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On Ξ-curvature of m-th root Finsler metrics

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Abstract. The notions of S-curvature and Ξ-curvature introduced by Shen that is very effective for understanding the other Riemannian and non-Riemannian geometric properties of Finsler metrics. Here, we study the S-curvature and Ξ-curvature of the class of cubic and quartic (α, β)-metrics. We prove that a third root (α, β)-metric of vanishing Ξ-curvature reduces to a $(-1/3)$-Kropina metric or it has vanishing S-curvature. Then, we prove that quartic (α, β)-metric of vanishing Ξ-curvature reduces to a special form of quartic (α, β)-metric or it has vanishing S-curvature.

Keywords: Ξ-curvature, (α, β)-metric, S-curvature, m–th root metric.

1. Introduction

In 1935, when Wegener investigated the 3-th root metrics of dimensions two an three, the m-th root metric theorem in Finsler geometry began with his studies. [25]. Matsumoto presented a more complete result of Wegener's work and revised some of calculations [10]. In 1961, Kropina considered Wegener's studies and investigated projectively flat two-dimensional cubic metrics [7]. In

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1979, Matumoto and Numata found the perfect forms of 3-th root metric on an $n(\geq 3)$ dimensional Finsler manifold (M, F) (see [12]). In 1978, Shimada studied 1-form metrics and in the following year he considered the generalization of 1-form metrics [16].

The m-th root Finsler metric $F = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}}$ is defined on the tangent bundle of a Finsler manifold M, where $a_{i_1 \dots i_m}$ is symmetric in all its indices. The Riemannian metrics are 2-th root Finsler metrics. Then,the study of m-th root metrics will raise our perception of Riemannian metrics. It soon became clear that m-th root metrics have interesting applications in Ecology, Biology, Seismic Ray Theory and etc [2].

To comprehend the structure of cubic Finsler metrics, the non-Riemannian curves of these metrics are studied. [21][22][23]. Among these quantities, the S-curvature and Ξ-curvature have important and deep relation with together. Let us give a brief explanation of their relation. The distortion $\tau = \tau(x, y)$ is a non-Riemannian quantity that specified by the Busemann-Hausdorff volume form. The horizontal derivations of distortion τ on each tangent space gives rise to the S-curvature $\mathbf{S} = \tau_{|k} y^k$, where "|" denotes the horizontal covariant derivative with respect to the Berwald connection of F [15]. The Ξ-curvature $\mathbf{\Xi} = \Xi_j dx^j$ defined by $\Xi_k := \mathbf{S}_{.k|m} y^m - \mathbf{S}_{|k}$, where "." is the vertical covariant derivative [15]. These non-Riemannian quantities were introduced by Shen to understanding the other Riemannian and non-Riemannian geometric properties of Finsler metrics.

It is clear that if $\mathbf{S} = 0$ then $\Xi = 0$. It is interesting to find some Finsler metrics that these two notions are equivalent for them. In this work, we calculated the Ξ-curvature of cubic Finsler metrics and prove the below.

Theorem 1.1. *Any cubic (α, β)-metric with vanishing Ξ-curvature is a $(-1/3)$-Kropina metric or it has vanishing S-curvature.*

In addition to the 3-th root Finsler metrics, the 4-th root Finsler metrics have an important and special role in Finsler geometry. A special form of the quartic metrics $F^4 = y^l y^m y^s y^p$, which is called Berwald-Moór metric, has a significant impact on other sciences[5, 6, 11]. The importance of the Berwald-Moór metric can be seen in the physical studies of Pavlov and Asano and their colleagues on the theory of space-time structure and gravity, also in the theory of unified field gauges [3, 13, 14]. In [4], Balan showed that for co-isotropic submanifolds of Berwald-Moór metrics acquired the Gauss-Codazzi, Gauss-Weingarten, Ricci-Kühne, Peterson-Mainardi equations and the Berwald-Moór metric is a pseudo-Finsler metric of Lorentz type. Tayebi and Najafi characterized locally dually flat and Antonelli m-th root Finsler metrics [21]. As well as, they proved that every m-th root Finsler metric of isotropic Landsberg curvature reduces to a Landsberg metric [22]. In [17], Also, he proved that every quartic metric of weakly isotropic flag curvature has vanishing scalar curvature. Then, he obtain the necessary and sufficient condition that under the conformal change of a quartic metric is isotropic scalar curvature [19]. In [20], Tayebi, Amini and Najafi searched for the necessary and sufficient condition under which a 4-th root (α, β)-metric is conformally Berwald. In [24], Tayebi and Razgordani showed that every conformally flat weakly Einstein quartic (α, β)-metric on an n$\geq$ 3-dimensional manifold M is either a locally Minkowski metric or a Riemannian metric. As well as, they showed that every conformally flat quartic (α, β)-metric of almost vanishing Ξ-curvature on an n$\geq$ 3-dimensional manifold M reduces to a locally Minkowski metric a Riemannian metric. For more progress, see [8] and [9].

In this paper, after computing the Ξ curvature for a fourth root (α, β)–metric, we state the following theorem and then prove it.

Theorem 1.2. *Suppose F be quartic (α, β)-metric on a Finsler manifold (M, F). Assume that F applies $\Xi = 0$. Then F is given by*

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2}, \tag{1.1}$$

where c_i are real constants, or F has vanishing S-curvature $\mathbf{S} = 0$.

2. Preliminaries

Assume that (M, F) be a Finsler manifold. Fundamental tensor be quadratic form $\mathbf{g}_y$ on T_eM

$$\mathbf{g}_y(v, w) = \frac{1}{2}\frac{\partial^2}{\partial m \partial s}\Big[F^2(y + mv + sw)\Big]|_{m=s=0}, \quad v, w \in T_eM.$$

Let $e \in M$ and $F := F|_{T_eM}$. One can define $\mathbf{C}_y : T_eM \times T_eM \times T_eM \longrightarrow \mathbb{R}$ via

$$\begin{aligned}\mathbf{C}_y(w, v, u) :=& \frac{1}{2}\frac{d}{ds}[\mathbf{g}_{y+su}(w, v)]_{s=0}\\ &= \frac{1}{4}\frac{\partial^3}{\partial r \partial m \partial s}[F^2(y + rw + mv + su)]_{r=m=s=0},\end{aligned}$$

where $w, v, u \in T_eM$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is the Cartan torsion where $\mathbf{C}_y$ is a symmetric trilinear form on T_eM.

For $y \in TM_0$, let $\mathbf{I}_y : T_eM \longrightarrow \mathbf{R}$ by follows

$$\mathbf{I}_y(w) = \sum_{i=1}^{n} g^{mt}(y)\mathbf{C}_y(w, \partial_m, \partial_t),$$

where $g^{mt} = (g_{mt})^{-1}$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in T_eM_0}$ is the mean Cartan torsion.

For a Finsler manifold (M, F) of dimension n, F induced spray $\mathbf{G}$ on TM_0. It is given by follows

$$\mathbf{G} = y^t\frac{\partial}{\partial x^t} - 2G^t\frac{\partial}{\partial y^t},$$

where $G^t = G^t(x,y)$ are local functions on TM_0 expressed by

$$G^t := \frac{1}{4} g^{ts} \Big\{ \frac{\partial^2 [F^2]}{\partial x^l \partial y^s} y^l - \frac{\partial [F^2]}{\partial x^s} \Big\}, \qquad y \in T_e M.$$

$\mathbf{G}$ is called the associated spray to Finsler manifold (M, F).

For a Finsler manifold (M, F), the Busemann-Hausdorf volume form $dV_F = \sigma_F(x) dx^1 ... dx^n$ define as below

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\big\{(y^t) \in \mathbb{R}^n | F(y^t \frac{\partial}{\partial x^t}|_x) < 1\big\}}.$$

Then, for $y = y^s \partial/\partial x^s|_e \in T_e M$, the S-curvature is expressed as follows

$$\mathbf{S}(y) := \frac{\partial G^s}{\partial y^s} - y^s \frac{\partial}{\partial x^s} \Big[ln \sigma_F(x) \Big]. \tag{2.1}$$

Shen introduced the S-curvature for a analogy theorem on Finsler manifold.

Assume that (M, F) of dimension n be a Finsler manifold. The Ξ-curvature $\mathbf{\Xi} = \Xi_j dx^j$ define as below

$$\Xi_j := \mathbf{S}_{.j|m} y^m - \mathbf{S}_{|j},$$

F is almost vanishing Ξ-curvature if

$$\Xi_j = -(n+1) F^2 \left(\frac{\rho}{F} \right)_{y^j},$$

where $\rho := t_s(x) y^s$ is a one-form on M.

Consider $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha^2 = a_{ns}(x) y^n y^s$ is a Riemannian metric and $\beta = b_j(x) y^j$ is a one-form over the manifold (M, F). For an (α, β)-metric, assume $b_{n|s}$ by $b_{n|s}\theta^s := db_n - b_s \theta^s_n$, where $\theta^s := dx^s$ and $\theta^s_n := \Gamma^s_{nl} dx^s$ denote the Levi-Civita connection form of α. Let

$$r_{kj} := \tfrac{1}{2}(b_{k|j} + b_{j|k}), \quad s_{kj} := \tfrac{1}{2}(b_{k|j} - b_{j|k}), \quad r_{k0} := r_{kj} y^j, \quad r_{00} := r_{kj} y^k y^j,$$
$$r_j := b^k r_{kj}, \quad s_{k0} := s_{kj} y^j, \quad s_j := b^k s_{kj}, \quad s^k{}_j = a^{km} s_{mj}, \quad s^k{}_0 = s^k{}_j y^j,$$
$$r_0 := r_j y^j, \quad s_0 := s_j y^j, \quad q_{kj} := r_{km} s^m{}_j, \quad q_{00} := q_{kj} y^k y^j, \quad q_j := b^k q_{kj},$$
$$t_{kj} := s_{km} s^m{}_j, \quad t_{00} := t_{kj} y^k y^j, \quad t_j := b^k t_{kj}.$$

where $a^{it} = (a_{it})^{-1}$ and $b^i := a^{it} b_t$. Put

$$Q := \frac{\phi'}{\phi^{\phi - s'}},$$
$$\Psi := \frac{1}{2}\frac{\phi''}{\phi''(B - s^2) - (s\phi' - \phi)}, \tag{2.2}$$
$$\Theta := \frac{s(\phi''\phi\phi'\phi') - \phi\phi'}{2\phi[(B - s^2)\phi'' - (\phi^{s'} - \phi)]}, \tag{2.3}$$

where $B := ||\beta||^2_\alpha$.

Let $G^t_\alpha = G^t_\alpha(x, y)$ and $G^t = G^t(x, y)$ denote the coefficients of α and F, respectively, in the same coordinate system. According to the definition, we see that

$$G^t = G^t_\alpha + \alpha Q {s^t}_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^t + \Psi b^t). \tag{2.4}$$

where

$$P := \Big[r_{00} - 2Q\alpha s_0\Big]\Theta\alpha^{-1}, \quad Q^t := \Psi\Big[r_{00} - 2\alpha Q s_0\Big]b^t + \alpha Q {s^t}_0.$$

Obviously, if β is parallel with respect to α then $P = 0$ and $Q^t = 0$. In other words, F is a Berwald metric, that is, $G^t = G^t_\alpha$ are quadratic in y. place

$$\Phi : = (sQ' - Q)\{n\Delta + Qs + 1\} - Q''(B - s^2)(sQ + 1),$$

where

$$\Delta := (b^2 - s^2)Q' + sQ + 1.$$

Assume F be an (α, β)-metric on an n-dimensional manifold (M, F). So the S-curvature of F is as follows

$$\mathbf{S} = \Big[2\Psi - \frac{f'(b)}{b\ f(b)}\Big](s_0 + r_0) + \frac{\Phi}{2\Delta^2\alpha}(2\alpha Q s_0 - r_{00}),$$

where

$$f(b) := \frac{\int_0^\pi T(b\cos t)\sin^{n-2} t dt}{\int_0^\pi \sin^{n-2} t dt},$$
$$T(s) := \big[\phi''(b^2 - s^2) - (s\phi' - \phi)\big](\phi - s\phi')^{n-2}\phi.$$

3. **Proof of Theorem 1.1**

we state and then prove the below theorem. Exactly, we give a general version of Theorem 1.1 and then prove it. Actually, we examine cubic (α, β)-metric of almost vanishing Ξ-curvature.

Theorem 3.1. *Suppose F be cubic (α, β)-metric on Finsler manifold (M, F). Presume that F has almost vanishing Ξ-curvature. So F reduces to*

$$F = \sqrt[3]{\alpha^2\beta}$$

which is a $(-1/3)$-Kropina metric or F has vanishing S-curvature.

To prove Theorem 1.1, we mention that in [12], Matsumoto-Numata considered the class of 3-th (α,β)-metrics and stated the below theorem.

Proposition 3.2. ([12]) *Let $F^3 = a_{ijk}y^iy^jy^k$ be a cubic Finsler metric on manifold M. So the following happens:*

(1): *If M be an 2-dimensional Finsler metric on manifold, thus via choosing a appropriate second-class form $\alpha^2 = \mathfrak{a}_{ks}(x)y^ky^s$ and 1-form $\beta = b_k(x)y^k$ on manifold (M,F) reduces to a $(-1/3)$-Kropina metric*

$$F^3 = \beta\alpha^2, \tag{3.1}$$

where α^2 may be degenerate.

(2): *If manifold M is $n(\geq 3)$-dimensional and F is a function of a non-degenerate second-class form $\alpha^2 = \mathfrak{a}_{ks}(x)y^ky^s$ and a 1-form $\beta = b_k(x)y^k$, then it is as follows*

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}, \tag{3.2}$$

with real constants c_i.

One can see that (3.1) is a special form of (3.2). Then, throughout this paper we consider the complete form (3.2). However, to prove Theorem 3.1 and apart from the corresponding metric form , the below lemma is presented and proved.

Lemma 3.3. *Let $F^m = a_{i_1\ldots i_m}(x)y^{i_1}y^{i_2}\ldots y^{i_m}$ be an m-th root Finsler metric. Assume that F is of almost vanishing Ξ-curvature. Then, F has vanishing Ξ-curvature.*

Proof. By assumption, the Ξ-curvature is as below

$$\Xi_j = -(n+1)F^2\Big(\frac{\rho}{F}\Big)_{y^j}. \tag{3.3}$$

where $\rho = t_j(x)y^j$ is a one-form on Finsler manifold (M,F). Here, the spray coefficients of an m-th root metric is a rational function in y[26]. we know that the Ξ-curvature and the S-curvature of F are rational functions in y. As we can see on the left side of (3.3) is a rational function in y, whereas the other side of (3.3) is an irrational function in y. So $\theta = 0$ and then $\Xi = 0$. □

Now, we calculate the Ξ-curvature of 3-th (α,β)-metrics. Hereupon, we state the below lemma and then prove it.

Lemma 3.4. *Let $F^3 = c_1\beta\alpha^2 + c_2\beta^3$ be the 3-th root (α,β)-metric on a manifold M. Assume that F satisfies $\Xi = 0$. Then F reduces to a $(-1/3)$-Kropina metric or β is a Killing 1-form.*

Proof. For any (α, β)-metric, the Ξ-curvature is given by

$$\Xi_j := H_{.j;t}y^t - H_{;j} - 2H_{.j.t}H^t, \tag{3.4}$$

where $H = \partial G^i/\partial y^i$, ";" indicates the horizontal covariant derivative with ratio to α and

$$H^t := Py^t + Rs^t{}_0 + Tb^t, \quad P := \frac{A}{\alpha}\Theta, \qquad A := r_{00} - 2\alpha Q s_0,$$
$$T := \Psi A, \qquad R := \alpha Q.$$

The right side of (3.4) is given by

$$H_{;j} := \frac{c_1}{\alpha}\ r_{00;j} + \frac{c_2}{\alpha}(r_{j0} - s_{j0}) + c_3 s_{0;j} + 2c_4(r_j + s_j) + 2\Psi r_{0;j},$$

where

$$\begin{aligned}
c_1 :=&(n+1)(\Psi' + \Theta),\\
c_2 :=&\Big\{(n+1)\Theta' + (B - s^2)\Psi'' - 2s\Psi'\Big\}\frac{A}{\alpha} + 2\Psi' r_0 + \Big\{ - 2(Q^{Q-}{}'s)\Psi - Qs\\
&- 2(n+1)\Theta Q' - Q'' - (Q' + 2Q'\Psi' + 2(B - s^2)\Psi Q'')\Big\}s_0,\\
c_3 :=&Q' - 2s\Psi Q - 2(n+1)Q\Theta - 2(B - s^2)(Q'\Psi + \Psi' Q),\\
c_4 :=&\Psi'\frac{A}{\alpha} - 2Q'\Psi s_0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
H_{.j;t}y^t :=& \ p_{5j}\frac{r_{00}}{\alpha} + p_6 s_{j;0} + 2p_{7j}(r_0 + s_0) + 2\Psi r_{j;0} + \frac{1}{\alpha^2}\Lambda_{;t}y^t(\alpha b_j - sy_j)\\
&+ \Lambda\Big(\frac{r_{j0} + s_{j0}}{\alpha} - \frac{r_{00}y_j}{\alpha^3}\Big),
\end{aligned}$$

where

$$\begin{aligned}
p_{5j} :=& 2\Psi' r_j + \Big\{Q - Q's - \big((B-s^2)Q'' + Q''\big)\Big\}s_j \\
& - \Big\{\Psi''(B-s^2) - 2s\Psi'(n+1)\Theta'\Big\}\Big(2Qs_j + r_{00}\frac{y_j}{\alpha^3} - \frac{2r_{j0}}{\alpha}\Big),\\
p_6 :=& Q' - (B-s^2)Q' - sQ,\\
p_{7j} :=& \Big(\frac{2r_{j0}}{\alpha} - r_{00}\frac{y_j}{\alpha^3} - 2Qs_j\Big)\Psi' + Q's_j,\\
\Lambda :=& \frac{A}{\alpha}\Big\{2(r_0 - s)\Psi' + (B-s^2)\Psi'' + (n+1)\Theta'\Big\} + \Big\{ - 2Q\Psi + Q'' \\
& + 2(\Psi Q' - Q\Psi')s - 2(n+1)Q'\Theta - 2(B-s^2)\Big(2\Psi'Q' - \Psi Q''\Big)\Big\}s_0,\\
\Lambda_{;t}y^t :=& p_{11}r_{00;0} + \big(\frac{1}{\alpha}p_{12} - 2p_{11}Q's_0\big)r_{00} + \big(Q'' - 2(n+1)Q'\Theta - 2\alpha Q p_{11}\big)s_{0;0}\\
& + 2\Psi''\frac{A}{\alpha}(r_0 + s_0) + 2\Psi' r_{0;0} - 2s\frac{\Psi'}{\alpha}\big(r_{00;0} - 2Q's_0r_{00} - 2\alpha Q s_{0;0}\big)\\
& + p_{21}\frac{r_{00}}{\alpha} + p_{22}s_{0;0} + 2p_{23}(r_0 + s_0) + p_{31}\frac{r_{00}}{\alpha} + p_{32}s_{0;0}\\
& - 4Q''\Psi Q''(r_0 + s_0)s_0,
\end{aligned}$$

$$\begin{aligned}
p_{11} :=& \frac{1}{\alpha}\Big\{(n+1)\Theta' + (B-s^2)\Psi''\Big\},\\
p_{12} :=& \frac{A}{\alpha}\Big\{(n+1)\Theta'' + (B-s^2)\Psi''' - 2s\Psi''\Big\}\\
& + \Big\{Q''' - 2(n+1)(Q'\Theta' + \Theta Q'')\Big\}s_0,\\
p_{21} :=& 2\Psi'' r_0 - 2\big\{3s\Psi'Q' + Q(s\Psi'' + \Psi')\big\}s_0 - 2\frac{A}{\alpha}(s\Psi'' + \Psi'),\\
p_{22} :=& -2\Psi'Qs - 4(B-s^2)\Psi'Q',\\
p_{23} :=& -4Q'\Psi's_0,\\
p_{31} :=& \big\{2(Q'\Psi' - Q\Psi' + 3Q''\Psi)s - 2(B-s^2)(Q''\Psi' + Q'''\Psi)\big\}s_0,\\
p_{32} :=& 2\big\{sQ' - Q - (B-s^2)Q''\big\}\Psi.
\end{aligned}$$

The sentence $H_{.j.t}H^t$ in (3.4) is given by

$$\begin{aligned}
H_{.j.t}H^t =& Q\Bigg\{c_{1j}s_0 + c_{2j}\alpha + c_{3j}\alpha^2 - \Lambda\Big(\frac{s_0}{\alpha^2}y_j + \frac{ss_{j0}}{\alpha}\Big) + \Lambda_{.m}s^m{}_0\Big(b_j - \frac{sy_j}{\alpha}\Big)\Bigg\}\\
& + A\Psi\Bigg\{\frac{c_{1j}(B-s^2)}{\alpha} + \Lambda\Big[(3\frac{s^2}{\alpha^3} - \frac{B}{\alpha^3})y_j - 2\frac{s}{\alpha^2}b_j\Big] + \Lambda_{.m}b^m\big(\frac{b_j}{\alpha} - \frac{sy_j}{\alpha^2}\big)\Bigg\},
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
\Lambda_{.t}{s^t}_0 :=& 2(\alpha p_{11} - 2\Psi' s)\Big(\frac{q_{00}}{\alpha} - Qt_0 - \frac{Q'}{\alpha}s_0^2\Big)\\
&+ \frac{1}{\alpha}(p_{12}+p_{21}+p_{31})s_0 + (p_{41}+p_{22}+p_{32})t_0 + 2\Psi' q_0,\\
\Lambda_{.t}b^t :=& (\alpha p_{11} - 2\Psi' s)\Big(2\frac{r_0}{\alpha} - \frac{sr_{00}}{\alpha^2} - 2(B-s^2)Q'\frac{s_0}{\alpha}\Big)\\
&+ \frac{(B-s^2)}{\alpha}(p_{12}+p_{21}+p_{31}) + 2\Psi' r,\\
p_{41} :=& Q'' - 2(n+1)Q'\Theta,\\
c_{1j} :=& 2\Psi' r_j + \Big[(n+1)\Theta' + (B-s^2)\Psi'' - 2s\Psi'\Big]\Big(2\frac{r_{j0}}{\alpha} - \frac{r_{00}y_j}{\alpha^3} - 2Qs_j\Big)\\
&+\Big\{Q'' - 2Q'\Big((n+1)\Theta + (B-s^2)\Psi'\Big) - \Big(Q + sQ' + Q''(B-s^2) - 2Q's\Big)\Big\}s_j,\\
c_{2j} :=& \Big\{(B-s^2)\Psi' + (n+1)\Theta\Big\}\Big(2\frac{q_{j0}}{\alpha} - 2\frac{q_{00}y_j}{\alpha^3} - \frac{r_{00}y_j}{\alpha^3}s_{j0}\Big),\\
c_{3j} :=& \Big\{(B-s^2)\Psi' + (n+1)\Theta\Big\}\Big(2\frac{r_j}{\alpha} - 2\frac{r_{j0}}{\alpha^2}s + 3\frac{r_{00}sy_j}{\alpha^4} - 2\frac{r_0y_j}{\alpha^3} - \frac{r_{00}b_j}{\alpha^3}\Big).
\end{aligned}$$

Let $F^3 = c_1\alpha^2\beta + c_2\beta^3$ be a cubic (α,β)-metric on a manifold M. Suppose that F has almost vanishing Ξ-curvature. By Lemma 3.3, we have $\Xi_j = 0$. Let us define $\Xi := \Xi_j b^j$. Multiplying (3.4) with b^j yields

$$\Xi = f_7\alpha^7 + f_6\alpha^6 + f_5\alpha^5 + f_4\alpha^4 + f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0 = 0, \tag{3.6}$$

where

$$
\begin{aligned}
f_7 :=& -576r_{00}^2c_1^3 - 32832c_2^2Br_{00}^2c_1 - 12420c_2^2B^2c_1r_{00}^2 + 7344c_2c_1^2r_{00}^2B\\
&+ 10368\beta^2c_2^3r_0s_0 - 19008\beta^2c_2^2s_0^2c_1 - 23040\beta c_2^3B^2s_0^2 - 4896\beta c_2^3r_{00}B^2\\
&- 2304\beta c_2r_{0|0}c_1^2 - 8640\beta c_2^3Br_0r_{00} - 640\beta c_2^3Br_{00}s_0 + 240\beta c_2^3s_0r_0B^2\\
&+ 4884\beta c_2^2c_1s_0^2B + 9260\beta c_2^2r_{00}s_0c_1 + 9216\beta c_2^2r_{0|0}c_1B + 1068\beta c_2^2r_{00}c_1\\
&- 1824\beta c_2r_{00}s_0c_1^2 - 4884\beta c_2^2s_0c_1r_0B + 460\beta c_2^2Br_{00}s_0c_1 + 416c_2^3B^2r_{00}^2\\
&+ 13824c_2c_1^2r_{00}^2 + 46656\beta^2c_2^3s_0^2 + 22032\beta^2c_2^3Bs_0^2 - 15360\beta c_2s_0^2c_1^2\\
&- 240\beta c_2^3B^2r_{00}s_0 + 1560\beta c_2s_0c_1^2r_0 + 3483c_2^3B^3r_{00}^2 + (n+1)\big(4248c_2^3B^2r_{00}^2\\
&- 2304c_2r_{00}^2c_1^2 - 25920\beta^2c_2^3s_0^2 - 5472c_2^2Br_{00}^2c_1 + 2896\beta c_2^3Br_{00}s_0\\
&- 308\beta c_2^2r_{00}s_0c_1\big),\\
f_6 :=& 768r_{00}^2c_1^3 + 4644c_2^2Br_{00}^2c_1 + 14688c_2^2B^2c_1r_{00}^2 - 27648\beta^2c_2^3Bs_0^2\\
&+ 2304\beta^2c_2^3rr_{00} - 22272\beta^2c_2^3s_{0|0}B + 21504\beta^2c_2^2s_{0|0}c_1 + 25344\beta^2c_2^2s_0^2c_1\\
&+ 2448\beta c_2^3r_{00|0}B^2 + 3456\beta c_2^2r_{00}^2c_1 + 2304\beta c_2r_{00|0}c_1^2 + 7680\beta^3c_2^3r_0B\\
&- 6144\beta^3c_2^2s_0c_1 - 6144\beta^3c_2^2r_0c_1 + 212\beta c_2^3Br_0r_{00} + 962\beta c_2^3Br_{00}s_0\\
&- 155136\beta c_2^2r_{00}s_0c_1 - 7488\beta c_2^2r_{00|0}c_1B - 31104\beta c_2^2r_0r_{00}c_1\\
&- 61440\beta c_2^2Br_{00}s_0c_1 - 4068c_2^3B^2r_{00}^2 - 3888c_2^3B^3r_{00}^2 - 22464c_2c_1^2r_{00}^2\\
&- 3072\beta^3c_2^3s_0 + 9216\beta^3c_2^3q_0 - 101952\beta^2c_2^3s_0^2 + 5760\beta^2c_2^3r_{00}s_0\\
&- 37248\beta^2c_2^3r_0s_0 - 1728\beta c_2^3Br_{00}^2 + 7680\beta^3c_2^3s_0B + 2976\beta c_2^3B^2r_{00}s_0\\
&+ 18432\beta c_2r_{00}s_0c_1^2 + 18432\beta^3c_2^3t_0 + 2304\beta^2c_2^3r_0r_{00}\\
&+ (n+1)\big(768\beta^2c_2^3r_{00} - 5796c_2^3B^2r_{00}^2 - 9216c_2c_1^2r_{00}^2B + 5312c_2r_{00}^2c_1^2\\
&- 2304\beta^2c_2^3r_0s_0 - 204\beta^2c_2^3r_{00}B - 2304\beta^2c_2^3q_{00}B + 1976\beta^2c_2^3s_{0|0}B\\
&+ 3072\beta^2c_2^2c_1q_{00} + 2048\beta^2c_2^2r_{00}c_1 - 13824\beta^2c_2^2s_{0|0}c_1 + 5856c_2^2Br_{00}^2c_1\\
&- 1152\beta c_2^3r_{00|0}B^2 - 204\beta c_2^2r_{00}^2c_1 - 180\beta c_2r_{00|0}c_1^2 - 4656\beta c_2^3Br_{00}s_0\\
&+ 302\beta c_2^2r_{00|0}c_1B + 738\beta c_2^2r_{00}s_0c_1 - 136\beta c_2^2r_0r_{00}c_1 - 1824\beta^3c_2^3t_0\\
&+ 204\beta^3c_2^3s_0 + 4860\beta^2c_2^3s_0^2 + 178\beta c_2^3Br_0r_{00} + 864\beta c_2^3Br_{00}^2\\
&- 108\beta^2c_2^3r_{00}s_0 - 4608\beta^3c_2^3q_0\big),\\
f_5 :=& 372\beta^3c_2^3r_{0|0}B + 346\beta^3c_2^3r_{00} - 11520\beta^3c_2^2r_{00}s_0c_1 + 11264\beta^3c_2^2s_0c_1r_0\\
&+ 276\beta^3c_2^3r_{00}s_0 + 744\beta^2c_2^2c_1r_{00}^2B - 1094\beta^2c_2^3Br_{00}^2 + 136\beta^3c_2^3s_0^2B
\end{aligned}
$$

$$
\begin{aligned}
&- 1356\beta^3 c_2^3 s_0 r_0 B + 1688\beta^3 c_2^3 B r_{00} s_0 - 204\beta^3 c_2^2 r_{0|0} - 128\beta^2 c_2 r_{00}^2\\
&- 1264\beta^3 c_2^2 s_0^2 c_1 + 1324\beta^2 c_2^2 c_1 r_{00}^2 - 440\beta^2 c_2^3 B^2 r_{00}^2 - 584\beta^4 c_2^3 s_0^2\\
&+ (n+1)\big(576\beta^2 c_2^3 B r_{00}^2 - 12096\beta^3 c_2^3 r_{00} s_0 - 1152\beta^2 c_2^2 r_{00}^2 c_1\big),\\
f_4 :=&1152\beta^3 c_2^3 r_{00}^2 - 2048\beta^5 c_2^3 r_0 - 2048\beta^5 c_2^3 s_0 - 10368\beta^3 c_2^3 r_0 r_{00}\\
&+ 489\beta^2 c_2^3 B^2 r_{00}^2 - 264\beta^2 c_2^2 c_1 r_{00}^2 + 204\beta^2 c_2 r_{00}^2 c_1^2 - 142 c_2^3 \beta^3 r_{00} s_0\\
&- 9216\beta^2 c_2^2 c_1 r_{00}^2 B - 2496\beta^3 c_2^3 r_{00|0} B + 2304\beta^3 c_2^2 r_{00|0} c_1\\
&+ 15648\beta^2 c_2^3 B r_{00}^2 - 51840\beta^3 c_2^3 r_{00} s_0 + 150 c_1 \beta^3 c_2^2 r_{00} s_0\\
&+ (n+1)\big(768\beta^4 c_2^3 r_{00} - 4608\beta^4 c_2^3 s_{0|0} - 1152\beta^3 c_2^3 r_{00}^2 + 28032\beta^3 c_2^3 r_{00} s_0\\
&+ 1920\beta^3 c_2^3 r_{00|0} B - 768\beta^3 c_2^3 r_0 r_{00} - 1536\beta^3 c_2^2 r_{00|0} c_1 - 1968\beta^2 c_2^3 B r_{00}^2\\
&+ 4224\beta^2 c_2^2 r_{00}^2 c_1 + 612\beta^4 c_2^3 s_0^2 + 644\beta^4 c_2^3 s_{0|0}\big),
\end{aligned}
$$

$$
\begin{aligned}
f_3 :=&408\beta^4 c_2^3 r_{00}^2 + 302\beta^5 c_2^3 s_0 r_0 - 302\beta^5 c_2^3 s_0^2 + 248\beta^4 c_2^3 r_{00}^2 + 768\beta^5 c_2^3 r_{00|0}\\
&- 346\beta^5 c_2^3 s_0 r_{00},\\
f_2 :=&408\beta^5 c_2^3 s_0 r_{00} - 748\beta^4 c_2^3 r_{00}^2 + 204\beta^4 c_2^2 r_{00}^2 c_1 - 302\beta^4 c_2^3 r_{00}^2 - 768\beta^5 c_2^3 r_{00}\\
&- 152\beta^4 c_2^3 r_{00}^2 - 178\beta^4 c_2^2 r_{00}^2 c_1 + (n+1)\big(280\beta^4 c_2^3 r_{00}^2 - 768\beta^5 c_2^3 r_{00|0}\big),\\
f_1 :=&576 c_2^3 \beta^6 r_{00}^2,\\
f_0 :=&768 c_2^3 \beta^6 r_{00}^2.
\end{aligned}
\tag{3.7}
$$

By (3.6), we get

$$
f_7\alpha^6 + f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \tag{3.8}
$$

$$
f_6\alpha^6 + f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \tag{3.9}
$$

(3.8) mention that there exists a function $\varrho = \varrho(x, y)$ where $\varrho \neq 0$, so that

$$
c_2^3 \beta^6 r_{00}^2 = \varrho\alpha^2. \tag{3.10}
$$

In like manner (3.9) mention that there exists a function $\zeta = \zeta(x, y)$ where $\zeta \neq 0$, so that the following is established

$$
c_2^3 \beta^6 {r_{00}}^2 = \zeta\alpha^2. \tag{3.11}
$$

Since $\varrho \neq \zeta$ and ϱ is not a multiplication of ζ, then by (3.10) and (3.11) we get

$$
c_2^3 \beta^6 {r_{00}}^2 = 0.
$$

If $c_2 = 0$, then F reduces to

$$
F^3 = c_1\alpha^2\beta
$$

which is a $(-1/3)$-Kropina metric. If $c_2 \neq 0$, then we get $r_{ij} = 0$. □

Now, using Lemma 3.4, the proof of Theorem 3.1 is stated.

Proof of Theorem 3.1: If F be a $(-1/3)$-Kropina metric, then the proof is done. Assume that F is not a $(-1/3)$-Kropina metric. So, by Lemma 3.4, we have $r_{ij}=0$ which yields $r_i=0$, $q_{ij}=0$ and $q_i=0$. Putting these in (3.6) yields

$$g_4\alpha^4+g_3\alpha^3+g_2\alpha^2+g_1\alpha+g_0=0, \tag{3.12}$$

where

$$\begin{aligned}
g_4:=&140c_2^2B^2s_0^2+960c_1^2s_0^2-3024c_2c_1s_0^2-2916c_2^2\beta s_0^2-1377c_2^2B\beta s_0^2\\
&+1620(n+1)c_2^2\beta s_0^2+1188c_1c_2\beta s_0^2,\\
g_3=&-936(n+1)c_2^2\beta s_{0|0}B-1344c_1c_2\beta s_{0|0}-1152c_2^2\beta^2t_0+192c_2^2\beta^2s_0\\
&-144(n+1)\beta^2c_2^2s_0+864(n+1)\beta^2c_2^2t_0+864(n+1)c_1c_2\beta s_{0|0}+632c_2^2\beta s_0^2\\
&+1392c_2^2\beta s_{0|0}+1728c_2^2\beta s_0^2-360(n+1)c_2^2\beta s_0^2-480c_2^2\beta^2s_0-1584c_1c_2\beta s_0^2\\
&+384c_1c_2\beta^2s_0\\
g_2:=&324c_2^2\beta^3s_0^2+704c_1c_2\beta^2s_0^2-816c_2^2B\beta^2s_0^2,\\
g_1:=&128s_0c_2^2\beta^4-384\beta^3c_2^2s_{0|0}-432s_0^2c_2^2\beta^3+288(n+1)c_2^2\beta^3s_{0|0},\\
g_0:=&192s_0^2c_2^2\beta^4.
\end{aligned}\tag{3.13}$$

By (3.12), we get

$$g_3\alpha^2+g_1=0, \tag{3.14}$$

$$g_4\alpha^4+g_2\alpha^2+g_0=0. \tag{3.15}$$

(3.15) implies that

$$\eta s_0^2=0,$$

where

$$\begin{aligned}
\eta:=&\Big(140c_2^2B^2+960c_1^2-304c_2c_1B-216\beta c_2^2-137\beta c_2^2B+120(n+1)\beta c_2^2\\
&+188\beta c_1c_2\Big)\alpha^4+\Big(324c_2^2\beta^3+704\beta^2c_1c_2-816\beta^2c_2^2B\Big)\alpha^2+192c_2^2\beta^4.
\end{aligned}\tag{3.16}$$

By (3.16), we get $\eta=0$ or $s_i=0$. Let $\eta(x,y)=0$ holds. One can rewrite it as follows

$$\theta\alpha^4+\gamma\alpha^2\beta^2+\varepsilon\beta^4=0, \tag{3.17}$$

where

$$\begin{aligned}
\gamma:=&\ 140c_2^2B^2+960c_1^2-304c_2c_1B-216\beta c_2^2-137\beta c_2^2B+120(n+1)\beta c_2^2\\
&+188\beta c_1c_2,\\
\theta:=&\ 324c_2^2\beta^3+704\beta^2c_1c_2-816\beta^2c_2^2B,\\
\varepsilon:=&\ 192c_2^2\beta^4.
\end{aligned}$$

(3.17) implies that

$$\alpha^2 = \left(\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\varepsilon}}{2\theta} \right) \beta^2. \tag{3.18}$$

Since α is positive-definite, then it is in contradiction with (3.18). So $\eta \neq 0$. Hence, we have $s_i = 0$. By putting $r_{ij} = 0$ and $s_i = 0$ in (2.5), we conclude that $\mathbf{S} = 0$.

□

4. Proof of Theorem 1.2

In this part of the work, we state the proof of Theorem 1.2. Indeed, we study quartic (α, β)-metrics with almost vanishing Ξ-curvature. For this purpose, we mention that in [1] The quartic (α, β)-metrics was characterized as follows by Abazari and Khoshdani.

Proposition 4.1. ([1]) *Assume $F = \sqrt[4]{a_{ijkl} y^i y^j y^k y^l}$ be a quartic metric on a Finsler manifold M with dimension n. So the following happens:*

(1): *If M be an 2-dimensional Finsler metric on manifold, so F is given by*

$$F = \sqrt[4]{d_1\alpha^4 + d_2\beta^2}\alpha^2,$$

where α^2 may be degenerate, d_1, d_2 be constants via choosing a appropriate second-class form $\alpha^2 = \mathfrak{a}_{ks}(x)y^k y^s$ and 1-form β.

(2): *If manifold M is $n(\geq 3)$-dimensional and F is a function of a non-degenerate second-class form $\alpha^2 = \mathfrak{a}_{ks}(x)y^k y^s$ and a 1-form $\beta = b_k(x)y^k$, then it is as follows*

$$F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2 + c_3\beta^4,$$

with real constants c_i.

It is easy to see that (4.1) is a complete form of (4.1) and throughout this paper we consider (4.1) as the quartic (α, β)-metric. First, The below lemma is proposed to prove Theorem 1.2.

Lemma 4.2. *Assume $F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2 + c_3\beta^4$ be a quartic (α, β)-metric on a Finsler manifold (M, F) of dimensional n. Assume that F satisfies $\Xi = 0$. Then F is given by*

$$F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2$$

Also, β is a Killing one-form or where c_i are real constants.

Proof. Let $F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2 + c_3\beta^4$ be a 4-th root (α,β)-metric on a manifold M. By assumption, we have $\Xi_j = 0$. Let us define $\Xi := \Xi_j b^j$. Multiplying (3.4) with b^j yields

$$\Xi = f_7\alpha^7 + f_6\alpha^6 + f_5\alpha^5 + f_4\alpha^4 + f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0 = 0, \qquad (4.1)$$

where

$$\begin{aligned}
f_7 :=\ & 2863125\beta c_3^2 c_2 s_0 B r_0 + 1715520\beta c_3^2 r_{00} s_0 B c_2 - 545004 c_3^2 c_2 r_{00}^2 B^2\\
&+ 89280 c_3^2 c_1 r_{00}^2 B - 70320 c_3 c_1 r_{00}^2 c_2 + 710826 c_3 c_2^2 r_{00}^2 B + 422400\beta^2 c_3^3 s_0^2 B\\
&- 537600\beta^2 c_3^2 s_0^2 c_2 + 945000\beta c_3^3 s_0^2 B^2 + 237600\beta c_3^3 B^2 r_{00} + 47250\beta c_3^2 r_{0|0} c_1\\
&+ 41225\beta c_3 c_2^2 r_{00} + 15762\beta c_3 c_2^2 s_0^2 - 945000\beta c_3^3 s_0 B^2 r_0 - 2052\beta c_3^3 r_{00} B r_0\\
&- 1335600\beta c_3^3 r_{00} s_0 B + 420300\beta c_3^2 r_{00} c_2 r_0 - 129600\beta c_3^2 r_{00} s_0 c_1\\
&- 262500\beta c_3^2 c_1 s_0 r_0 - 2863125\beta c_3^2 c_2 s_0^2 B - 747675\beta c_3^2 c_2 r_{0|0} B\\
&- 1576250\beta c_3 c_2^2 s_0 r_0 + 91152 c_3^3 B^3 r_{00}^2 + 77760 c_3^3 r_{00}^2 B^2 + 270000 c_3^2 r_{00}^2 c_1\\
&+ 1227660 c_3 r_{00}^2 c_2^2 + 880\beta^2 c_3^3 s_0^2 - 2118 c_2^3 r_{00}^2 - 927440\beta c_3 r_{00} s_0 c_2^2\\
&+ 3225900\beta c_3^2 r_{00} s_0 c_2 + 21600\beta^2 c_3^3 s_0 r_0 - 127080 c_3^2 r_{00}^2 c_2 B + 262500\beta c_3^2 c_1 s_0^2\\
&- 580\beta c_3^3 B^2 r_{00} s_0 + (n+1)\Big(420\beta c_3^3 B r_{00} s_0 - 120\beta c_3^2 r_{00} s_0 c_2 + 120 c_3^3 r_{00}^2 B^2\\
&+ 205200 c_3^2 r_{00}^2 c_1 - 242160 c_3 r_{00}^2 c_2^2 - 480000\beta^2 c_3^3 s_0^2 - 38520 c_3^2 r_{00}^2 c_2 B\Big),\\
f_6 :=\ & 12440 c_3^2 c_2 r_{00}^2 B^2 - 1461600 c_3^2 r_{00}^2 c_2 B - 165975 c_3 c_1 r_{00}^2 c_2 + 7380\beta^2 c_3^3 s_0 r_0\\
&+ 54000\beta^2 c_3^3 r_{00} r + 54000\beta^2 c_3^3 r_{00} r_0 + 913500\beta^2 c_3^3 s_{0|0} B - 12000\beta^2 c_3^2 q_{00} c_2\\
&- 1500\beta^2 c_3^2 c_2 s_{0|0} - 101250\beta c_3^3 B^2 r_{00|0} - 37800\beta c_3^3 r_{00}^2 B - 78750\beta c_3^2 c_1 r_{00|0}\\
&- 3855\beta c_3 c_2^2 r_{00|0} - 337500\beta^3 c_3^3 s_0 B - 337500\beta^3 c_3^3 r_0 B + 450000\beta^3 c_3^2 c_2 s_0\\
&- 904500\beta c_3^3 r_{00} B r_0 - 316800\beta c_3^3 B^2 r_{00} s_0 - 1811700\beta c_3^3 r_{00} s_0 B\\
&- 78000\beta c_3^2 r_{00} s_0 c_1 + 3534300\beta c_3^2 r_{00} s_0 c_2 + 553500\beta c_3^2 c_2 r_{00|0} B\\
&+ 198720 c_3^3 B^3 r_{00}^2 + 35100 c_3^3 r_{00}^2 B^2 + 337500 c_3^2 r_{00}^2 c_1 + 1581300 c_3 r_{00}^2 c_2^2\\
&+ 1800\beta^3 c_3^3 q_0 + 123750\beta^3 c_3^3 s_0 + 270000\beta^2 c_3^3 s_0^2 - 5405 c_2^3 r_{00}^2 + 18000 c_3^3 \beta^2 q_{00}\\
&+ 676000\beta^2 c_3^2 s_0^2 c_2 + 13700\beta c_3^2 r_{00}^2 c_2 + 450000\beta^3 c_3^2 c_2 r_0 + 1957500\beta c_3^2 r_{00} c_2 r_0\\
&- 587550\beta c_3 r_{00} s_0 c_2^2 + 330000\beta^3 c_3^3 t_0 + (n+1)\Big(2050\beta c_3^3 r_{00} r_0\\
&- 573300\beta c_3^2 r_{00} c_2 r_0 - 2400 c_3^3 \beta^3 t_0 - 33750\beta^2 c_3^3 r_{00} - 54000 c_3^3 \beta^2 q_{00} - 10230 c_2 c_3^2 r_{00}^2\\
&+ 421200\beta c_3^3 B r_{00} s_0 - 393750\beta c_3^2 c_2 r_{00|0} B - 408300\beta c_3^2 r_{00} s_0 c_2\\
&+ 1028400\beta c_3^2 r_{00} s_0 B c_2 + 259200 c_3^3 r_{00}^2 B^2 + 75050 c_3 r_{00}^2 c_2^2 + 1185 c_3 c_2^2 r_{00}^2 B\\
&- 90000\beta^3 c_3^3 s_0 - 90000\beta^3 c_3^3 q_0 - 504000\beta^2 c_3^3 s_0^2 B + 360000\beta^2 c_3^3 s_0^2
\end{aligned}$$

$$
\begin{aligned}
&- 216000\beta^2 c_3^3 s_0 r_{00} + 10800\beta^2 c_3^3 B r_{00} - 612000\beta^2 c_3^3 s_{0|0} B - 5400\beta^2 c_3^3 s_0 r_0 \\
&- 173250\beta^2 c_3^2 c_2 r_{00} + 948000\beta^2 c_3^2 c_2 s_{0|0} + 36000\beta^2 c_3^2 q_{00} c_2 + 21600\beta c_3^3 r_{00}^2 B \\
&+ 54000\beta c_3^3 r_{00|0} B^2 - 56250\beta c_3^2 c_1 r_{00|0} - 126900\beta c_3^2 r_{00}^2 c_2 + 345375\beta c_3 c_2^2 r_{00|0}\Big), \\
f_5 :&= 552600\beta^2 c_3^2 r_{00}^2 c_2 - 506250\beta^3 c_3^3 s_0^2 B + 506250\beta^3 c_3^3 s_0 B r_0 + 621000\beta^3 c_3^3 r_{00} s_0 \\
&+ 637500\beta^3 c_3^2 c_2 s_0^2 + 316800\beta^3 c_3^3 r_{00} s_0 B - 637500\beta^3 c_3^2 c_2 s_0 r_0 + 8100\beta^3 c_3^3 r_{00} r_0 \\
&+ 3760\beta^2 c_3^2 c_2 r_{00}^2 - 14150\beta^3 c_3^3 r_{00} B - 391200\beta^3 c_3^2 r_{00} s_0 c_2 - 162795\beta^2 c_3 c_2^2 r_{00}^2 \\
&- 248400\beta^2 c_3^3 r_{00}^2 B - 104220\beta^2 c_3^3 r_{00}^2 B^2 - 96000 c_3^3 \beta^4 s_0^2 - 1980 c_1 \beta^2 c_3^2 r_{00}^2 \\
&+ 17875 c_2 \beta^3 c_3^2 r_{00} + (n+1)\Big(10800\beta^2 c_3^3 r_{00}^2 B - 252000\beta^3 c_3^3 r_{00} s_0 \\
&- 110700\beta^2 c_3^2 r_{00}^2 c_2\Big), \\
f_4 :&= 84375\beta^5 c_3^3 r_0 + 84375\beta^5 c_3^3 s_0 + 630000\beta^3 c_3^3 r_{00} s_0 + 108000 B\beta^3 c_3^3 r_{00|0} \\
&- 4250\beta^2 c_3^2 r_{00}^2 c_1 - 13250\beta^3 c_3^2 c_2 r_{00|0} - 283500\beta^2 c_3^3 r_{00}^2 B - 23760\beta^2 c_3^3 r_{00}^2 \\
&- 247500\beta^4 c_3^3 s_{0|0} - 4125\beta^2 c_3 c_2^2 r_{00}^2 + 18900\beta^3 c_3^3 r_{00} s_0 B - 24600\beta^3 c_3^2 r_{00} s_0 c_2 \\
&+ 120000\beta^4 c_3^3 s_0^2 + 27000\beta^3 c_3^3 r_{00}^2 + 3700\beta^3 c_3^3 r_{00} r_0 + 6750\beta^2 c_3^2 r_{00}^2 c_2 \\
&+ 746\beta^2 c_3^2 c_2 r_{00}^2 + (n+1)\Big(180000\beta^4 c_3^3 s_{0|0} - 33750\beta^4 c_3^3 r_{00} - 2700\beta^3 c_3^3 r_{00}^2 \\
&- 1890\beta^2 c_3^3 r_{00}^2 B - 650\beta^3 c_3^3 r_{00} s_0 - 1250\beta^3 c_3^3 r_{00} r_0 - 87750\beta^3 c_3^3 r_{00|0} B \\
&+ 1550\beta^3 c_3^2 c_2 r_{00|0} + 357750\beta^2 c_3^2 r_{00}^2 c_2\Big), \\
f_3 :&= 5940\beta^4 c_3^3 r_{00}^2 B - 700\beta^5 c_3^3 r_{00} s_0 + 112500 s_0^2 c_3^3 \beta^5 - 112500\beta^5 c_3^3 s_0 r_0 \\
&- 7110\beta^4 c_3^2 r_{00}^2 c_2 + 3750\beta^5 c_3^3 r_{00} + (n+1)\Big(180\beta^4 c_3^3 r_{00}^2 - 2700\beta^4 c_3^3 r_{00}^2\Big), \\
f_2 :&= 1350\beta^4 c_3^3 r_{00}^2 - 3750\beta^5 c_3^3 r_{00|0} + 141750\beta^4 c_3^3 r_{00}^2 B - 450\beta^5 c_3^3 r_{00} s_0 \\
&+ (n+1)\Big(650\beta^4 c_3^3 r_{00}^2 - 178875\beta^4 c_3^2 r_{00}^2 c_2 + 33750\beta^5 c_3^3 r_{00|0}\Big), \\
f_1 :&= -13500 c_3^3 \beta^6 r_{00}^2, \\
f_0 :&= -33750 c_3^3 \beta^6 r_{00}^2. \qquad (4.2)
\end{aligned}
$$

By (4.1), we get

$$f_7\alpha^6 + f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \qquad (4.3)$$

$$f_6\alpha^6 + f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \qquad (4.4)$$

(4.3)mention that there exists a function $\varrho = \varrho(x, y)$ where $\varrho \neq 0$, so that

$$c_3^3 r_{00}^2 \beta^6 = \varrho\alpha^2. \qquad (4.5)$$

In like manner (4.4) mention that there exists a function $\zeta = \zeta(x, y)$ where $\zeta \neq 0$, so that the following is established

$$c_3^3 r_{00}^2 \beta^6 = \zeta\alpha^2. \qquad (4.6)$$

Since $\varrho \neq \zeta$ and ϱ is not a multiplication of ζ, then by (4.5) and (4.6) we get

$$c_3^3 {r_{00}}^2 \beta^6 = 0. \tag{4.7}$$

If $c_3 = 0$, then F reduces to $F^4 = c_1\alpha^4 + c_2\alpha^2\beta^2$. If $c_3 \neq 0$, So we conclude that $r_{ij} = 0$ that means β is a Killing one-form. □

Proof of Theorem 1.2: Lemma 4.2 stated that $r_{ij} = 0$, which yields $r_i = 0$, $q_{ij} = 0$ and $q_i = 0$. Putting these relations in (4.2) yields

$$g_4\alpha^4 + g_3\alpha^3 + g_2\alpha^2 + g_1\alpha + g_0 = 0. \tag{4.8}$$

where

$$\begin{aligned}
g_4 :=\ & 1686c_3^2B\beta s_0^2 + 3520c_3^2\beta s_0^2 - 21504c_2c_3\beta s_0^2 + 37800c_3^2B^2s_0^2 - 1425c_2c_3Bs_0^2 \\
& + 1000c_1c_3s_0^2 + 63050c_2^2s_0^2 - 19200(n+1)c_3^2\beta s_0^2, \\
g_3 :=\ & 4950c_3^2\beta^2 s_0 + 130c_3^2\beta^2 t_0 - 1350c_3^2B\beta^2 s_0 + 1800c_2c_3\beta^2 s_0 + 1080c_3^2\beta s_0^2 \\
& + 3650c_3^2B\beta s_{0|0} - 2060c_3^2B\beta s_0^2 - 5060c_2c_3\beta s_{0|0} + 2040c_2c_3\beta s_0^2 \\
& + (n+1)\big(3720c_3c_2\beta s_{0|0} + 1400c_3^2\beta s_0^2 \\
& \qquad - 9600c_3^2\beta^2 t_0 - 3600c_3^2\beta^2 s_0 - 24480c_3^2\beta s_{0|0}B\big), \\
g_2 :=\ & c_3\big(25500c_2 - 3840c_3\beta - 20250c_3B\big)\beta^2 s_0^2, \\
g_1 :=\ & 3375c_3^2\beta^4 s_0 + 4800c_3^2\beta^3 s_0^2 + c_3^2\big(7200(n+1) - 9900\big)\beta^3 s_{0|0}, \\
g_0 :=\ & 4500c_3^2\beta^4 s_0^2.
\end{aligned}$$

By (4.8), we get

$$g_3\alpha^2 + g_1 = 0, \tag{4.9}$$

$$g_4\alpha^4 + g_2\alpha^2 + g_0 = 0. \tag{4.10}$$

(4.10) implies that

$$\rho s_0^2 = 0,$$

where

$$\begin{aligned}
\rho :=\ & \Big\{1686\beta c_3^2B + 3520\beta c_3^2 - 21504\beta c_2c_3 + 37800c_3^2B^2 - 114525c_3c_2B \\
& - 19200(n+1)c_3^2\beta\Big\}\alpha^4 + \Big\{25500c_2c_3\beta^2 - 3840c_3^2\beta^3 - 20250c_3^2B\beta^2\Big\}\alpha^2 \\
& + 4500c_3^2\beta^4 + 1000c_1c_3 + 63050c_2^2.
\end{aligned}$$

By (4.11), we get $\rho = 0$ or $s_i = 0$. One can rewrite $\rho = 0$ as

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0,$$

where

$\gamma := 1686\beta c_3^2 B + 3520\beta c_3^2 - 21504\beta c_2 c_3 + 37800 c_3^2 B^2 - 114525 c_3 c_2 B + 1000 c_1 c_3$
$\quad + 63050 c_2^2 - 19200(n+1) c_3^2 \beta,$

$\theta := 25500 c_2 c_3 \beta^2 - 3840 c_3^2 \beta^3 - 20250 c_3^2 B \beta^2,$

$\varepsilon := 4500 c_3^2 \beta^4.$

Solving (4.11) implies that

$$\alpha^2 = \left(\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\varepsilon}}{2\theta} \right) \beta^2. \tag{4.11}$$

Since α is positive-definite, then it is in contradiction with (4.11). So $\rho \neq 0$. Hence, we have $s_i = 0$. By putting $r_{ij} = 0$ and $s_i = 0$ in (2.5), we get $\mathbf{S} = 0$. The proof is complete. □

By using Lemma 3.3, one can give the below generalized version of Theorem 1.2.

Corollary 4.3. *Assume $F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2 + c_3\beta^4$ be quartic (α, β)-metric on an n-dimensional Finsler manifold (M, F). Assume that F has almost vanishing Ξ-curvature. Therefore, F is as follows*

$$F^4 = c_1\alpha^4 + c_2\beta^2\alpha^2,$$

whereF has vanishing S-curvature or c_i are real constants.

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