4\alpha^{14}\beta^2 - 840b^2\alpha^{12}\beta^4 - 1414b^2\alpha^{12}\beta^4 - 72b^4\alpha^{12}\beta^4 + 1206\alpha^{10}\beta^6 \\ &+660b^2\alpha^{10}\beta^6 + 92b^4\alpha^{10}\beta^6 - 1036\alpha^8\beta^8 - 2606b^2\alpha^8\beta^8 - 132b^4\alpha^8\beta^8 \\ &+1726\alpha^6\beta^{10} + 512b^2\alpha^6\beta^{10} + 60b^4\alpha^6\beta^{10} - 473\alpha^4\beta^{12} - 188b^2\alpha^4\beta^{12} \\ &-8b^4\alpha^4\beta^{12} + 147\alpha^2\beta^{14} + 24b^2\alpha^2\beta^{14} - 18\beta^{16},\end{aligned} \tag{4.6}$$

$$B = \alpha^2\Big[\beta r_{00}\{-24\alpha^{12} + 96\alpha^{10}\beta^2 - 32\alpha^8\beta^4 + 4\alpha^6\beta^6 + 12\alpha^4\beta^8 - 14\alpha^2\beta^{10} + 3\beta^{12}\}$$
$$+2s_0(24\alpha^{14} + 48b^2\alpha^{14} - 72\alpha^{12}\beta^2 - 60b^2\alpha^{12}\beta^2 + 132\alpha^{10}\beta^4$$
$$+36b^2\alpha^{10}\beta^4 - 142\alpha^8\beta^6 - 46b^2\alpha^8\beta^6 + 160\alpha^6\beta^8 + 66b^2\alpha^6\beta^8$$
$$-148\alpha^4\beta^{10} - 30b^2\alpha^4\beta^{10} + 53\alpha^2\beta^{12} + 4b^2\alpha^2\beta^{12} - 6\beta^{14})\Big].$$

Transvection of (4.5) by $b_i y_j$ yields

$$As_0 + B_1(b^2\alpha^2 - \beta^2) = 0, \tag{4.7}$$

where

$$B_1 = \beta r_{00}(-24\alpha^{12} + 96\alpha^{10}\beta^2 - 32\alpha^8\beta^4 + 4\alpha^6\beta^6 + 12\alpha^4\beta^8$$
$$-14\alpha^2\beta^{10} + 3\alpha^{12}) + 2s_0(24\alpha^{14} + 48b^2\alpha^{14}$$
$$-72\alpha^{12}\beta^2 - 60b^2\alpha^{12}\beta^2 + 132\alpha^{10}\beta^4 + 36b^2\alpha^{10}\beta^4$$
$$-142\alpha^8\beta^6 - 46b^2\alpha^8\beta^6 + 160\alpha^6\beta^8 + 66b^2\alpha^6\beta^8$$
$$-148\alpha^4\beta^{10} - 30b^2\alpha^4\beta^{10} + 53\alpha^2\beta^{12} + 4b^2\alpha^2\beta^{12} - 6\beta^{14}).$$

The term of (4.7) which does not contain α^2 is found in $-3\beta^{15}(r_{00} + 2\beta s_0)$. As a result, there exists $hp(15) : V_{15}$ such that

$$\beta^{15}(r_{00} + 2\beta s_0) = \alpha^2 V_{15}. \tag{4.8}$$

Then it would be wiser to divide our examination into three situations, as follows:

(1) $V_{15} = 0$,
(2) $V_{15} \neq 0, \alpha^2 \not\equiv 0 \pmod{\beta}$,
(3) $V_{15} \neq 0, \alpha^2 \equiv 0 \pmod{\beta}$.

Case (1):

For $V_{15} = 0$: from (4.8), $r_{00} = -2\beta s_0$, that is, $r_{ij} = -(b_i s_j + b_j s_i)$. Using $r_{00} = -2\beta s_0$ in (4.7), we obtain

$$s_0\{A + 2B_1^{'}(b^2\alpha^2 - \beta^2)\} = 0, \tag{4.9}$$

where

$$B_1^{'} = 24\alpha^{14} + 48b^2\alpha^{14} - 48\alpha^{12}\beta^2 - 60b^2\alpha^{12}\beta^2 + 36\alpha^{10}\beta^4 + 36b^2\alpha^{10}\beta^4 \tag{4.10}$$
$$-110\alpha^8\beta^6 - 46b^2\alpha^8\beta^6 + 156\alpha^6\beta^8 + 66b^2\alpha^6\beta^8 - 160\alpha^4\beta^{10}$$
$$-30b^2\alpha^4\beta^{10} + 67\alpha^2\beta^{12} + 4b^2\alpha^2\beta^{12} - 9\beta^{14}.$$

If $A + 2B_1^{'}(b^2\alpha^2 - \beta^2) = 0$ in (4.9), then we obtain

$$A + 2B_1^{'}(b^2\alpha^2 - \beta^2) = \alpha^2 A_1$$

where

$$A_1 = -24\alpha^{14} - 48b^2\alpha^{14} + 216\alpha^{12}\beta^2 + 288b^2\alpha^{12}\beta^2 - 744\alpha^{10}\beta^4 - 1222b^2\alpha^{10}\beta^4 + 1134\alpha^8\beta^6 + 368b^2\alpha^8\beta^6 - 816\alpha^6\beta^8 - 2202b^2\alpha^6\beta^8 + 1414\alpha^4\beta^{10} + 60b^2\alpha^4\beta^{10} - 153\alpha^2\beta^{12} + 6b^2\alpha^2\beta^{12} + 13\beta^{14} - 2b^2\beta^{14}.$$

then the expression $A_1 = 0$ is an expression that does not contain α^2 is $(13 - 2b^2)\beta^{14}$. Therefore, there exixts $hp(12) : V_{12}$ such that

$$(13 - 2b^2)\beta^{14} = \alpha^2 V_{12}.$$

where we suppose $b^2 \neq 13/2$. Hence we have $V_{12} = 0$. This leads to a contradiction. Therefore

$$A + 2B_1^{'}(b^2\alpha^2 - \beta^2) \neq 0.$$

As a result of (4.9), $s_0 = 0$ yields $r_{00} = 0$. Substituting $s_0 = 0$ and $r_{00} = 0$ in (4.5). We have,

$$A(s_0^i y^j - s_0^j y^i) = 0. \tag{4.11}$$

If $A = 0$, we have from (4.6)

$$A = -24\alpha^{16} - 96b^2\alpha^{16} - 96b^4\alpha^{16} + 264\alpha^{14}\beta^2 + 480b^2\alpha^{14}\beta^2 + 120b^4\alpha^{14}\beta^2 - 840b^2\alpha^{12}\beta^4 - 1414b^2\alpha^{12}\beta^4 - 72b^4\alpha^{12}\beta^4 + 1206\alpha^{10}\beta^6 + 660b^2\alpha^{10}\beta^6 + 92b^4\alpha^{10}\beta^6 - 1036\alpha^8\beta^8 - 2606b^2\alpha^8\beta^8 - 132b^4\alpha^8\beta^8 + 1726\alpha^6\beta^{10} + 512b^2\alpha^6\beta^{10} + 60b^4\alpha^6\beta^{10} - 473\alpha^4\beta^{12} - 188b^2\alpha^4\beta^{12} - 8b^4\alpha^4\beta^{12} + 147\alpha^2\beta^{14} + 24b^2\alpha^2\beta^{14} - 18\beta^8 = 0. \tag{4.12}$$

The term of (4.12) that seems not to include α^2 is $-18\beta^{16}$. Thus, there exists $hp(14) : V_{14}$ such that $-18\beta^{16} = \alpha^2 V_{14}$. This equation yields $V_{14} = 0$. This leads to a contradiction. Therefore $A \neq 0$, Thus we have from (4.11)

$$s_0^i y^j - s_0^j y^i = 0. \tag{4.13}$$

Transvection (4.13) by y_j yields

$$s_0^i = 0.$$

Finally, $r_{ij} = s_{ij} = 0$, implying $b_{i;j} = 0$.

Case (2):

In case of $V_{15} \neq 0; \alpha^2 \not\equiv 0(mod\beta)$: In this situation, (4.8) proves that there exists a function $h = h(x)$ obtaining

$$r_{00} + 2\beta s_0 = h(x)\alpha^2. \tag{4.14}$$

Substituting (4.14) into (4.7),

$$s_0\Big\{-24\alpha^{14}-48b^2\alpha^{14}+216\alpha^{12}\beta^2+288b^2\alpha^{12}\beta^2-744\alpha^{10}\beta^4-1222b^2\alpha^{10}\beta^4$$
$$+1134\alpha^8\beta^6+368b^2\alpha^8\beta^6-816\alpha^6\beta^8-2202b^2\alpha^6\beta^8+1414\alpha^4\beta^{10}+60b^2\alpha^4\beta^{10}$$
$$-153\alpha^2\beta^{12}+6b^2\alpha^2\beta^{12}+13\beta^{14}-2b^2\beta^{14}\Big\}$$
$$+h\beta(b^2\alpha^2-\beta^2)\Big\{-24\alpha^{12}+96\alpha^{10}\beta^2-32\alpha^8\beta^6+4\alpha^6\beta^6+12\alpha^4\beta^8-14\alpha^2\beta^{10}$$
$$+3\beta^{12}\Big\}=0.$$

The term of (4.15) that seems to not include α^2 is $\{(13-2b^2)s_0-3h\beta\}\beta^{14}$. Hence there exists $hp(13):V_{13}$ such that $\{(13-2b^2)s_0-3h\beta\}\beta^{14}=\alpha^2V_{13}$. $\alpha^2\not\equiv 0(mod\beta)$ implies that $V_{13}=0$. Therefore, we have,

$$\big\{(13-2b^2)s_0-3h\beta\big\}\,\beta^{14}=0.$$

which indicates

$$s_0=\frac{3h(x)}{(13-2b^2)}\beta. \tag{4.15}$$

From (4.16), we get $s_i=\dfrac{3h(x)b_i}{(13-2b^2)}$. Transvecting by b^i yields $h(x)b^2=0$. Hence $h(x)=0$. Substituting $h(x)=0$ into (4.14) and (4.16) yields $s_0=0$ and $r_{00}=0$. Therefore (4.5) simplifies to $A(s^i_0y^j-s^j_0y^i)=0$. Since $A\neq 0$, we get $s^i_0y^j-s^j_0y^i=0$. Transvection of this equation by y_j yields $s^i_0=0$. Finally, $r_{ij}=s_{ij}=0$ are concluded, that is, $b_{i;j}=0$.

Case (3):

In case of $V_{15}\neq 0;\alpha^2\equiv 0(mod\beta)$: In this case, lemma (2.2) indicates that $n=2$, $b^2=0$ and $\alpha^2=\beta\delta$, where $\delta=d_i(x)y^i$. From (4.8) we have $\beta^{14}(r_{00}+2\beta s_0)=\delta V_{15}$, which must be reduced to

$$r_{00}+2\beta s_0=\delta V,$$

where $V=V_i(x)y^i$. Using (4.16) we obtain $s_0=\dfrac{3h(x)}{13}\beta$ easily.

Transvection of $r_{00}+2\beta s_0=\delta V$ by b^i yields

$$r_{00}b^i=2Vy^i. \tag{4.16}$$

Again Transvection (4.17) by b_i yields

$$r_{00}b^2=2\beta V. \tag{4.17}$$

$V=0$ contradicts $V=V_i(x)y^i$. It is conceivable when $s_0=0$ and $r_{00}=0$. Substitute $s_0=0$ and $r_{00}=0$ in (4.5), we have $A(s^i_0y^j-s^j_0y^i)=0$. Because $A\neq 0$, we get $s^i_0y^j-s^j_0y^i=0$. Transvection this equation by y_j yields $s^i_0=0$. Thus $r_{ij}=s_{ij}=0$, implying that $b_{i;j}=0$.

Conversely if $b_{i;j} = 0$, then we get $B^{ij} = 0$ from (4.1). Hence, F^n is a Douglas space. Consequently, we have

Theorem 4.1. *An n-dimensional Finsler space F^n with deformed Berwald-Matsumoto metric is a Douglas space, if and only if*

(1) $\alpha^2 \not\equiv 0(mod\beta)$: $b_{j;i} = 0$.
(2) $\alpha^2 \equiv 0(mod\beta)$: $n = 2$, $b^2 = 0$ *and* $b_{j;i} = 0$, *where* $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ *and* $h = h(x)$.

Based on Theorems 3.1 and 4.1, we have the following.

Theorem 4.2. *If an n-dimensional Finsler space F^n with deformed Berwald-Matsumoto metric is a Douglas space, then F^n is also a Berwald space.*

5. Conclusion

In this study, we analyzed the Finsler space with the Berwald-Matsumoto metric and established the circumstances under which the Finsler space F^n will be a Berwald and Douglas space. The requirements are presented in theorems (3.1), (3.2), (4.1), and (4.2), respectively. This is an essential combination of two exceptional (α, β)-metrics; therefore, in future work, we examine additional significant Finsler features such as reducibility, main scalars in two and three dimensions, Landsberg space, etc., using this metric.

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