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Complex Lagrange spaces with a special $(\gamma, \beta)-$metric

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Abstract. In this paper we study the complex Lagrange space with a special $(\gamma, \beta)-$metric and determined the fundamental metric tensor, its inverse Euler-Lagrange equation, complex semi-spray coefficient, complex non-linear connection as well as Chern-Lagrange connections for Lagrange space with the mentioned special metric.

Keywords: $(\gamma, \beta)-$metric, inverse Euler-Lagrange equation, complex semi-spray coefficient, Chern-Lagrange connections for Lagrange space.

1. **Introduction**

Finsler space with (α, β) metric were studied by several geometers such as Hashiguchi, M. Matsumoto [7] and Kitayama [8]. The notion of (α, β) metric

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was generalised by to a space is called Lagrange space and by the authors Miron [10], Nicolaescu [1, 2] and Suresh K. Shukla and P. N. Pandey [13].

In 2011 Pandey and Chaubey [14] discussed Lagranges Spaces with (γ, β) and obtained several interesting result. In 2009 N. Aldea and G. Munteanu [9] introduced the notion of Complex Finsler space with (α, β) metric. In 2019, Sweta Kumari and P. N. Pandey worked on complex Randers space [12]. G. Munteanu [6] obtained various important result of complex Lagrange space [4]. Sweta Kumari and P. N. Pandey [11] studied the complex Lagrange space with (γ, β) metric in 2023.

In present paper, we studied the complex Lagrange space with a special (γ, β) metric

$$L = (\gamma + |\beta|)^2, \tag{1.1}$$

where

$$\gamma = \sqrt[3]{a_{\bar{i}\bar{j}\bar{k}}\eta^i \bar{\eta}^j \bar{\eta} k}, \tag{1.2}$$

$$|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\overline{\beta(z,\eta)}} \quad \text{with} \quad \beta(z,\eta) = b_i(z)\eta^i \tag{1.3}$$

γ is a cubic root metric and β is a differential $(1,0)$-form and determined the fundamental metric tensor, its inverse Euler-Lagrange equations, complex semi-spray coefficient, Complex non-linear connection is as well as Chern-Lagrange connections for complex Lagrange space with a special (γ, β) metric (1.1).

2. **Preliminaries**

Let M be a complex manifold of dimension n. Let $(z^k), \quad k = \overline{1,n}$ be local coordinates in a chart (U, z^k) and $T'M$ be its holomorphic tangent bundle. $T'M$ has a natural structure of complex manifold such that (z^k, η^k) are local coordinates in a chart on U belongs to $T'M$. A complex Lagrangian L [7] on $T'M$ is a real valued smooth function $L : T'M \to \mathbb{R}$

$$g_{i\bar{j}} = \frac{\partial^2}{\partial \eta^i \partial \eta^{\bar{j}}} L \tag{2.1}$$

is a non-degenerated metric and resolve a Hermitian metric structure. A complex Lagrange space is a pair $L^n = \{M, L(z,\eta)\}$. The presence of a complex Lagrange function L concern the study of the variational problems on curves. Let $c : [0,1] \to M$. be holomorphic curve and $L(z,\eta)$ be the complex Lagrangian on $T'M$. The Euler-Lagrange equations for a geodesic are

$$E_i(L) \equiv \frac{\partial}{\partial z^i} L - \frac{d}{dt}\left(\frac{\partial}{\partial \eta^i} L\right) = 0. \tag{2.2}$$

The coefficients of the complex semi-spray S of a complex Lagrange space $L^n = \{M, L(z,\eta)\}$ are

$$G^k(z,\eta) = \frac{1}{2} g^{\bar{i}k}\left(\frac{\partial^2}{\partial \eta^j \partial \eta^{\bar{i}}} L\right)\eta^j. \tag{2.3}$$

The Cartan-connection [6] of a complex Lagrange space $L^n = \{M, L(z,\eta)\}$ is

$$\overbrace{N_j^k}^{C} = \frac{\partial}{\partial \eta^j} G^i. \tag{2.4}$$

And the Chern-Lagrange connection $\overbrace{N_j^k}^{CL}$ [6] given by

$$\overbrace{N_j^k}^{CL} = g^{\bar{i}k} \frac{\partial^2}{\partial \eta^j \partial \eta^{\bar{i}}} L. \tag{2.5}$$

And above connections are associated by

$$\overbrace{N_j^k}^{C} = \frac{1}{2} \frac{\partial}{\partial \eta^j} \overbrace{N_0^k}^{CL}. \tag{2.6}$$

3. **Fundamental Metric Tensor**

Differentiating partially equation (1.2) with respect to η^l and $\bar{\eta}^m$ and using the symmetry of $a_{\bar{i}\bar{j}\bar{k}}$ in its indices, we obtain

$$\frac{\partial}{\partial \eta^l} \gamma = \frac{a_l}{3\gamma^2}, \tag{3.1}$$

$$\frac{\partial}{\partial \eta^{\bar{m}}} \gamma = \frac{2a_{\bar{m}}}{3\gamma^2}, \tag{3.2}$$

where $a_l = a_{\bar{i}\bar{j}\bar{k}} \eta^{\bar{j}} \eta^{\bar{k}}$ and $a_{\bar{m}} = a_{\bar{i}\bar{j}\bar{m}} \eta^i \eta^{\bar{j}}$.

Now, differentiating partially equation (3.1) with respect to $\bar{\eta}^p$, we have

$$\frac{\partial^2}{\partial \eta^l \partial \eta^{\bar{p}}} \gamma = \frac{2a_l \bar{p}}{3\gamma^2} - \frac{4a_l \bar{p}}{9\gamma^5} \tag{3.3}$$

where $a_l \bar{p} = a_{l\bar{p}\bar{k}} \bar{\eta} k$. Differentiating partially equation (1.3) with respect to η^l and $\bar{\eta}^m$ we get,

$$\frac{\partial}{\partial \eta^l} |\beta| = \frac{\bar{\beta} b_l}{2|\beta|} \tag{3.4}$$

$$\frac{\partial}{\partial \eta^{\bar{m}}} |\beta| = \frac{\beta b_{\bar{m}}}{2|\beta|}. \tag{3.5}$$

Again differentiating partially equation (3.1) with respect to $\bar{\eta}^p$, we find

$$\frac{\partial^2}{\partial \eta^l \partial \eta^{\bar{p}}} |\beta| = \frac{b_l b_{\bar{P}}}{4|\beta|}. \tag{3.6}$$

Then, we have the following.

Proposition 3.1. *In a complex Lagrange space with a special* (γ, β) *metric satisfy the equations* $(3.1), (3.2), (3.3), (3.4), (3.5) and (3.6)$.

The moments of Lagrangian $L = (\gamma + |\beta|)^2$ are defined as

$$p_i = \frac{1}{2}\frac{\partial}{\partial \eta^i}(\gamma + |\beta|)^2.$$

By using the equations (3.1) and (3.4), we get

$$p_i = \frac{(\gamma + |\beta|)}{3\gamma^2}a_i + \frac{(\gamma + |\beta|)\bar{\beta}}{2|\beta|}b_i. \tag{3.7}$$

Theorem 3.2. *In a complex Lagrange space L^n with a special (γ, β), the moments of Lagrangian $L = (\gamma + |\beta|)^2$ are as follows*

$$p_i = \rho a_i + \rho_1 b_i,$$

where

$$\rho = \frac{(\gamma + |\beta|)}{3\gamma^2} \tag{3.8}$$

$$\rho_1 = \frac{(\gamma + |\beta|)\bar{\beta}}{2|\beta|}. \tag{3.9}$$

Here, the scalars ρ and ρ_1 are called the principal invariants of the space L^n. Differentiating partially equations (3.8) and (3.9) with respect to η^j and $\bar{\eta}^l$ we get,

$$\frac{\partial}{\partial \eta^j}\rho = \frac{1}{9}\gamma^{-4}\{1 - 2\gamma^{-1}(\gamma + |\beta|)\}a_j + \frac{1}{6}\bar{\beta}|\beta|^{-1}\gamma^{-2}b_j.$$
$$\frac{\partial}{\partial \eta^{\bar{j}}}\rho = \frac{2}{9}\gamma^{-4}\{1 - 2\gamma^{-1}(\gamma + |\beta|)\}a_{\bar{j}} + \frac{1}{6}\bar{\beta}|\beta|^{-1}\gamma^{-2}b_{\bar{j}}.$$
$$\frac{\partial}{\partial \eta^j}\rho_1 = \frac{1}{6}\bar{\beta}|\beta|^{-1}\gamma^{-2}a_j + \frac{1}{4}\bar{\beta}|\beta|^{-1}\{1 + |\beta|^{-1}(\gamma + |\beta|)\}b_j.$$
$$\frac{\partial}{\partial \eta^{\bar{j}}}\rho_1 = \frac{1}{3}\bar{\beta}|\beta|^{-1}\gamma^{-2}a_{\bar{j}} + \frac{1}{4}|\beta|^{-1}\{1 + |\beta|^{-1}(\gamma + |\beta|)\}b_{\bar{j}}.$$

Thus, we get

Theorem 3.3. *The derivatives of the principal invariants of a complex Lagrange space L^n with a special (γ, β) metric are given by*

$$\frac{\partial}{\partial \eta^j}\rho = \rho_{-2}a_j + \bar{\beta}\beta^{-1}\rho_{-1}b_j. \tag{3.10}$$

$$\frac{\partial}{\partial \eta^{\bar{j}}}\rho = \rho_{-2}a_{\bar{j}} + \rho_{-1}b_{\bar{j}}. \tag{3.11}$$

$$\frac{\partial}{\partial \eta^j}\rho_1 = \bar{\beta}\beta^{-1}(\rho_{-1}a_j + \rho_0 b_j). \tag{3.12}$$

$$\frac{\partial}{\partial \eta^{\bar{j}}}\rho_1 = 2\bar{\beta}\beta^{-1}(\rho_{-1}a_{\bar{j}} + \rho_0 b_{\bar{j}}) \tag{3.13}$$

where

$$\rho_{-2} = \frac{1}{9}\gamma^{-4}\Big\{1 - 2\gamma^{-1}(\gamma + |\beta|)\Big\}. \tag{3.14}$$

$$\rho_{-1} = \frac{1}{6}\beta|\beta|^{-1}\gamma^{-2}. \tag{3.15}$$

$$\rho_0 = \frac{1}{4}\Big\{1 + |\beta|^{-1}(\gamma + |\beta|)\Big\}. \tag{3.16}$$

The energy of the complex Lagrangian L is given by

$$E_L = \eta^i \frac{\partial}{\partial \eta^j} L - L. \tag{3.17}$$

Put $L = (\gamma + |\beta|)^2$ in equation (3.17), we get

$$E_L = \eta^i \Big\{ \frac{\partial}{\partial \gamma}(\gamma + |\beta|)^2 \frac{\partial}{\partial \eta^i}\gamma + \frac{\partial}{\partial |\beta|}(\gamma + |\beta|)^2 \frac{\partial}{\partial \eta^i}|\beta| \Big\} - (\gamma + |\beta|)^2. \tag{3.18}$$

We know that γ and $|\beta|$ positively homogeneous of degree 1 in η^i, therefore by Euler's theorem on homogeneous functions, we come to a result

$$\eta^i \frac{\partial}{\partial \eta^i}\gamma = \frac{\gamma}{3} \tag{3.19}$$

$$\eta^i \frac{\partial}{\partial \eta^i}|\beta| = \frac{|\beta|}{2}. \tag{3.20}$$

With the help of equations (3.19) and (3.20), the equation (3.18) becomes

$$E_L = \frac{2\gamma}{3}(\gamma + |\beta|) + |\beta|(\gamma + |\beta|) - (\gamma + |\beta|)^2 \tag{3.21}$$

$$E_L = -\frac{\gamma}{3}(\gamma + |\beta|). \tag{3.22}$$

This leads to

Theorem 3.4. *The energy of the Lagrangian L in a complex Lagrange space with a special (γ, β) metric is given by* (3.22).

Now, we find the fundamental metric tensor $g_{i\bar{j}}(z, \eta)$ of a complex Lagrange space with a special (γ, β) metric. By using equation (1.1) in equation (2.1), we get

$$\begin{aligned} g_{\bar{i}\bar{j}} =& \frac{2(\gamma + |\beta|)}{3\gamma^2} a_{\bar{i}\bar{j}} + \Big\{ \frac{2}{9\gamma^4} - \frac{4(\gamma + |\beta|)}{9\gamma^5} \Big\} a_i a_{\bar{j}} \\ &+ \left(\frac{\bar{\beta}}{3\gamma^2 |\beta|} a_{\bar{j}} b_i + \frac{\beta}{6\gamma^2 |\beta|} a_i b_{\bar{j}} \right) + \Big\{ \frac{\beta\bar{\beta}}{4|\beta|^2} + \frac{(\gamma + |\beta|)}{4|\beta|} \Big\}. \end{aligned} \tag{3.23}$$

With the help of equations $(3.8), (3.14), (3.15)$ and (3.16) the equation (3.23) becomes

$$g_{\bar{i}\bar{j}} = 2\rho a_{\bar{i}\bar{j}} + \rho_{-2} a_i a_{\bar{j}} + \beta^{-1}\rho_{-1}(2\bar{\beta} a_{\bar{j}} b_i + \beta a_i b_{\bar{j}}) + \rho_0 b_i b_{\bar{j}}. \tag{3.24}$$

A normal calculation shows that

$$(2\bar{\beta} a_{\bar{j}} b_i + \beta a_i b_{\bar{j}}) = \frac{3\gamma^2 |\beta|}{2(\gamma + |\beta|)^2} \eta_i \bar{\eta}_j - \frac{3\gamma^2 |\beta|}{2} b_i b_{\bar{j}} - \frac{4|\beta|}{3\gamma^2} a_i a_{\bar{j}}, \tag{3.25}$$

where $\eta_i = \dot{\partial}_i L$ and $\bar{\eta}_j = \dot{\partial}_{\bar{j}} L$. By using equation (3.25), equation (3.24) becomes

$$g_{i\bar{j}} = 2\rho a_{i\bar{j}} + q_{-2} a_i a_{\bar{j}} + q_{-1}\eta_i\bar{\eta}_j + q_0 b_i b_{\bar{j}}, \tag{3.26}$$

where

$$q_{-2} = 2\left(\rho_{-2} - \frac{2|\beta|\rho_{-1}}{3\gamma^2\beta}\right),$$
$$q_{-1} = \frac{3\gamma^2|\beta|\rho_{-1}}{\beta(\gamma+|\beta|)^2}$$
$$\text{and} \quad q_0 = \rho_0 - \frac{3\gamma^2|\beta|\rho_{-1}}{2\beta}.$$

Equation (3.26) can be written as

$$g_{i\bar{j}} = 2\rho a_{i\bar{j}} + c_i c_{\bar{j}}, \tag{3.27}$$

where

$$c_i = r_{-1} a_i + r_0 b_i, \quad r_0 r_{-1} = q_{-1}, \quad (r_{-1})^2 = q_{-2}, (r_0)^2 = q_0.$$

Thus, we obtain

Theorem 3.5. *The expression for the fundamental metric tensor $g_{i\bar{j}}$ of a complex Lagrange space with a special (γ, β) metric is given by equation (3.27).*

With the help of proposition given by D. Bao, S. S. Chern and Z. Shen [3], the inverse $g^{\bar{j}i}$ of the fundamental metric tensor $g_{i\bar{j}}$ is obtained as

$$g^{\bar{j}i} = \frac{1}{2\rho}(a^{\bar{j}i} - \frac{1}{2\rho + c^2} c^i c^{\bar{j}}) \tag{3.28}$$

where $c^i = a^{\bar{j}i} c_{\bar{j}}$, $c^2 = a^{\bar{j}i} c_i c_{\bar{j}}$. This gives:

Theorem 3.6. *The inverse $g^{\bar{j}i}$ of the fundamental metric tensor $g_{i\bar{j}}$ of a complex Lagrange space with a special (γ, β)-metric is given by (3.28).*

4. **Euler-Lagrange Equations**

By using equation (1.1), the equation (2.2) becomes

$$\begin{aligned} E_i(L) =& 2(\gamma+|\beta|)E_i(\gamma) + 2(\gamma+|\beta|)E_i(|\beta|) - \left(2\frac{d\gamma}{dt} + 2\frac{d|\beta|}{dt}\right)\frac{\partial\gamma}{\partial\eta^i} \\ & - \left(2\frac{d\gamma}{dt} + 2\frac{d|\beta|}{dt}\right)\frac{\partial|\beta|}{\partial\eta^i} = 0 \end{aligned} \tag{4.1}$$

and

$$E_i(\gamma^3) = 3\gamma^2 E_i(\gamma) - 3\frac{\partial\gamma}{\partial\eta^i}\frac{d\gamma^2}{dt}, \tag{4.2}$$

$$E_i(|\beta|^2) = 2|\beta|E_i(|\beta|) - 2\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt}. \tag{4.3}$$

Putting values of $E_i(\gamma)$ and $E_i(|\beta|)$ from equations (4.2) and (4.3) in equation (4.1), we get

$$\begin{aligned} E_i(L) \equiv & 2\rho E_i(\gamma^3) + \frac{2}{\bar{\beta}}\rho_1 E_i(|\beta|^2) + 6\rho \frac{\partial\gamma}{\partial\eta^i}\frac{d\gamma^2}{dt} + \frac{4}{\bar{\beta}}\rho_1 \frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} \\ & - \frac{\partial\gamma}{\partial\eta^i}\left(2\frac{d\gamma}{dt} + 2\frac{d|\beta|}{dt}\right) - \frac{\partial|\beta|}{\partial\eta^i}\left(2\frac{d\gamma}{dt} + 2\frac{d|\beta|}{dt}\right) = 0. \end{aligned} \tag{4.4}$$

This gives

Theorem 4.1. *The Euler-Lagrange equations of a complex Lagrange space with a special (γ, β)-metric is given by equation (4.4).*

For the natural parameterization of the curve $c : t \in [1, 0] \to z^i(t) \in M$ with respect to the cubic-root metric γ, we have

$$\gamma\left(z, \frac{dz}{dt}\right) = 1.$$

Thus, we find

Theorem 4.2. *The natural parameterization, the Euler–Lagrange equations of a complex Lagrange space with a special (γ, β)-metric is*

$$E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\bar{\beta}}\rho_1 E_i(|\beta|^2) + \frac{4}{\bar{\beta}}\rho_1 \frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} - 2\frac{\partial\gamma}{\partial\eta^i}\frac{d|\beta|}{dt} - 2\frac{\partial|\beta|}{\partial\eta^i}\frac{d|\beta|}{dt} = 0.$$

If $|\beta|$ is constant along the integral curve of the Euler-Lagrange equations with natural parameterization, then the Euler-Lagrange equations of a complex Lagrange space with a special (γ, β) metric is given by

$$E_i(L) \equiv 2\rho E_i(\gamma^3) + \frac{2}{\bar{\beta}}\rho_1 E_i(|\beta|^2) = 0. \tag{4.5}$$

This gives

Theorem 4.3. *If $|\beta|$ is constant along the integral curve of the Euler-Lagrange equations with natural parameterization, then the Euler-Lagrange equation of a complex Lagrange space with a special (γ, β) metric are given by equation (4.5).*

5. **Complex Canonical Semi-spray**

The coefficients of the complex canonical semi-spray of a complex Lagrange space with a special (γ, β)-metric is given by equation (2.3) together with equation (1.1).
Differentiating partially equations (1.2) and (1.3) with respect to z^h, we get

$$\frac{\partial}{\partial z^h}\gamma = A_h \gamma^{-2} \tag{5.1}$$

$$\frac{\partial}{\partial z^h}|\beta| = \frac{\bar{\beta}}{2|\beta|}B_h + \frac{\beta}{2|\beta|}C_h, \tag{5.2}$$

where

$$A_h = \partial_h(a_{\overline{ijk}})\eta^i\bar{\eta}^j\bar{\eta}^k, \quad B_h = (\partial_h b_i)\eta^i, \quad C_h = (\partial_h b_{\bar{j}})\bar{\eta}^j.$$

Putting equations (4.5), (5.1) and (5.2) in

$$\frac{\partial}{\partial\eta^k}L = L_\gamma \partial_k\gamma + L_{|\beta|}\partial_k|\beta|,$$

we obtain

$$\frac{\partial}{\partial\eta^k}(\gamma + |\beta|)^2 = 6\rho A_k + 2\rho_1(B_k + \frac{\beta}{\bar{\beta}}C_k). \tag{5.3}$$

Differentiating partially equation (5.3) with respect to $\bar{\eta}^h$, we get

$$\begin{aligned}\frac{\partial^2}{\partial\bar{\eta}^h\partial\eta^k}(\gamma + |\beta|)^2 = &\left(6\rho_{-2}A_k + \frac{4\bar{\beta}}{\beta}\rho_{-1}B_k + 4\rho_{-1}C_k\right)a_{\bar{h}} \\ &+ \left(6\rho_{-1}A_k + 2\rho_0 B_k + \frac{2\beta}{\bar{\beta}}\rho_0 C_k - 2\rho_1\frac{2\beta}{\bar{\beta}^2}C_k\right)b_{\bar{h}} \\ &+ \left(6\rho A_{k\bar{h}} + 2\rho_1 B_{k\bar{h}} + \frac{2\beta}{\bar{\beta}}\rho_1 C_{k\bar{h}}\right).\end{aligned} \tag{5.4}$$

where

$$A_{k\bar{h}} = \frac{\partial}{\partial\bar{\eta}^h}A_k, \quad B_{k\bar{h}} = \frac{\partial}{\partial\bar{\eta}^h}B_k, \quad C_{k\bar{h}} = \frac{\partial}{\partial\bar{\eta}^h}C_k. \tag{5.5}$$

$$\begin{aligned}\frac{\partial^2}{\partial\bar{\eta}^h\partial\eta^k}(\gamma + |\beta|)^2\eta^k = &\left(6\rho_{-2}A_0 + \frac{4\bar{\beta}}{\beta}\rho_{-1}B_0 + 4\rho_{-1}C_0\right)a_{\bar{h}} \\ &+ \left(6\rho_{-1}A_0 + 2\rho_0 B_0 + \frac{2\beta}{\bar{\beta}}\rho_0 C_0 - 2\rho_1\frac{2\beta}{\bar{\beta}^2}C_0\right)b_{\bar{h}} \\ &+ \left(6\rho A_{0\bar{h}} + 2\rho_1 B_{0\bar{h}} + \frac{2\beta}{\bar{\beta}}\rho_1 C_{0\bar{h}}\right).\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}&A_0 = A_k(z,n)\eta^k, \quad A_0 = A_k(z,n)\eta^k, \quad B_0 = B_k(z,n)\eta^k, \quad C_0 = C_k(z,n)\eta^k \\ &A_{0\bar{h}} = A_{k\bar{h}}(z,n)\eta^k, \quad B_{0\bar{h}} = B_{k\bar{h}}(z,n)\eta^k, \quad C_{0\bar{h}} = C_{k\bar{h}}(z,n)\eta^k.\end{aligned} \tag{5.7}$$

Substituting equation (5.6) in equation (2.3), we find

$$\begin{aligned}G^i = g^{\bar{h}i}[&\left(3\rho_{-2}A_0 + \frac{2\bar{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\right)a_{\bar{h}} \\ &+ \left(3\rho_{-1}A_0 + \rho_0 B_0 + \frac{\beta}{\bar{\beta}}\rho_0 C_0 - \rho_1\frac{\beta}{\bar{\beta}^2}C_0\right)b_{\bar{h}} \\ &+ \left(3\rho A_{0\bar{h}} + \rho_1 B_{0\bar{h}} + \frac{\beta}{\bar{\beta}}\rho_1 C_{0\bar{h}}\right)].\end{aligned} \tag{5.8}$$

Hence, we have

Theorem 5.1. *The coefficient of the complex canonical semi-spray of a complex Lagrange space with a special (γ, β) metric is given by (5.8).*

6. **Canonical Complex Nonlinear Connection and Chern-Lagrange Connection**

Now we find the coefficients of the nonlinear connection $\overbrace{N_j^k}^{C}$ and Chern-Lagrange connection $\overbrace{N_j^k}^{CL}$ of a complex Lagrange space with a special (γ, β)-metric. Partial differentiation of $g^{\bar{h}i} g_{\bar{h}i} = \delta_j^i$ with respect to η^j, gives

$$\frac{\partial}{\partial \eta^j} g^{\bar{h}i} = -g^{\bar{h}r} C_{rj}^i. \tag{6.1}$$

Differentiating partially equations (3.14), (3.15), (3.16) and (5.7) with respect to η^j, we obtain

$$\frac{\partial}{\partial \eta^j} \rho_{-2} = \mu_{-3} a_j + \mu_{-2} b_j, \tag{6.2}$$

$$\frac{\partial}{\partial \eta^j} \rho_{-1} = \frac{1}{2} \beta \bar{\beta}^{-1} \mu_{-2} a_j + \mu_{-1} b_j \tag{6.3}$$

$$\frac{\partial}{\partial \eta^j} \rho_0 = \mu_{-1} a_j + \mu_0 b_j, \tag{6.4}$$

$$\frac{\partial}{\partial \eta^j} A_0 = A_j + A_{0j}, \tag{6.5}$$

$$\frac{\partial}{\partial \eta^j} C_0 = C_j, \tag{6.6}$$

$$\frac{\partial}{\partial \eta^j} A_{0\bar{h}} = 2A_{0\bar{h}j} + A_{j\bar{h}} \tag{6.7}$$

$$\frac{\partial}{\partial \eta^j} C_{0\bar{h}} = C_{j\bar{h}}, \tag{6.8}$$

$$\frac{\partial}{\partial \eta^j} a_{\bar{h}} = 2a_{j\bar{h}} \tag{6.9}$$

$$\frac{\partial}{\partial \eta^j} B_0 = \mathfrak{S}_{kj} \{ \frac{\partial}{\partial \eta^k} b_j \} \eta^k, \tag{6.10}$$

$$\frac{\partial}{\partial \eta^j} B_{0\bar{h}} = B_{j\bar{h}}. \tag{6.11}$$

where

$$\mu_{-3} = \frac{2}{27} \gamma^{-8} \{ -6\gamma + 10(\gamma + |\beta|) \},$$

$$\mu_{-2} = \frac{-2}{9} \gamma^{-5} \bar{\beta} |\beta|^{-1},$$

$$\mu_{-1} = \frac{1}{12} \gamma^{-2} |\beta|^{-1},$$

$$\mu_0 = \frac{1}{8}\bar{\beta}|\beta|^{-1}\{|\beta| - (\gamma + |\beta|)\},$$
$$A_{0\bar{h}j} = A_{r\bar{h}j}\eta^r,$$
$$A_{r\bar{h}j} = \frac{\partial}{\partial\eta^j}a_{\bar{h}j},$$

and $\mathfrak{S}_{kj}$ shows that the interchange of the indices k and j, and addition. By applying equation (5.8) in equation (2.4), we find

$$\overbrace{N^i_j}^{C} = \frac{1}{2}\frac{\partial}{\partial\eta^j}g^{\bar{h}i}\Bigg[\Big(3\rho_{-2}A_0 + \frac{2\bar{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\Big)a_{\bar{h}}$$
$$+ \Big(3\rho_{-1}A_0 + \rho_0 B_0 + \frac{\beta}{\bar{\beta}}\rho_0 C_0 - \rho_1\frac{\beta}{\bar{\beta}^2}C_0\Big)b_{\bar{h}} + \Big(3\rho A_{0\bar{h}} + \rho_1 B_{0\bar{h}} + \frac{\beta}{\bar{\beta}}\rho_1 C_{0\bar{h}}\Big)\Bigg]$$
$$+ g^{\bar{h}i}\frac{\partial}{\partial\eta^j}\Bigg[\Big(3\rho_{-2}A_0 + \frac{2\bar{\beta}}{\beta}\rho_{-1}B_0 + 2\rho_{-1}C_0\Big)a_{\bar{h}} + \Big(3\rho_{-1}A_0 + \rho_0 B_0$$
$$+ \frac{\beta}{\bar{\beta}}\rho_0 C_0 - \rho_1\frac{\beta}{\bar{\beta}^2}C_0\Big)b_{\bar{h}} + \Big(3\rho A_{0\bar{h}} + \rho_1 B_{0\bar{h}} + \frac{\beta}{\bar{\beta}}\rho_1 C_{0\bar{h}}\Big)\Bigg]. \tag{6.12}$$

With the help of (3.10), (3.11), (3.12), (3.13), (5.5), (5.7), (6.1) and (6.2) in (6.3) and simplifying, we get

$$\overbrace{N^i_j}^{C} = C^i_{rj}G^r + g^{\bar{h}i}\Big[\rho_{-2}\{3(A_{0j} + A_j)a_{\bar{h}} + 6A_0 a_{j\bar{h}} + \frac{3}{2}a_{0\bar{h}}a_j\}\Big]$$
$$\rho_{-1}\Big\{(3A_{0j} + 3A_j - a_j\bar{\beta}^{-1}C_0)b_{\bar{h}} + 4(\bar{\beta}\beta^{-1}B_0 + C_0)a_{j\bar{h}}$$
$$+(3\bar{\beta}\beta^{-1}A_{0\bar{h}} - \bar{\beta}\beta^{-2}B_0 a_{\bar{h}})b_j + 2(\bar{\beta}\beta^{-1}\mathfrak{S}_{kj}\eta^k\frac{\partial}{\partial\eta^k}b_j + C_j)a_{\bar{h}}$$
$$+\bar{\beta}\beta^{-1}(B_{0\bar{h}} + \beta^{-1}\beta C_{0\bar{h}})a_j\Big\} + \rho_0\Big\{(\mathfrak{S}_{kj}\eta^k\frac{\partial}{\partial\eta^k}b_j + \bar{\beta}\beta^{-1}C_j)b_{\bar{h}}$$
$$+\bar{\beta}\beta^{-1}(B_{0\bar{h}} + \bar{\beta}^{-1}\beta C_{0\bar{h}})b_j\Big\} + \rho_1\Big\{\bar{\beta}^{-1}\beta C_{0\bar{h}}b_j - \beta^{-2}(b_j C_0 - \beta C_j)b_{\bar{h}}$$
$$+\bar{\beta}^{-1}\beta C_{j\bar{h}} + B_{j\bar{h}}\Big\} + 3\rho(2A_{0\bar{h}}j + A_{j\bar{h}}) + 3\mu_{-3}a_j a_{\bar{h}}A_0$$
$$+\mu_{-2}\Big\{(B_0 + \bar{\beta}^{-1}\beta C_0)a_j a_{\bar{h}} + A_0 b_j a_{\bar{h}}\Big\} + \mu_{-1}\Big\{(B_0 + \bar{\beta}^{-1}\beta C_0)a_j b_{\bar{h}}$$
$$+2(\bar{\beta}\beta^{-1}B_0 + C_0)b_j a_{\bar{h}} + 3A_0 b_j b_{\bar{h}}\Big\}. \tag{6.13}$$

Using equation (5.4) in equation (2.5), we get

$$\overbrace{N_j^i}^{CL} = g^{\bar{i}k}[(6\rho_{-2}A_j + \frac{4\bar{\beta}}{\beta}\rho_{-1}B_j + 4\rho_{-1}C_0)a_{\bar{i}} + (6\rho_{-1}A_j + 2\rho_0 B_j + \frac{2\beta}{\bar{\beta}}\rho_0 C_j - 2\rho_1\frac{2\beta}{\bar{\beta}^2}C_j)b_{\bar{i}} + (6\rho A_{j\bar{i}} + 2\rho_1 B_{j\bar{i}} + \frac{2\beta}{\bar{\beta}}\rho_1 C_{j\bar{i}})]. \tag{6.14}$$

Theorem 6.1. *The coefficients of complex nonlinear connection and Chern–Lagrange connection of a complex Lagrange space with a special (γ, β)-metric are given by equation (6.4) and equation (6.5) respectively.*

7. Conclusions

The theory of Complex-Lagrange spaces developed with metric (1.1) plays a crucial role in further study of works of G. Muntanu [5, 6]. The several results obtained in this paper will be applicable in extensions work of connections, holomorphic curvature and torsions. The results regarding complex canonical spray, complex non-linear connections and Chern Lagrange connections obtained in the paper can be used in geodesic correspondence between any two complex Lagrange spaces developed by two different (γ, β) metrics.

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