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Research Paper

STUDY OF η-EINSTEIN SOLITON ON α-SASAKIAN MANIFOLD ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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ABSTRACT

The purpose of the present paper is to study some properties of α -Sasakian manifolds with respect to Schouten-van Kampen connection. We study η-Einstein soliton on pseudo-projectively flat α-Sasakian manifolds with respect to Schouten-van Kampen connection. Further, we discuss η-Einstein soliton on quasi-concircularly flat and W_i-flat α-Sasakian manifolds with respect to this connection.

1. Introduction

R. S. Hamilton was the first who introduced the concept of Ricci flow in differential geometry in 1982. Hamilton [7] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian

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manifold by smoothing out the irregularities. The Ricci flow equation is an evolution equation for metrices on a Riemannian manifold defined as follows:

$$\frac{\partial g}{\partial t} = -2S,$$

where g is a Riemannian metric, S is Ricci curvature tensor and t is time. The solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\lambda g = 0,$$

where $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. The vector field V is called potential vector field and if it is gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton. Ricci soliton was further studied by many researchers. For instance, we see [8, 15, 18, 22] and their references.

Catino and Mazzieri [5] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure (ϕ, ξ, η, g) is said to have an Einstein soliton (g, V, λ) if

$$L_V g + 2S + (2\lambda - r)g = 0, \tag{1.1}$$

holds, where r being the scalar curvature. The Einstein soliton (g, V, λ) is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation

$$\frac{\partial g}{\partial t} = -2S + rg.$$

Again as a generalization of Einstein soliton the η-Einstein soliton on manifold $M\,(\phi, \xi, \eta, g)$ was introduced by A. M. Blaga [4] and it is given by

$$L_V g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where, μ is some constant. When $\mu = 0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\mu \neq 0$, the data (g, V, λ, μ) is called proper η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$.

In 2012, G. Ingalahalli and C. S. Bagewadi [6] introduced α-Sasakian manifold as generalization of Sasakian manifold and proved that a symmetric parallel second order covariant tensor in an α-Sasakian manifold is a constant multiple of the metric tensor. Further, he studied Ricci soliton on this manifold and showed that a Ricci soliton in an n-dimensional α-Sasakian manifold cannot be steady. This manifold is further studied by many authors, for instance we see [2, 19].

The notion of Schouten-van Kampen connection (shortly, SVK-connection) was introduced in the third decade of last century for a study of non-holomorphic manifolds [16, 23]. In 2006 Bejancu [3] studied Schouten-van Kampen connection on Foliated manifolds. Recently, A. Biswas and K. K. Baisya investigated some properties of pesudo symmetric Sasakian manifolds with respect to SVK-connectiopn. SVK-connection for an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g)

consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi, \tag{1.3}$$

for all X, $Y \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M.

In Riemannian manifold of dimension $n(> 2)$, the pseudo-projective curvature tensor was introduced by B. Prasad [12] in 2002. In [9], H. G. Nagaraja and G. Somashekhara showed that every pseudo-projectively flat and pseudo-projective semi symmetric Sasakian manifolds are locally isomorphic to unit sphere. The properties of this curvature tensor was further studied by many researchers. For details we refer [10, 17, 20, 21] and the references therein. In a Riemannian manifold M, the pseudo-projective curvature tensor P of of rank three is given by

$$P(X,Y)Z = aR(X,Y)Z \quad + b[S(Y,Z)X - S(X,Z)Y] \\ + cr[g(Y,Z)X - g(X,Z)Y],$$

for all X, Y, $Z \in \chi(M)$, set of all vector fields of the manifold M, where the non zero constants a, b and c are related as

$$c = -\frac{1}{n}\left(\frac{a}{n-1} + b\right)$$

and r is the scalar curvature, $R(X,Y)Z$ denotes the Riemannian curvature tensor, S denotes the Ricci tensor of type $(0,2)$ and g is a Riemannian metric.

As a generalization of concircular curvature tensor, the quasi-concircular curvature tensor $\mathcal{W}$ [1, 11, 13] on an n-dimensional Riemannian manifold M is given by

$$\begin{aligned} \mathcal{W}(X,Y)Z &= \delta R(X,Y)Z \\ &\quad -\frac{r}{n}\left(\frac{\delta}{n-1} + 2\sigma\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.4}$$

for all X, Y, $Z \in \chi(M)$, where δ and σ are constants such that $\delta, \sigma \neq 0$ and R is the Riemannian curvature tensor and r is the scalar curvature.

The W_i-curvature tensors $(i = 0, 1, 2...9)$ are viewed as a particular case of τ-Tensor introduced by M. M. Tripathi and P. Gupta [21]. Some of the W_i-curvature tensors were previously introduced by Pokhariyal [14]. The W_i-curvature tensors $(i = 1, 2...9)$ of type (1,3) are defined as

$$\begin{aligned} W_i(X,Y)Z &= a_0 R(X,Y)Z + a_1 S(Y,Z)X \\ &\quad + a_2 S(X,Z)Y + a_3 S(X,Y)Z + a_4 g(Y,Z)QX \\ &\quad + a_5 g(X,Z)QY + a_6 g(X,Y)QZ, \end{aligned} \tag{1.5}$$

for all $X, Y, Z \in \chi(M)$, where R, S and Q are Riemannian curvature tensor, Ricci tensor and Ricci operator, respectively. The expressions for $W_0, W_1....W_9$ curvature tensors are given by

TABLE 1

Value of a's	W_i *− curvature tensors*	*i's value*
$a_0 = 1, a_1 = -a_5 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY]$	$i = 0$
$a_0 = 1, a_1 = -a_2 = \frac{1}{n-1}$ *all other* $a_i = 0$	$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$	$i = 1$
$a_0 = 1, a_4 = -a_5 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY]$	$i = 2$
$a_0 = 1, a_2 = -a_4 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_3(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Z)Y - g(Y,Z)QX]$	$i = 3$
$a_0 = 1, a_5 = -a_6 = \frac{1}{n-1}$ *all other* $a_i = 0$	$W_4(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ]$	$i = 4$
$a_0 = 1, a_2 = -a_5 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_5(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Z)Y - g(X,Z)QY]$	$i = 5$
$a_0 = 1, a_1 = -a_6 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_6(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Y)QZ]$	$i = 6$
$a_0 = 1, a_1 = -a_4 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_7(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(Y,Z)QX]$	$i = 7$
$a_0 = 1, a_1 = -a_3 = -\frac{1}{n-1}$ *all other* $a_i = 0$	$W_8(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Y)Z]$	$i = 8$
$a_0 = 1, a_3 = -a_4 = \frac{1}{n-1}$ *all other* $a_i = 0$	$W_9(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX]$	$i = 9$

Definition 1.1. An α-Sasakian manifold M is said to be η-Einstein manifold if the Ricci tensor of type (0,2) is of the form:

$$S(X,Y) = k_1 g(X,Y) + k_2 \eta(X)\eta(Y),$$

for all $X,\ Y \in \chi(M)$, where $k_1,\ k_2$ are scalars.

Definition 1.2. An n-dimensional α-Sasakian manifold M is said to be pseudo-projectively flat if $P(X,Y)Z = 0,\ \textit{for all}\ X,\ Y,\ Z \in \chi M)$.

Definition 1.3. An n-dimensional α-Sasakian manifold M is said to be quasi-concircularly flat if $\mathcal{W}(X,Y)Z = 0,\ \textit{for all}\ X,\ Y,\ Z \in \chi(M)$.

Definition 1.4. An n-dimensional α-Sasakian manifold M is said to be W_i- flat if $W_i(X,Y)Z = 0,\ \textit{for all}\ X,\ Y,\ Z \in \chi(M)$

This paper has been organized as follows:
After introduction, a short description of α-Sasakian manifold has been given in **Section-2**. In **Section-3**, we establish some properties of α-Sasakian manifold with respect to SVK-connection. **Section-4** contains η-Einstein soliton on α-Sasakian manifold with respect to the SVK-connection. **Section-5** concerns with η-Einstein soliton on pseudo-projectively flat α-Sasakian manifold with respect to SVK-connection. **Section-6** deals with η-Einstein soliton on quasi-concircularly flat α-Sasakian manifold with respect to SVK-connection. **Section-7** is related to η-Einstein soliton on W_i-flat α-Sasakian manifold with respect to SVK-connection.

2. Preliminaries

This section devoted to some basic definitions and results on para-contact metric manifolds and α-Sasakian manifolds. Also, all manifolds are assumed to be connected and smooth. If an n-dimensional differentiable manifold M equipped with a metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g, which satisfies the relations:

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \eta(X) = g(X, \xi), \tag{2.3}$$

for all X, $Y \in \chi(M)$, then the manifold M is called an almost contact metric manifold. Again an almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be α-Sasakian manifold if the following conditions hold:

$$(\nabla_X \phi)Y = \alpha\left[g(X,Y)\xi - \eta(Y)X\right], \tag{2.4}$$

$$\nabla_X \xi = -\alpha\phi X, \quad (\nabla_X \eta)Y = \alpha g(X, \phi Y), \tag{2.5}$$

where α is a non-zero real constant on M.
In an α-Sasakian manifold, the following relations also hold [2, 6]:

$$\eta(R(X,Y)Z) = \alpha^2\left[\eta(X)g(Y,Z) - g(X,Z)\eta(Y)\right], \tag{2.6}$$

$$R(X,Y)\xi = \alpha^2\left[\eta(Y)X - \eta(X)Y\right], \tag{2.7}$$

$$R(\xi, X)Y = \alpha^2\left[g(X,Y)\xi - \eta(Y)X\right], \tag{2.8}$$

$$S(X,\xi) = \alpha^2(n-1)\eta(X), \tag{2.9}$$

$$S(\xi,\xi) = \alpha^2(n-1), Q\xi = \alpha^2(n-1)\xi. \tag{2.10}$$

In view of (1.3), (2.4) and (2.5), we get the expression for SVK-connection in α-Sasakian manifold as

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha g(X, \phi Y)\xi + \alpha\eta(Y)\phi X, \tag{2.11}$$

with torsion tensor

$$\overline{T}(X,Y) = 2\alpha g(X,\phi Y)\xi + \alpha\left[\eta(X)\phi Y - \eta(Y)\phi X\right].$$

On an α-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\overline{\nabla}_X \xi = 0, \left(\overline{\nabla}_X g\right)(Y,Z) = 0, \left(\overline{\nabla}_X \eta\right)Y = 0. \tag{2.12}$$

Proposition 2.1. *SVK-connection on α-Sasakian manifold is a metric compatible linear connection and its torsion is of the form:*

$$\overline{T}(X,Y) = 2\alpha g(X,\phi Y)\xi + \alpha\left[\eta(X)\phi Y - \eta(Y)\phi X\right].$$

Proposition 2.2. *In α-Sasakian maifold, ξ and g are parallel with respect to SVK-connection.*

Proposition 2.3. *In α-Sasakian manifold, the integral curve of ξ with respect to SVK-connection is a geodesic.*

3. Some properties of α-Sasakian manifold with respect to $\overline{\nabla}$

Let us denote the Riemannian curvature tensor with respect to SVK-connection by $\overline{R}$ and it is defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \tag{3.1}$$

for all X, Y, $Z \in \chi(M)$.
In reference to (2.1), (2.4), (2.5) and (2.11) we have

$$\begin{aligned}\overline{\nabla}_X\overline{\nabla}_Y Z &= \nabla_X\nabla_Y Z + \alpha g(\nabla_X Y, \phi Z)\xi + \alpha^2 g(X,Z)\eta(Y)\xi \\ &\quad -\alpha^2 g(X,Y)\eta(Z)\xi + \alpha g(Y, \phi\nabla_X Z)\xi - \alpha^2 g(Y,\phi Z)\phi X \\ &\quad +\alpha^2 g(X,\phi Z)\phi Y + \alpha\eta(\nabla_X Z)\phi Y + \alpha^2 g(X,Y)\eta(Z)\xi \\ &\quad -\alpha^2\eta(Y)\eta(Z)X + \alpha\eta(Z)\phi(\nabla_X Y) + \alpha g(X,\phi\nabla_Y Z)\xi \\ &\quad -\alpha^2 g(X,Y)\eta(Z)\xi + \alpha^2\eta(X)\eta(Y)\eta(Z)\xi \\ &\quad +\alpha\eta(\nabla_Y Z)\phi X + \alpha^2 g(Y,\phi Z)\phi X.\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\overline{\nabla}_{[X,Y]}Z &= \nabla_{[X,Y]}Z + \alpha g(\nabla_X Y, \phi Z)\xi - \alpha g(\nabla_Y X, \phi Z)\xi \\ &\quad +\alpha\eta(Z)\phi(\nabla_X Y) - \alpha\eta(Z)\phi(\nabla_Y X).\end{aligned} \tag{3.3}$$

Interchanging X and Y in (3.2) and using it along with (2.12), (3.2) and (3.3) in (3.1), we obtain the Riemannian curvature tensor in α-Sasakian manifold M with respect to SVK-connection as

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + \alpha^2 g(X,\phi Z)\phi Y - \alpha^2 g(Y,\phi Z)\phi X \\ &\quad +\alpha^2 g(X,Z)\eta(Y)\xi - \alpha^2 g(Y,Z)\eta(X)\xi \\ &\quad +\alpha^2\eta(X)\eta(Z)Y - \alpha^2\eta(Y)\eta(Z)X.\end{aligned} \tag{3.4}$$

Taking inner product with a vector field V of (3.4) and contracting over X and V we have

$$\overline{S}(Y,Z) = S(Y,Z) - \alpha^2(n-1)\eta(Y)\eta(Z). \tag{3.5}$$

Consequently, one can easily bring out the following results:

$$\eta(\overline{R}(X,Y)Z) = \eta(\overline{R}(X,Y)\xi) = \eta\left(\overline{R}(\xi,Y)Z\right) = 0, \tag{3.6}$$

$$\overline{R}(X,Y)\xi = 0, \overline{R}(\xi,Y)Z = 0, \overline{R}(X,\xi)Z = 0, \tag{3.7}$$

$$\overline{Q}Y = QY - \alpha^2(n-1)\eta(Y)\xi, \tag{3.8}$$

$$\overline{S}(Y,\xi) = \overline{S}(\xi,Z) = 0, \overline{Q}\xi = 0, \tag{3.9}$$

$$\overline{r} = r - \alpha^2(n-1). \tag{3.10}$$

Thus we can state the following propositions:

Proposition 3.1. *Let M be an n-dimensional α-Sasakian manifold admitting SVK-connection $\overline{\nabla}$, then*
(i) The curvature tensor $\overline{R}$ of $\overline{\nabla}$ is given by (3.4),

(ii) The Ricci tensor $\overline{S}$ of $\overline{\nabla}$ is given by (3.5),
(iii) The scalar curvature $\overline{r}$ of $\overline{\nabla}$ is given by (3.10),
(iv) The Ricci tensor $\overline{S}$ of $\overline{\nabla}$ is symmetric.

The pseudo-projective curvature tensor with respect to SVK-connection is given by

$$\overline{P}(X,Y)Z = a\overline{R}(X,Y)Z + b\left[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\right] + c\overline{r}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{3.11}$$

for all X, Y, $Z \in \chi(M)$, where $\overline{P}$, $\overline{R}$ and $\overline{S}$ are the pseudo-projective curvature tensor, Riemannian curvature tensor and Ricci curvature tensor with respect to $\overline{\nabla}$, respectively.
The quasi-concircular curvature tensor with respect to SVK-connection is given by

$$\begin{aligned}\overline{\mathcal{W}}(X,Y)Z &= \delta\overline{R}(X,Y)Z \\ &\quad -\frac{\overline{r}}{n}\left(\frac{\delta}{n-1}+2\sigma\right)\left[g(Y,Z)X - g(X,Z)Y\right],\end{aligned} \tag{3.12}$$

for all X, Y, $Z \in \chi(M)$, where $\overline{\mathcal{W}}$, is the quasi-concircular curvature tensor with respect to $\overline{\nabla}$.
The W_i-curvature tensor with respect to SVK-connection is given by

$$\begin{aligned}\overline{W}_i(X,Y)Z &= a_0\overline{R}(X,Y)Z + a_1\overline{S}(Y,Z)X \\ &\quad + a_2\overline{S}(X,Z)Y + a_3\overline{S}(X,Y)Z + a_4 g(Y,Z)\overline{Q}X \\ &\quad + a_5 g(X,Z)\overline{Q}Y + a_6 g(X,Y)\overline{Q}Z,\end{aligned} \tag{3.13}$$

for all X, Y, $Z \in \chi(M)$, where $\overline{W}_i$ denotes W_i-curvature tensor with respect to SVK-connection.

4. η-Einstein soliton on α-Sasakian manifold with respect to $\overline{\nabla}$

The equation (1.2) with respect to SVK-connection on an α-Sasakian manifold M may be written as

$$(\overline{L}_V g)(X,Y) + 2\overline{S}(X,Y) + (2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{4.1}$$

where $\overline{L}_V g$ denote Lie derivative of g with respect to $\overline{\nabla}$ along the vector field V and $\overline{S}$ is the Ricci curvature tensor of M with respect to $\overline{\nabla}$.
After expanding (4.1) and using (2.11), (3.5) we have

$$\begin{aligned}&(\overline{L}_v g)(X,Y) + 2\overline{S}(X,Y) + (2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y) \\ =\ & g(\overline{\nabla}_X V, Y) + g(X, \overline{\nabla}_Y V) + 2\overline{S}(X,Y) \\ & +(2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y) \\ =\ & (L_V g)(X,Y) + 2S(X,Y) + (2\lambda - r)g(X,Y) + 2\mu\eta(X)\eta(Y) \\ & -2\alpha^2(n-1)\eta(X)\eta(Y) - \alpha^2(n-1)g(X,Y) \\ & +\alpha g(X,\phi V)\eta(Y) + \alpha g(Y,\phi V)\eta(X),\end{aligned} \tag{4.2}$$

which gives the following theorem:

Theorem 4.1. *An η-Einstein soliton (g, V, λ, μ) with respect to $\overline{\nabla}$ is invariant on an n-dimensional α-Sasakian manifold M if and only if*

$$\begin{aligned} 0 &= -2\alpha^2(n-1)\eta(X)\eta(Y) - \alpha^2(n-1)g(X,Y) \\ &\quad +\alpha g(X,\phi V)\eta(Y) + \alpha g(Y,\phi V)\eta(X), \end{aligned}$$

holds for all X, Y, $Z \in \chi(M)$.

Setting $V = \xi$ in (4.1) and using (2.12) we get

$$\begin{aligned} 0 &= (\overline{L}_\xi g)(X,Y) + 2\overline{S}(X,Y) + (2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y) \\ &= 2\overline{S}(X,Y) + (2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y), \end{aligned} \tag{4.3}$$

which gives

$$\overline{S}(X,Y) = -\left(\lambda - \frac{\overline{r}}{2}\right) g(X,Y) - \mu\eta(X)\eta(Y). \tag{4.4}$$

Setting $Y = \xi$ in (4.4) we get

$$\overline{S}(X,\xi) = -\left(\lambda - \frac{\overline{r}}{2}\right) \eta(X) - \mu\eta(X). \tag{4.5}$$

Using (3.9) and (3.10) in (4.5) we get

$$\lambda = \frac{r - \alpha^2(n-1)}{2} - \mu. \tag{4.6}$$

Using (3.5) in (4.4) we obtain

$$S(X,Y) = -\left[\lambda - \frac{r - \alpha^2(n-1)}{2}\right] g(X,Y) + \left[\alpha^2(n-1) - \mu\right] \eta(X)\eta(Y).$$

Thus we have the following theorem:

Theorem 4.2. *If an n-dimensional α-Sasakian manifold M contains an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$ then M becomes an η-Einstein manifold.*

Contracting (4.4) over X and Y using (3.10) we obtain

$$r = \frac{2(n\lambda + \mu)}{n-2} + \alpha^2(n-1). \tag{4.7}$$

Corollary 4.3. *If an α-Sasakian manifold M contains an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$, then the scalar curvature of M is given by (4.7).*

Consider the distribution D on α-Sasakian manifold M as $D = \ker\eta$. If $V \in D$, then

$$\eta(V) = 0.$$

Taking covariant derivative with respect to ξ and using $(\nabla_\xi \eta) V = 0$, we get

$$\eta(\nabla_\xi V) = 0. \tag{4.8}$$

In view of (2.11) and (4.8) it follows that

$$\eta\left(\overline{\nabla}_\xi V\right) = 0. \tag{4.9}$$

Equation (4.1) gives

$$\begin{aligned} 0 &= g(\overline{\nabla}_X V, Y) + g(X, \overline{\nabla}_Y V) + 2\overline{S}(X,Y) \\ &\quad +(2\lambda - \overline{r})g(X,Y) + 2\mu\eta(X)\eta(Y). \end{aligned} \tag{4.10}$$

Setting $X = Y = \xi$ and using (3.9), (4.9) in (4.10) we obtain

$$0 = 2\lambda - r + \alpha^2(n-1) + 2\mu.$$

This gives the following theorem:

Theorem 4.4. *If an α-Sasakian manifold M contains an η-Einstein soliton (g, V, λ, μ) with respect to $\overline{\nabla}$ such that $V \in D = \ker\eta$, then the soliton constants are related by*

$$\lambda + \mu = \frac{r - \alpha^2(n-1)}{2}.$$

5. η-Einstein soliton on pseudo-projectively flat α-Sasakian manifold with respect to $\overline{\nabla}$

Let an α-Sasakian manifold M be pseudo-projectively flat with respect to $\overline{\nabla}$ the the condition that must be satisfied by $\overline{P}$ is

$$\overline{P}(X,Y)Z = 0.$$

In view of (3.11) we have

$$\begin{aligned} 0 = a\overline{R}(X,Y)Z \quad &+b\left[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y\right] \\ &+c\overline{r}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{5.1}$$

Taking inner product of (5.1) with a vector field V we have

$$\begin{aligned} 0 &= ag\left(\overline{R}(X,Y)Z, V\right) + b\left[\overline{S}(Y,Z)g(X,V) - \overline{S}(X,Z)g(Y,V)\right] \\ &\quad +c\overline{r}\left[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)\right], \end{aligned} \tag{5.2}$$

for all vector fields X, Y, Z, V on M.
Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point of the manifold M. Then putting $X = V = e_i$ in the equation (5.2) and taking summation over i $(1 \le i \le n)$ we get

$$\overline{S}(Y,Z) = -\frac{c\overline{r}(n-1)}{[a + b(n-1)]}g(Y,Z). \tag{5.3}$$

Using (3.5) in (5.3)we obtain

$$S(Y,Z) = -\frac{c\left[r - \alpha^2(n-1)\right](n-1)}{[a+b(n-1)]}g(Y,Z) + \alpha^2(n-1)\eta(Y)\eta(Z).$$

This leads to the following theorem:

Theorem 5.1. *A pseudo-projectively flat α-Sasakian manifold with respect to SVK-connection is η-Einstein.*

In view of (4.4) and (5.3) we get

$$\left(\lambda - \frac{\overline{r}}{2}\right)g(Y,Z) + \mu\eta(Y)\eta(Z) = \frac{c\overline{r}(n-1)}{[a+b(n-1)]}g(Y,Z). \tag{5.4}$$

Setting $Z=\xi$ and using (3.10) in (5.4) we get

$$\lambda+\mu=\frac{\left[r-\alpha^2(n-1)\right]\left[a+(2c+b)(n-1)\right]}{2\left[a+b(n-1)\right]}.$$

This leads to the following theorem:

Theorem 5.2. *If a pseudo-projectively flat α-Sasakian manifold M contains an η-Einstein soliton (g,ξ,λ,μ) with respect to $\overline{\nabla}$, then the soliton constants are given by the equation*

$$\lambda+\mu=\frac{r-\alpha^2(n-1)\left[a+(2c+b)(n-1)\right]}{2\left[a+b(n-1)\right]}.$$

6. η-Einstein soliton on quasi-concircularly flat α-Sasakian manifold with respect to $\overline{\nabla}$

The condition that must be satisfied by $\mathcal{W}$ is

$$\overline{\mathcal{W}}(X,Y)Z=0$$

for all X, Y, $Z\in\chi(M)$.
Equation (3.12) gives

$$0=\delta\overline{R}\left(X,Y\right)Z-\frac{\overline{r}}{n}\left(\frac{\delta}{n-1}+2\sigma\right)\left[g\left(Y,Z\right)X-g\left(X,Z\right)Y\right]. \tag{6.1}$$

Taking inner product of (6.1) with a vector field V we obtain

$$\begin{aligned}0 &= g(\overline{R}\left(X,Y\right)Z,V)\\ &\quad-\frac{\overline{r}}{n}\left(\frac{\delta}{n-1}+2\sigma\right)\left[g\left(Y,Z\right)g\left(X,V\right)-g\left(X,Z\right)g\left(Y,V\right)\right].\end{aligned} \tag{6.2}$$

Contracting (6.2) over X and V we get

$$\overline{S}\left(Y,Z\right)=\frac{\overline{r}}{n}(n-1)(\delta+2\sigma)g\left(Y,Z\right). \tag{6.3}$$

Using (3.5) in (6.3)we obtain

$$S\left(Y,Z\right)=\frac{r-\alpha^2(n-1)}{n}(n-1)(\delta+2\sigma)g\left(Y,Z\right)+\alpha^2(n-1)\eta\left(Y\right)\eta\left(Z\right),$$

which gives the following theorem:

Theorem 6.1. *An quasi-concircularly flat α-Sasakian manifold with respect to SVK-connection is η-Einstein manifold.*

In reference to (4.4) and (6.3) we get

$$\begin{aligned}0 &= \frac{r-\alpha^2(n-1)}{n}(n-1)(\delta+2\sigma)g\left(X,Y\right)\\ &\quad+\left[\lambda-\frac{r-\alpha^2(n-1)}{2}\right]g(X,Y)+\mu\eta(X)\eta(Y)\end{aligned} \tag{6.4}$$

Setting $Y=\xi$ in (6.4) we get

$$\lambda+\mu=\frac{1}{2n}\left[r-\alpha^2(n-1)\right]\left[n-2(n-1)(\delta+2\sigma)\right].$$

Therefore, we have the following theorem:

Theorem 6.2. *If an quasi-concircularly flat α-Sasakian manifold M contains an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$, then the soliton constants are given by the equation*

$$\lambda + \mu = \frac{1}{2n}\left[r - \alpha^2(n-1)\right]\left[n - 2(n-1)(\delta + 2\sigma)\right].$$

Now, if M be ξ-quasi-concircularly flat with respect to $\overline{\nabla}$, then from (3.7) and (3.12) we have

$$\begin{aligned} 0 &= \frac{\overline{r}}{n}\left(\frac{\delta}{n-1} + 2\sigma\right)\left[\eta(Y)X - \eta(X)Y\right]. \\ &= \frac{r - \alpha^2(n-1)}{n}\left(\frac{\delta}{n-1} + 2\sigma\right) R(X,Y)\xi. \end{aligned} \tag{6.5}$$

Since $R(X,Y)\xi \neq 0$ in M we have

$$\delta + 2\sigma(n-1) = 0,$$

if $r \neq \alpha^2(n-1)$.
This leads the following theorem:

Theorem 6.3. *If an α-Sasakian manifold is ξ-quasi-concircularly flat with respect to SVK-connection, then*

$$\delta + 2\sigma(n-1) = 0,$$

provided $r \neq \alpha^2(n-1)$.

7. η-Einstein soliton on $\overline{W}_i$-flat α-Sasakian manifold

The condition must be satisfied by W_i-curvature tensor is

$$\begin{aligned} 0 &= a_0\overline{R}(X,Y)Z + a_1\overline{S}(Y,Z)X \\ &\quad + a_2\overline{S}(X,Z)Y + a_3\overline{S}(X,Y)Z + a_4 g(Y,Z)\overline{Q}X \\ &\quad + a_5 g(X,Z)\overline{Q}Y + a_6 g(X,Y)\overline{Q}Z, \end{aligned} \tag{7.1}$$

for all X, Y, $Z \in \chi(M)$.
Taking inner product of (7.1) with a vector field V we get

$$\begin{aligned} 0 &= a_0 g(\overline{R}(X,Y)Z,V) + a_1\overline{S}(Y,Z)g(X,V) \\ &\quad + a_2\overline{S}(X,Z)g(Y,V) + a_3\overline{S}(X,Y)g(Z,V) + a_4 g(Y,Z)\overline{S}(X,V) \\ &\quad + a_5 g(X,Z)\overline{S}(Y,V) + a_6 g(X,Y)\overline{S}(Z,V), \end{aligned} \tag{7.2}$$

Contracting (7.2) over X and V we obtain

$$0 = \left[a_0 + na_1 + a_2 + a_3 + a_5 + a_6\right]\overline{S}(Y,Z) + \overline{r}a_4 g(Y,Z),$$

which gives

$$\overline{S}(Y,Z) = -\frac{\overline{r}a_4}{a} g(Y,Z), \tag{7.3}$$

where $a = a_0 + na_1 + a_2 + a_3 + a_5 + a_6$ and a vanishes for the curvature tensors W_0, W_6, W_8.
By the help of (4.4) and (7.3) we get

$$0 = \left(\lambda - \frac{\overline{r}}{2} - \frac{\overline{r}a_4}{a}\right) g(Y,Z) + \mu\eta(Y)\eta(Z). \tag{7.4}$$

Setting $Y = \xi$ in (7.4) we have

$$\lambda + \mu = \frac{(a + 2a_4)\left[r - \alpha^2(n-1)\right]}{2a}.$$

Therefore, we have the following theorem and corollaries:

Theorem 7.1. *If an α-Sasakian manifold containing an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$ be $\overline{W}_i$-flat (i=1,2,3,4,5,7,9), the relation between soliton constants is*

$$\lambda + \mu = \frac{(a + 2a_4)\left[r - \alpha^2(n-1)\right]}{2a},$$

where $a = a_0 + na_1 + a_2 + a_3 + a_5 + a_6$.

Corollary 7.2. *If an α-Sasakian manifold M be $\overline{W}_0$-flat, or $\overline{W}_6$-flat, or $\overline{W}_8$-flat, then the scalar curvature of M with respect to SVK-connection vanishes.*

Corollary 7.3. *Let an α-Sasakian manifold M admits an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$. If M be $\overline{W}_1$-flat or $\overline{W}_4$-flat, or $\overline{W}_5$-flat, then the soliton constants are given by*

$$\lambda + \mu = \frac{1}{2}\left[r - \alpha^2(n-1)\right].$$

Corollary 7.4. *Let an α-Sasakian manifold M admits an η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$. If M be $\overline{W}_2$-flat, or $\overline{W}_9$-flat, then the soliton constants are given by*

$$\lambda + \mu = \left(\frac{n-2}{2n}\right)\left[r - \alpha^2(n-1)\right].$$

Corollary 7.5. *If an α-Sasakian manifold M containing η-Einstein soliton (g, ξ, λ, μ) with respect to $\overline{\nabla}$ be $\overline{W}_7$-flat, then the soliton constants are given by*

$$\lambda + \mu = -\frac{1}{2}\left[r - \alpha^2(n-1)\right].$$

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