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Research Paper

SECURE MONOPHONIC DOMINATION NUMBER OF SUBDIVISION OF GRAPHS

K. SUNITHA[1], D. JOSEPHINE DIVYA[2,*]

[1]Department of Mathematics, Scott Christian College(Autonomous), Tamil Nadu, India, ksunithasam@gmail.com

[2]Department of Mathematics, Scott Christian College(Autonomous), Nagercoil, Tamilnadu, India, (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, India.), divyajosephine1999@gmail.com

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ABSTRACT

Let $G = (V, E)$ be a connected graph. A monophonic dominating set M is said to be a secure monophonic dominating set (abbreviated as SMD set) of G if for each $v \in V \backslash M$ there exists $u \in M$ such that v is adjacent to u and $\{M \backslash (u)\} \bigcup \{v\}$ is a monophonic dominating set. The minimum cardinality of a secure monophonic dominating set of G is the secure monophonic domination number of G and is denoted by $\gamma_{sm}(G)$. In this paper, we investigate the secure monophonic domination number of subdivision of graphs such as subdivision of Path graph $S(P_n)$, subdivision of Cycle graph $S(C_n)$, subdivision of Star graph $S(K_{1,n-1})$, subdivision Bistar graph $S(B_{m,n})$ and subdivision of Y- tree graph $S(Y_{n+1})$.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. A chord of a path P is an edge which connects two non-consecutive vertices of P. For two vertices u and v, the closed interval $J[u, v]$ consists of all the vertices lying in a $u - v$ monophonic path including the vertices u and v. For a set M of vertices, let $J[M] = \bigcup_{u,v \in M} J[u, v]$. Then certainly $M \subseteq J[M]$. A set $M \subseteq V(G)$ is called a monophonic set of G if $J[M] = V$. In [3], Haynes introduced the concept of domination in graphs.The secure dominating set was introduced by Cockayne et al in [2]. In 2012. John et al [4]

*Address correspondence to D. Josephine Divya; E-mail: divyajosephine1999@gmail.com

introduced the concept of monophonic domination number of a graph. In this sequel, we introduced the secure monophonic domination number of graphs. For basic graph theoretic terminolgy, we refer [1].

Definition 1.1. [5] The Subdivision of a graph G, $S(G)$, is obtained from G by inserting a new vertex in the middle of every edge of G.

Definition 1.2. The Bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendent edges to one end of K_2 and n pendent edges to the other end of K_2.

Definition 1.3. A Y-tree graph Y_{n+1} is obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end point.

Definition 1.4. [10] A monophonic dominating set M is said to be a secure monophonic dominating set S_m(abbreviated as SMD set) of Gif for each $v \in V \backslash M$ there exists $u \in M$ such that v is adjacent to u and $S_m = \{M \backslash (u)\} \bigcup \{v\}$ is a monophonic dominating set. The minimum cardinality of a secure monophonic dominating set of G is the secure monophonic domination number of G and is denoted by $\gamma_{sm}(G)$.

Observation

• Each end vertex of a connected graph G belongs to every SMD set of G.

2. Main Results

Theorem 2.1. *For the Subdivision of Path graph $G = S(P_n)$, $n \geq 2$,*

$$\gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 2 \\ n & if \quad 3 \leq n \leq 6 \\ \lceil \frac{6n+5}{7} \rceil & if \quad n \equiv 0,1,3,4,5,6 (mod 7) \\ \lceil \frac{6n+5}{7} \rceil + 1 & if \quad n \equiv 2 (mod 7) \end{cases}, n \geq 7.$$

Proof: *Let $f_i, 1 \leq i \leq 2n-1$ be the vertices of G. If $n = 2$ then $G = S(P_2)$,$S_m = \{f_1, f_2, f_3\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 3$. If $n = 3$ then $G = S(P_3)$, $S_m = \{f_1, f_3, f_5\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 3$. If $n = 4$ then $G = S(P_4)$, $S_m = \{f_1, f_3, f_5, f_7\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$. If $n = 5$ then $G = S(P_5)$, $S_m = \{f_1, f_3, f_5, f_7, f_9\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 5$. If $n = 6$ then $G = S(P_6)$, $S_m = \{f_1, f_3, f_5, f_7, f_9, f_{11}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 6$. Hence*

$$\gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 2 \\ n & if \quad 3 \leq n \leq 6 \end{cases}$$

Let $n \geq 7$. Now we consider the following cases.
case (a) subcase(i): $n \equiv 0 (mod 7)$
We take $G = S(P_7)$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}\}$. Remove any vertex $f_i \in S_m$ and add another vertex $f_j \in V - S_m$ to S_m such that f_i is adjacent to f_j. Hence the set S_m is again a secure dominating set of G.Also the monophonic path exists and it contain all the vertices of G.Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}\}$ is a minimum SMD set of G.In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}\}$ is a minimum SMD set of G.Therefore

$|S_m| = \lceil \frac{6n+5}{7} \rceil$.

subcase(ii): $n \equiv 1 (mod 7)$

We take $G = S(P_8)$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}\}$. *Then by similar argument as in subcase(i).Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil$.

subcase(iii): $n \equiv 3 (mod 7)$

We take $G = S(P_{10})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}\}$. *Then by similar argument as in subcase(i). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil$.

subcase(iv): $n \equiv 4 (mod 7)$

We take $G = S(P_{11})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}\}$.*Then by similar argument as in subcase(i). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil$.

subcase(v): $n \equiv 5 (mod 7)$

We take $G = S(P_{12})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}\}$.*Then by similar argument as in subcase(i). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}, f_{2n-9}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil$.

subcase(vi): $n \equiv 6 (mod 7)$

We take $G = S(P_{13})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}\}$.*Then by similar argument as in subcase(i). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}, f_{2n-9}, f_{2n-11}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil$.

case b: $n \equiv 2 (mod 7)$

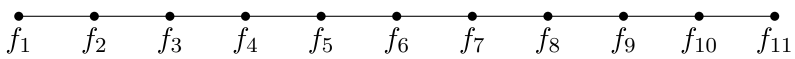
We take $G = S(P_9)$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}, f_{17}\}$. *Then by similar argument as in subcase(i). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}, f_{17}\}$ *is a minimum SMD set of* G.*In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}\}$ *is a minimum SMD set of* G.*Therefore* $|S_m| = \lceil \frac{6n+5}{7} \rceil + 1$. *Finally we conclude that* $S_m =$

$$\{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}\} \bigcup \begin{cases} \phi & for \quad r = 0 \\ f_{2n-3} & for \quad r = 1 \\ f_{2n-5}, f_{2n-3} & for \quad r = 2, 3 \\ f_{2n-7}, f_{2n-5}, f_{2n-3} & for \quad r = 4 \\ f_{2n-9}, f_{2n-7}, f_{2n-5}, f_{2n-3} & for \quad r = 5 \\ f_{2n-11}, f_{2n-9}, f_{2n-7}, f_{2n-5}, f_{2n-3} & for \quad r = 6 \end{cases}$$

$$\text{Hence } \gamma_{sm}(G) = \begin{cases} \lceil \frac{6n+5}{7} \rceil & if \quad n \equiv 0,1,3,4,5,6(mod7) \\ \lceil \frac{6n+5}{7} \rceil + 1 & if \quad n \equiv 2(mod7) \end{cases}.$$

$$\text{Therefore } \gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 2 \\ n & if \quad 3 \leq n \leq 6 \\ \lceil \frac{6n+5}{7} \rceil & if \quad n \equiv 0,1,3,4,5,6(mod7) \\ \lceil \frac{6n+5}{7} \rceil + 1 & if \quad n \equiv 2(mod7) \end{cases}, n \geq 7$$

Example 2.2. The secure monophonic domination number of the Subdivision of Path graph $S(P_6)$ in Figure 1



$\gamma_{sm}(S(P_6)) = 6.$

Figure 1

Theorem 2.3. *For the subdivision of cycle graph $G = S(C_n)$, $n \geq 3$,*
$$\gamma_{sm}(G) = \begin{cases} n & if \quad 3 \leq n \leq 6 \\ n-k & if \quad n \equiv 1,2,3,4,5,6(mod7) \\ 6k & if \quad n \equiv 0(mod7) \end{cases}, \ n \geq 7,$$ *where k is the quotient when n divided by 7.*

Proof: *Let $f_i, 1 \leq i \leq 2n$ be the vertices of G. If $n = 3$ then $G = S(C_3)$,$S_m = \{f_1, f_3, f_5\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 3$. If $n = 4$ then $G = S(C_4)$,$S_m = \{f_1, f_3, f_5, f_7\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$. If $n = 5$ then $G = S(C_5)$,$S_m = \{f_1, f_3, f_5, f_7, f_9\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 5$. If $n = 6$ then $G = S(C_6)$,$S_m = \{f_1, f_3, f_5, f_7, f_9, f_{11}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 6$. Hence $\gamma_{sm}(G) = \{\ n \quad if \quad 3 \leq n \leq 6\ \}$. Let $n \geq 7$.Now we consider the following cases.*

case a: $n \equiv 0(mod7)$

We take $G = S(C_7)$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}\}$. Remove any vertex $f_i \in S_m$ and add another vertex $f_j \in V - S_m$ to S_m such that f_i is adjacent to f_j. Hence the set S_m is again a secure dominating set of G.Also the monophonic path exists and it contain all the vertices of G.Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}\}$ is a minimum SMD set of G.In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\}$ is a minimum SMD set of G.Therefore $|S_m| = 6k$, where k is the quotient when n divided by 7.

case (b) subcase(i): $n \equiv 1(mod7)$

We take $G = S(C_8)$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}\}$. Then by similar argument as in case(a).Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}\}$ is a minimum SMD set of G.In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-2}\}$ is a minimum SMD set of G.Therefore $|S_m| = n - k$, where k is the quotient when n divided by 7.

subcase(ii): $n \equiv 2(mod7)$

We take $G = S(C_9)$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}\}$ is a minimum SMD set of G.In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-4}, f_{2n-2}\}$ is a minimum SMD set of G.Therefore $|S_m| = n - k$, where k is the quotient when n divided by 7.

subcase(iii): $n \equiv 3(mod7)$

We take $G = S(C_{10})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}\}$.*Then by similar argument as in case(a). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}\}$ *is a minimum SMD set of G.In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-2}, f_{2n-4}, f_{2n-6}\}$ *is a minimum SMD set of G.Therefore* $|S_m| = n - k$, *where k is the quotient when n divided by 7.*

subcase(iv): $n \equiv 4(mod7)$

We take $G = S(C_{11})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}\}$. *Then by similar argument as in case(a).Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}\}$ *is a minimum SMD set of G.In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8}\}$ *is a minimum SMD set of G.Therefore* $|S_m| = n - k$, *where k is the quotient when n divided by 7.*

subcase(v): $n \equiv 5(mod7)$

We take $G = S(C_{12})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$. *Then by similar argument as in case(a). Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$ *is a minimum SMD set of G.In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8}, f_{2n-10}\}$ *is a minimum SMD set of G.Therefore* $|S_m| = n - k$, *where k is the quotient when n divided by 7.*

subcase(vi): $n \equiv 6(mod7)$

We take $G = S(C_{13})$. *Choose* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}, f_{24}\}$. *Then by similar argument as in case(a).Therefore* $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}, f_{24}\}$ *is a minimum SMD set of G.In General* $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8}, f_{2n-10}, f_{2n-12}\}$ *is a minimum SMD set of G.Therefore* $|S_m| = n - k$, *where k is the quotient when n divided by 7. Finally we conclude that*

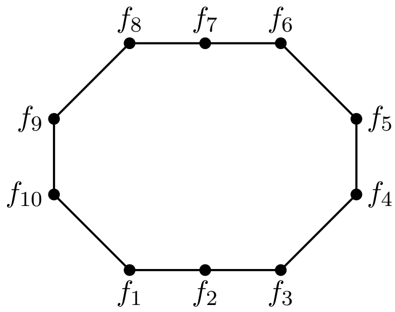
$$S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \begin{cases} \phi & for \quad r = 0 \\ f_{2n-2} & for \quad r = 1 \\ f_{2n-2}, f_{2n-4} & for \quad r = 2 \\ f_{2n-2}, f_{2n-4}, f_{2n-6} & for \quad r = 3 \\ f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8} & for \quad r = 4 \\ f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8}, f_{2n-10} & for \quad r = 5 \\ f_{2n-2}, f_{2n-4}, f_{2n-6}, f_{2n-8}, f_{2n-10}, f_{2n-12} & for \quad r = 6 \end{cases}$$

Hence $\gamma_{sm}(G) = \{ \begin{matrix} n-k & if & n \equiv 1,2,3,4,5,6(mod7) \\ 6k & if & n \equiv 0(mod7) \end{matrix}$, *where k is the quotient when n divided by 7.*

Therefore $\gamma_{sm}(G) = \begin{cases} n & if \quad 3 \leq n \leq 6 \\ n-k & if \quad n \equiv 1,2,3,4,5,6(mod7) \\ 6k & if \quad n \equiv 0(mod7) \end{cases}$, $n \geq 7$, *where k is the quotient when n divided by 7.*

Example 2.4. The Secure monophonic domination number of the Subdivision of Cycle graph $S(C_5)$ in Figure 2

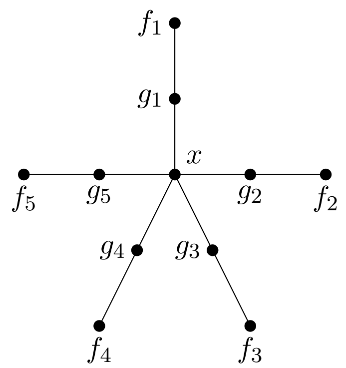
Theorem 2.5. *For the Subdivision of Star graph* $S(K_{1,n-1}), \gamma_{sm}(S(K_{1,n-1})) = n, n \geq 3$.



$\gamma_{sm}(S(C_5)) = 5$
Figure 2

Proof: *Let* $\{x, f_i, 1 \leq i \leq n-1\}$ *be the vertices of* $K_{1,n-1}$. *Subdivide each edge of* $K_{1,n-1}$ *by vertices* $g_i, 1 \leq i \leq n-1$. *The resultant graph is* $S(K_{1,n-1})$ *whose vertex set* $V(S(K_{1,n-1})) = \{x \cup \{f_i/1 \leq i \leq n-1\} \cup \{g_i/1 \leq i \leq n-1\}\}$ *and edge set* $E(S(K_{1,n-1})) = \{(xg_i/1 \leq i \leq n-1) \cup (g_i f_i/1 \leq i \leq n-1)\}$ *such that* $|V(S(K_{1,n-1}))| = 2n-1$ *and* $|E(S(K_{1,n-1}))| = 2n-2$.*Let* $Z = \{f_i, 1 \leq i \leq n-1\}$ *be the end vertices of* $S(K_{1,n-1})$ *. By observation ,* Z *is a subset of every SMD set of* $S(K_{1,n-1})$. *Since* x *is not dominated by any vertex of* Z, Z *is not a SMD set of* $S(K_{1,n-1})$. *Therefore* $\gamma_{sm}(S(K_{1,n-1})) \geq n$.
Let $Z' = Z \cup \{x\}$. *Clearly* $J[Z'] = V(S(K_{1,n-1}))$ *and every element of* $V(S(K_{1,n-1})) - Z'$ *is dominated by atleast one element of* Z'. *Therefore* Z' *is a SMD set of* $S(K_{1,n-1})$, *so that* $\gamma_{sm}(S(K_{1,n-1})) = n$

Example 2.6. The Secure monophonic domination number of Subdivision of Star graph $S(K_{1,6-1})$ in Figure 3

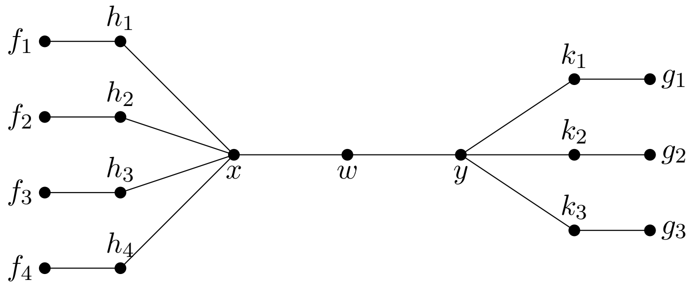


$\gamma_{sm}(S(K_{1,6-1})) = 6.$
Figure 3

Theorem 2.7. *For the subdivision of Bistar graph $S(B_{m,n})$, $\gamma_{sm}(S(B_{m,n})) = m + n + 2, m, n \geq 2$.*

Proof: *Let $\{x, y, f_i(1 \leq i \leq m), g_j(1 \leq j \leq n)\}$ be the vertices of $B_{m,n}$. Subdivide each edge of $B_{m,n}$ with vertices $\{w, h_i(1 \leq i \leq m), k_i(1 \leq i \leq n)\}$ into each edge of $B_{m,n}$. The resultant graph $S(B_{m,n})$ whose vertex set is $V(S(B_{m,n})) = \{(x, w, y) \cup \{f_i, h_i/1 \leq i \leq m\} \cup \{g_i, k_i/1 \leq i \leq n\}\}$ and edge set is $E(S(B_{m,n})) = \{(xw, wy) \cup \{f_ih_i, xh_i/1 \leq i \leq m\} \cup \{yk_i, k_ig_i/1 \leq i \leq n\}\}$ such that $|V(S(B_{m,n}))| = 2m + 2n + 3$ and $|E(S(B_{m,n}))| = 2m + 2n + 2$ and let $Z = \{f_i, 1 \leq i \leq m, g_j, 1 \leq j \leq n\}$ be the $m + n$ end vertices of $S(B_{m,n})$. By observation, Z is a subset of every SMD set of $S(B_{m,n})$. Since $\{x, w, y\}$ is not dominated by any vertex of Z, Z is not a SMD set of $S(B_{m,n})$.Therefore $\gamma_{sm}(S(B_{m,n})) \geq m + n + 2$.*
Let $Z' = Z \cup \{x, y\}$. $J[Z'] = V(S(B_{m,n}))$ and every element of $V(S(B_{m,n})) - Z'$ is dominated by atleast one element of Z'. Therefore Z' is a SMD set of $S(B_{m,n})$, so that $\gamma_{sm}(S(B_{m,n})) = m + n + 2$

Example 2.8. The Secure monophonic domination number of the Bistar graph $B_{4,3}$ in Figure 4



$\gamma_{sm}(S(B_{4,3})) = 9$
Figure 4

Theorem 2.9. *For the Subdivision of Y-tree graph $G = S(Y_{n+1})$, $n \geq 3$,*

$$\gamma_{sm}(G) = \begin{cases} n+1 & if \quad 3 \leq n \leq 6 \\ \lceil \frac{6n+7}{7} \rceil & if \quad n \equiv 3(mod7) \\ \lceil \frac{6n+7}{7} \rceil + 1 & if \quad n \equiv 0,1,2,4,5,6(mod7) \end{cases}, n \geq 7.$$

Proof: *Let $f_i, 1 \leq i \leq n+1$ be the vertices of the Y-tree graph Y_{n+1}. Subdivide each edge of Y_{n+1}. The resultant graph is $S(Y_{n+1})$ whose vertex set $V(S(Y_{n+1})) = \{f_i/1 \leq i \leq 2n+1\}$ and edge set $E(S(Y_{n+1})) = \{f_if_{i+1}/1 \leq i \leq 2n-2\} \cup \{f_if_{i+3}, f_{i+3}f_{i+4}/i = 2n-3\}$ such that $|V(S(Y_{n+1}))| = 2n+1$ and $|E(S(Y_{n+1}))| = 2n$*
If $n = 3$ then $G = S(Y_{3+1})$,$S_m = \{f_1, f_2, f_3, f_7\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$. If $n = 4$ then $G = S(Y_{4+1})$,$S_m = \{f_1, f_3, f_5, f_7, f_9\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 5$. If $n = 5$ then $G = S(Y_{5+1})$,$S_m = \{f_1, f_3, f_5, f_7, f_9, f_{11}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 6$. If $n = 6$ then $G = S(Y_{6+1})$,$S_m = \{f_1, f_3, f_5, f_7, f_9, f_{11}, f_{13}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 7$. Hence $\gamma_{sm}(G) = \{\ n+1 \quad if \quad 3 \leq n \leq 6\ \}$. Let $n \geq 7$. Now we consider the following cases.
case a: $n \equiv 3(mod7)$
We take $G = S(Y_{10+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}\}$. Remove any vertex $f_i \in S_m$ and add another vertex $f_j \in V - S_m$ to S_m such that f_i is adjacent to f_j.

Hence the set S_m is again a secure dominating set of G. Also the monophonic path exists and it contain all the vertices of G. Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n+1}, f_{2n-1}, f_{2n-3}, f_{2n-5}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil$.

case (b) subcase(i): $n \equiv 0(mod7)$

We take $G = S(Y_{7+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{11}, f_{13}, f_{15}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{11}, f_{13}, f_{15}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-2} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-6}, f_{2n-4}, f_{2n-3}, f_{2n-1}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$.

subcase(ii): $n \equiv 1(mod7)$

We take $G = S(Y_{8+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}, f_{17}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{13}, f_{15}, f_{17}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$.

subcase(iii): $n \equiv 2(mod7)$

We take $G = S(Y_{9+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{15}, f_{17}, f_{19}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{14}, f_{15}, f_{17}, f_{19}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-4}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$.

subcase(iv): $n \equiv 4(mod7)$

We take $G = S(Y_{11+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$.

subcase(v): $n \equiv 5(mod7)$

We take $G = S(Y_{12+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}, f_{2n-9}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$.

subcase(vi): $n \equiv 6(mod7)$

We take $G = S(Y_{13+1})$. Choose $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}, f_{27}\}$. Then by similar argument as in case(a). Therefore $S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}, f_{21}, f_{23}, f_{25}, f_{27}\}$ is a minimum SMD set of G. In General $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \{f_{2n-1}, f_{2n-3}, f_{2n-5}, f_{2n-7}, f_{2n-9}, f_{2n-11}, f_{2n+1}\}$ is a minimum SMD set of G. Therefore $|S_m| = \lceil \frac{6n+7}{7} \rceil + 1$. Finally we conclude that

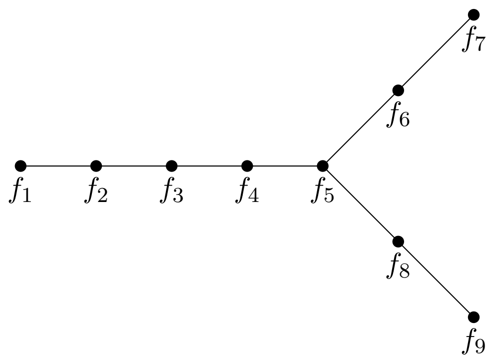
$$S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \begin{cases} f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 1 \\ f_{2n-4}, f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 2 \\ f_{2n-5}, f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 3 \\ f_{2n-7}, f_{2n-5}, f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 4 \\ f_{2n-9}, f_{2n-7}, f_{2n-5}, f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 5 \\ f_{2n-11}, f_{2n-9}, f_{2n-7}, f_{2n-5}, f_{2n-3}, f_{2n-1}, f_{2n+1} & for \quad r = 6 \end{cases}$$

$$and \ \{\bigcup_{j=0}^{2k-2} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \bigcup \left\{ f_{2n-6}, f_{2n-4}, f_{2n-3}, f_{2n-1}, f_{2n+1} \quad for \quad r = 0 \right\}$$

$$Hence \ \gamma_{sm}(G) = \begin{cases} \lceil \frac{6n+7}{7} \rceil & if \quad n \equiv 3(mod7) \\ \lceil \frac{6n+7}{7} \rceil + 1 & if \quad n \equiv 0,1,2,4,5,6(mod7) \end{cases}.$$

$$Therefore \ \gamma_{sm}(G) = \begin{cases} n+1 & if \quad 3 \leq n \leq 6 \\ \lceil \frac{6n+7}{7} \rceil & if \quad n \equiv 3(mod7) \\ \lceil \frac{6n+7}{7} \rceil + 1 & if \quad n \equiv 0,1,2,4,5,6(mod7) \end{cases}, n \geq 7.$$

Example 2.10. The secure monophonic domination number of the Subdivision of Y-tree graph $S(Y_{4+1})$ in Figure 5



$\gamma_{sm}(S(Y_{4+1})) = 5$
Figure 5

3. Conclusions

In this paper, we investigated the secure monophonic domination number of graphs such as subdivision of Path graph, subdivision of Cycle graph, subdivision of Star graph, subdivision Bistar graph, subdivision of Y- tree graph.

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