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Research Paper

COMPUTING SOME BOND ADDITIVE INDICES OF CERTAIN CLASS OF NANOSTRUCTURES

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ABSTRACT

Carbon nanosheets are nanomaterials consisting of two-dimensional circular arrangements of carbon atoms with nanoscale diameters. A C_4C_8 nanosheet is a lattice obtained from a trivalent arrangement of carbon atoms into alternating squares C_4 and octagons C_8. The two important class of nanosheets made by C_4C_8 decorations are $T^1UC_4C_8[p,q]$ and $T^2UC_4C_8[p,q]$. In nanotechnology, topological indices are used to quantify the structural properties of nanoparticles. In this paper, we investigate the application of topological indices, specifically weighted Mostar indices, to characterize the structures of nanosheets. We employ a variant of the cut method to determine explicit expressions for the additively weighted Mostar index and multiplicatively weighted Mostar index for $T^1UC_4C_8[p,q]$ and $T^2UC_4C_8[p,q]$ nanosheets.

1. Introduction

Nanotechnology is a relatively new branch of science which deals with the study and analysis of materials which are of nanoscale size. Nanotechnology found several applications in wide variety of fields ranging from space explorations to medicinal technology. Carbon

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nanomaterials are distinct class of nanomaterials consisting of trivalent decorations of carbon atoms. Carbon nanosheets (CNSs) are two diamensional structures consisting of carbon atoms. The study of properties of nanomaterials from the underlying structure is an important area in chemical graph theory. Topological indices are the most commonly used tool for such study and analysis. According to the IUPAC definition [?], a topological index is a numerical value associated with chemical constitution for correlation of chemical structure with various physical properties, chemical reactivity or biological activity. Several topological indices have been defined and many of them have found applications as a means to model chemical, pharmaceutical and other properties of molecules [?, ?, ?, ?]. The Mostar index is a recently introduced distance based bond-additive invariant that quantifies the peripherality of edges and graphs. For a graph $G = (V, E)$ the Mostar index [?, ?] is defined as,

$$Mo(G) = \sum_{e=uv\in E} |n_u(e|G) - n_v(e|G)|$$

where $n_u(e|G)$ is the number of vertices closer to vertex u than v. In recent years, these indices have gained significant attention due to their potential application in drug discovery, material science and network analysis. For more on Mostar indices, see [?, ?, ?, ?, ?, ?, ?, ?, ?]. Recently, in this direction, new topological indices called Additively weighted Mostar index and Multiplicatively weighted Mostar index have been proposed in [?, ?, ?]

$$Mo_A(G) = \sum_{e=uv\in E} (d(u) + d(v)|n_u(e|G) - n_v(e|G)|$$
$$Mo_M(G) = \sum_{e=uv\in E} (d(u)d(v)|n_u(e|G) - n_v(e|G)|,$$

For more literature on weighted Mostar indices, see [?, ?, ?, ?]. $T^1UC_4C_8[p, q]$ and $T^2UC_4C_8[p, q]$ are the two types of nanosheets made by C_4C_8 nanosheets. $T^1UC_4C_8[p, q]$ and $T^2UC_4C_8[p, q]$ nanosheets are obtained by the positioning of C_4 squares and C_8 octagons. $T^1UC_4C_8[p, q]$ consist of p octagons in each row and q octagons in each column connected by C_4's, see Figure **??**. Similarly, $T^2UC_4C_8[p, q]$ nanosheet consist of p octagons and p+1 -C_4 in each row and q octagons and $q + 1$ - C_4's in each column, see Figure **??**. A cut method is a computational technique used in chemical graph theory to compute the Szeged type topological indices in linear or polynomial time. The standard form of cut method was introduced in 1995 by Sandi Klavzar [?]. Several recent works regarding the determination of topological indices for different classes of nanostrutures are available in literature, see [?, ?, ?, ?, ?, ?, ?]. In this paper, the expressions of additively weighted Mostar index and multiplicatively weighted Mostar indices of $T^1UC_4C_8[p, q]$, $T^2UC_4C_8[p, q]$ nanosheets are determined using cut method.

2. Main Results

In this section we obtain explicit expressions of bond additive indices of the of Mostar type of some classes of nanostructures.

Theorem 2.1. *For the graph $T^1UC_4C_8[p, q]$, the weighted versions of Mostar indices are as follows*

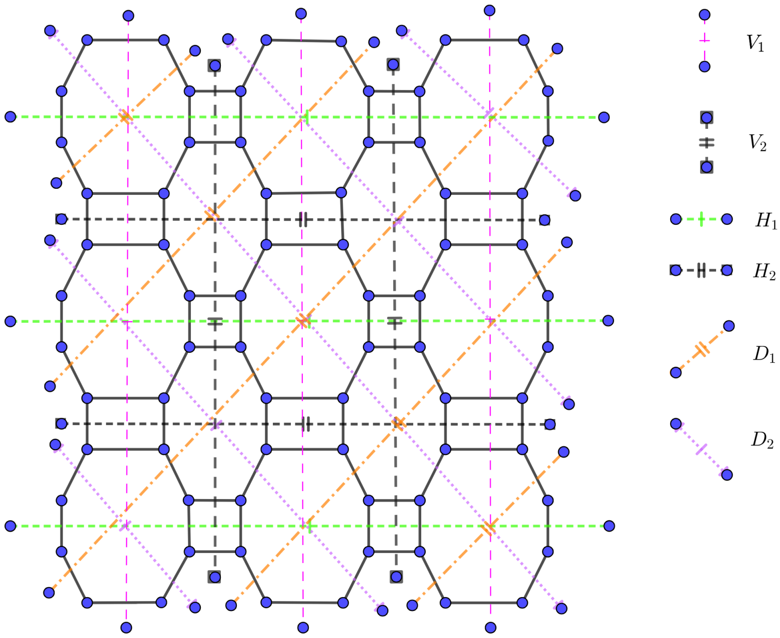

FIGURE 1. The different cuts of $T^1UC_4C_8[p,q]$.

(a.) *When p,q are even*

$$\begin{aligned}Mo_A(T^1UC_4C_8[p,q]) &= 192p^2q^2 - 112p^2q - 112pq^2 \\ &- \frac{40}{3}\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\ &+ 4\left(-\frac{3p^4}{2} + 2p^3(3q+5) + 3p^2(5q^2-14q-6)\right) \\ &+ 4\left(p(6q^3-42q^2+60q+8) - \frac{3q^4}{2} + 10q^3 - 18q^2 + 8q\right) - 4I\end{aligned}$$

$$Where\ I = \begin{cases} (8pq - 8p^2)) & if\ p < q \\ (8pq - 8q^2)) & if\ q < p \\ 0 & if\ p = q \end{cases}$$

(b.) *When p,q are odd*

$$\begin{aligned}Mo_A(T^1UC_4C_8[p,q]) &= 192p^2q^2 - 112p^2q - 112pq^2 + 16p + 16q \\ &- \frac{40}{3}\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\ &+ 4\left(-\frac{3p^4}{2} + 2p^3(3q+5) + 3p^2(5q^2-14q-6)\right) \\ &+ 4\left(p(6q^3-42q^2+60q+8) - \frac{3q^4}{2} + 10q^3 - 18q^2 + 8q\right) - 4I\end{aligned}$$

where I is defined as previously.

(c.) *When p even q odd*

$$\begin{aligned} Mo_A(T^1UC_4C_8[p,q]) &= 192p^2q^2 - 112p^2q - 112pq^2 + 16p - 6p^4 + 8p^3(3q+2) + 12p^2(5q^2 - 12q + 1) \\ &- \frac{40}{3}\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) + 8p\left(3q^3 - 18q^2 + 21q - 2\right) \\ &- 6q^4 + 16q^3 + 12q^2 - 16q - 6 - 4I \end{aligned}$$

(d.) *When p odd q even*

$$\begin{aligned} Mo_A(T^1UC_4C_8[p,q]) &= 192p^2q^2 - 112p^2q - 112pq^2 + 16q - 6p^4 + 8p^3(3q+2) + 12p^2(5q^2 - 12q + 1) \\ &- \frac{40}{3}\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) + 8p\left(3q^3 - 18q^2 + 21q - 2\right) \\ &- 6q^4 + 16q^3 + 12q^2 - 16q - 6 - 4I \end{aligned}$$

(e.) *When p, q are even*

$$\begin{aligned} Mo_M(T^1UC_4C_8[p,q]) &= 288p^2q^2 - 184p^2q - 184pq^2 \\ &- 16\left(p^3 - 3p^2(3q+1) + p(-9q^2 + 18q + 2) + q(q^2 - 3q + 2)\right) - 8I \\ &- 3\left(3p^4 - 12p^3q - 20p^3 - 30p^2q^2 + 84p^2q + 36p^2 - 12pq^3\right) \\ &- 3\left(84pq^2 - 120pq - 16p + 3q^4 - 20q^3 + 36q^2 - 16q\right) \end{aligned}$$

(f.) *When p, q are odd*

$$\begin{aligned} Mo_M(T^1UC_4C_8[p,q]) &= 288p^2q^2 - 184p^2q - 184pq^2 + 40p + 40q \\ &- 16\left(p^3 - 3p^2(3q+1) + p(-9q^2 + 18q + 2) + q(q^2 - 3q + 2)\right) - 8I \\ &- 3\left(3p^4 - 12p^3q - 20p^3 - 30p^2q^2 + 84p^2q + 36p^2 - 12pq^3\right) \\ &- 3\left(84pq^2 - 120pq - 16p + 3q^4 - 20q^3 + 36q^2 - 16q\right) \end{aligned}$$

(g.) *When p even and q odd*

$$\begin{aligned} Mo_M(T^1UC_4C_8[p,q]) &= 288p^2q^2 - 184p^2q - 184pq^2 + 40p \\ &- 16\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) - 8I \\ &- 3\left(3p^4 - 4p^3(3q+2) - 6p^2(5q^2 - 12q + 1)\right) \\ &- 3\left(p(-12q^3 + 72q^2 - 84q + 8) + 3q^4 - 8q^3 - 6q^2 + 8q + 3\right) \end{aligned}$$

(h.) *When p odd and q even*

$$\begin{aligned} Mo_M(T^1UC_4C_8[p,q]) &= 288p^2q^2 - 184p^2q - 184pq^2 + 40q \\ &- 16\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) - 8I \\ &- 3\left(3p^4 - 4p^3(3q+2) - 6p^2(5q^2 - 12q + 1)\right) \\ &- 3\left(p(-12q^3 + 72q^2 - 84q + 8) + 3q^4 - 8q^3 - 6q^2 + 8q + 3\right) \end{aligned}$$

Proof. We use a variant of cut method to determine the explicit expressions of the indices. We partition the edges of the graph into five different sets according to the different cuts on the graph. We use two different types of vertical cuts, namely V_1 and V_2 (see Figure **??**) and two horizontal cuts, namely H_1 and H_2. Also there are two different types of diagonal cuts, namely D_1 and D_2 (see Figure **??**) to determine the expressions. Let L_1, L_2, L_3, L_4, L_5 and L_6 denote the contribution of the edges in the V_1, V_2, H_1, H_2, D_1 and D_2 cuts to the additively weighted Mostar index. Similarly, let $L'_1, L'_2, L'_3, L'_4, L'_5$ and L'_6 denote the contribution of the edges in the V_1, V_2, H_1, H_2, D_1 and D_2 cuts to the multiplicative weighted Mostar index The following tables gives the summary of the contribution of the edges in the V_1 cuts.

Table 1: Details regarding the contribution of the edges to the additively weighted Mostar index for the edges in the vertical cut V_1.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u) + d(v)$	No. of edges
p	$2q$	$\lvert 8pq - (2(8ql - 4q))\rvert$	4	2
			6	$2q - 2$

Table 1': Details regarding the contribution of the edges to the multiplicative weighted Mostar index for the edges in the vertical cut V_1.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u)d(v)$	No. of edges
p	$2q$	$\lvert 8pq - (2(8ql - 4q))\rvert$	4	2
			9	$2q - 2$

Therefore, the contribution of the edges corresponding to the edges in V_1 cut is as follows. When p is even

$$\begin{aligned}
L_1 &= \sum_{e=uv\in V1} (d(u) + d(v))|n_u(e|G) - n_v(e|G)| \\
&= 2\left(\sum_{l=1}^{\frac{p}{2}} 8(8pq - (2(8ql - 4q)))\right) + 2\left(\sum_{l=1}^{\frac{p}{2}} 6(2q-2)(8pq - (2(8ql - 4q)))\right) \\
&= 32p^2q + 48p^2(q-1)q
\end{aligned}$$

$$\begin{aligned}
L'_1 &= \sum_{e=uv\in V1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| \\
&= 2\left(\sum_{l=1}^{\frac{p}{2}} 8(8pq - (2(8ql - 4q)))\right) + 2\left(\sum_{l=1}^{\frac{p}{2}} 9(2q-2)(8pq - (2(8ql - 4q)))\right) \\
&= 32p^2q + 72p^2(q-1)q
\end{aligned}$$

When p is odd

$$\begin{aligned}
L_1 &= \sum_{e=uv\in V1} (d(u) + d(v))|n_u(e|G) - n_v(e|G)| \\
&= 2\left(\sum_{l=1}^{\frac{p-1}{2}} 8(8pq - (2(2l-1)4q)\right) + 2\left(\sum_{l=1}^{\frac{p-1}{2}} 6(2q-2)(8pq - (2(2l-1)4q)\right) \\
&= 32(p^2-1)q + 48(p^2-1)(q-1)q
\end{aligned}$$

$$\begin{aligned} L_1' &= \sum_{e=uv\in V1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| \\ &= 2\left(\sum_{l=1}^{\frac{p-1}{2}} 8(8pq - (2(2l-1)4q)\right) + 2\left(\sum_{l=1}^{\frac{p-1}{2}} 9(2q-2)(8pq - (2(2l-1)4q)\right) \\ &= 32(p^2-1)q + 72(p^2-1)(q-1)q \end{aligned}$$

The following table gives the summary of the contribution of the edges in the V_2 cuts.

Table 2: Details regarding the contribution of the edges to the additively weighted Mostar index for the edges in the vertical cut V_2.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u) + d(v)$	No. of edges
$p-1$	$2q$	$\lvert 8pq - 16lq\rvert$	6	$2q$

Table 2': Details regarding the contribution of the edges to the multiplicative weighted Mostar index for the edges in the vertical cut V_2.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u)d(v)$	No. of edges
$p-1$	$2q$	$\lvert 8pq - 16lq\rvert$	9	$2q$

Therefore, the contribution of the edges corresponding to the edges in V_2 cut is as follows. When p is even

$$L_2 = \sum_{e=uv\in V_2} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| = 2\left(\sum_{l=1}^{\frac{p-2}{2}} 12q(8pq - 16lq)\right) = 48(p-2)pq^2$$

$$L_2' = \sum_{e=uv\in V_2} (d(u)d(v))|n_u(e|G) - n_v(e|G)| = 2\left(\sum_{l=1}^{\frac{p-2}{2}} 18q(8pq - 16lq)\right) = 72(p-2)pq^2$$

When p is odd

$$L_2 = \sum_{e=uv\in V2} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| = 2\left(\sum_{l=1}^{\frac{p-1}{2}} 12q(8pq - 16lq)\right) = 48(p-1)^2q^2$$

$$L_2' = \sum_{e=uv\in V2} (d(u)d(v))|n_u(e|G) - n_v(e|G)| = 2\left(\sum_{l=1}^{\frac{p-1}{2}} 18q(8pq - 16lq)\right) = 72(p-1)^2q^2$$

The following table gives the summary of the contribution of the edges in the H_1 cuts.

Table 3: Details regarding the contribution of the edges to the additively weighted Mostar index for the edges in the horizontal cut H_1.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u) + d(v)$	No. of edges
q	$2p$	$\lvert 8pq - (2l-1)8p\rvert$	4	2
			6	$2p-2$

Table 3': Details regarding the contribution of the edges to the multiplicative weighted Mostar index for the edges in the horizontal cut H_1.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u)d(v)$	No. of edges
q	$2p$	$\lvert 8pq-(2l-1)8p\rvert$	4	2
			9	$2p-2$

Now, the the contribution of the edges corresponding to the edges in $H1$ cut is as follows. When q is even

$$\begin{aligned} L_3 &= \sum_{e=uv\in H_1} (d(u)+d(v))|n_u(e|G)-n_v(e|G)| \\ &= 16\left(\sum_{l=1}^{\frac{q}{2}}(8pq-8p(2l-1))\right)+12(2p-2)\left(\sum_{l=1}^{\frac{q}{2}}(8pq-8p(2l-1))\right) \\ &= 32pq^2+48p(p-1)q^2 \end{aligned}$$

$$\begin{aligned} L_3' &= \sum_{e=uv\in H_1} (d(u)d(v))|n_u(e|G)-n_v(e|G)| \\ &= 16\left(\sum_{l=1}^{\frac{q}{2}}(8pq-8p(2l-1))\right)+18(2p-2)\left(\sum_{l=1}^{\frac{q}{2}}(8pq-8p(2l-1))\right) \\ &= 32pq^2+72p(p-1)q^2 \end{aligned}$$

When q is odd

$$\begin{aligned} L_3 &= \sum_{e=uv\in H1} (d(u)+d(v))|n_u(e|G)-n_v(e|G)| \\ &= 16\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq-8p(2l-1))\right)+12(2p-2)\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq-8p(2l-1))\right) \\ &= 32p(q^2-1)+48p(p-1)(q^2-1) \end{aligned}$$

$$\begin{aligned} L_3' &= \sum_{e=uv\in H1} (d(u)+d(v))|n_u(e|G)-n_v(e|G)| \\ &= 16\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq-8p(2l-1))\right)+18(2p-2)\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq-8p(2l-1))\right) \\ &= 32p(q^2-1)+72p(p-1)(q^2-1) \end{aligned}$$

The following table gives the summary of the contribution of the edges in the H_2 cuts.

Table 4: Details regarding the contribution of the edges to the additively weighted Mostar index for the edges in the horizontal cut H_2.

No. of cuts	No. of edges in each cut	$\lvert n_u(e\lvert G) - n_v(e\lvert G)\rvert$	$d(u)+d(v)$	No. of edges
$q-1$	$2p$	$\lvert 8pq-16lp\rvert$	6	$2p$

Table 4': Details regarding the contribution of the edges to the multiplicative weighted Mostar index for the edges in the horizontal cut H_2.

No. of cuts	No. of edges in each cut	$\vert n_u(e\vert G) - n_v(e\vert G)\vert$	$d(u)d(v)$	No. of edges
$q-1$	$2p$	$\vert 8pq - 16lp\vert$	9	$2p$

Therefore, the contribution of the edges corresponding to the edges in H_2 cut is as follows. When q is even

$$L_4 = \sum_{e=uv\in H_2} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| = 12(2p)\left(\sum_{l=1}^{\frac{q-2}{2}}(8pq - 16lp)\right) = 48p^2(q-2)q$$

$$L'_4 = \sum_{e=uv\in H_1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| = 18(2p)\left(\sum_{l=1}^{\frac{q-2}{2}}(8pq - 16lp)\right) = 72p^2(q-2)q$$

When q is odd

$$L_4 = \sum_{e=uv\in H_2} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| = 12(2p)\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq - 16lp)\right) = 48p^2(q-1)^2$$

$$L'_4 = \sum_{e=uv\in H_1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| = 18(2p)\left(\sum_{l=1}^{\frac{q-1}{2}}(8pq - 16lp)\right) = 72p^2(q-1)^2$$

Now, in the case of diagonal cuts, D_1 and D_2, the contribution is same. When $p < q$, note that every $l-$th diagonal cut except the $p-$th and $(q-1)-$th cut contains exactly $2l-2$ edges having degree $d(u)+d(v) = 6, (d(u)d(v) = 9)$ and 2 edge having degree $d(u)+d(v) = 5, (d(u)d(v) = 6)$. In the case of $p-$th and $(q-1)-$th cut there are $2l-2$ edges having degree $d(u)+d(v) = 6, (d(u)d(v) = 9)$ and one edge having degree $d(u)+d(v) = 5, (d(u)d(v) = 6)$ and one other edge having degree $d(u)+d(v) = 4, (d(u)d(v) = 4)$(similarly for $q < p$). When $p = q$ the middle diagonal cut contains exactly two edges with degree $d(u)+d(v) = 4, (d(u)d(v) = 4)$. The following table gives the summary of the contribution of the edges in the $D_1(D_2)$ cuts.

Table 5: Details regarding the contribution of the edges to the additively weighted Mostar index for the edges in the diagonal cut D_1 (or D_2).

No. of cuts	No. of edges in each cut	$\vert n_u(e\vert G) - n_v(e\vert G)\vert$	$d(u)+d(v)$	No. of edges
$p+q-1$	$2l$	$\vert 8pq - 8l^2\vert$	5	1 or 2
			4	1 or 2
			6	$2l-2$

Table 5': Details regarding the contribution of the edges to the multiplicative weighted Mostar index for the edges in the diagonal cut D_1 (or D_2).

No. of cuts	No. of edges in each cut	$\vert n_u(e\vert G) - n_v(e\vert G)\vert$	$d(u)d(v)$	No. of edges
$p+q-1$	$2l$	$\vert 8pq - 8l^2\vert$	6	2
			4	1 or 2
			9	1 or 2

Therefore, the contribution of the edges corresponding to the edges in D_1 (or D_2) cut is as follows.
When p, q are of same parity with $p < q$

$$\begin{aligned}
L_5 = L_6 &= \sum_{e=uv\in H1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| \\
&= 2\left(\sum_{l=1}^{\frac{p+q-2}{2}} 10(8pq-8l^2)\right) + 2\left(\sum_{l=1}^{\frac{p+q-2}{2}} 6(2l-2)(8pq-8l^2)\right) - 2\left((8pq-8p^2)\right) \\
&= -\frac{20}{3}\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\
&+ 2\left(-\frac{3p^4}{2} + 2p^3(3q+5) + 3p^2(5q^2-14q-6)\right) \\
&+ 2\left(p(6q^3-42q^2+60q+8) - \frac{3q^4}{2} + 10q^3 - 18q^2 + 8q\right) - 2(8pq-8p^2)
\end{aligned}$$

Similarly, when p, q are of same parity with $q < p$

$$\begin{aligned}
L_5 = L_6 = &-\frac{20}{3}\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\
&+ 2\left(-\frac{3p^4}{2} + 2p^3(3q+5) + 3p^2(5q^2-14q-6)\right) \\
&+ 2\left(p(6q^3-42q^2+60q+8) - \frac{3q^4}{2} + 10q^3 - 18q^2 + 8q\right) - 2(8pq-8q^2)
\end{aligned}$$

when p, q are of same parity with $p = q$

$$\begin{aligned}
L_5 = L_6 = &-\frac{20}{3}\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\
&+ 2\left(-\frac{3p^4}{2} + 2p^3(3q+5) + 3p^2(5q^2-14q-6)\right) \\
&+ 2\left(p(6q^3-42q^2+60q+8) - \frac{3q^4}{2} + 10q^3 - 18q^2 + 8q\right)
\end{aligned}$$

When p, q are of same parity with $p < q$

$$\begin{aligned}
L'_5 = L'_6 &= \sum_{e=uv\in H1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| \\
&= 2\left(\sum_{l=1}^{\frac{p+q-2}{2}} 12(8pq-8l^2)\right) + 2\left(\sum_{l=1}^{\frac{p+q-2}{2}} 6(2l-2)(8pq-8l^2)\right) - 4\left((8pq-8p^2)\right) \\
&= -4(8pq-8p^2) - 8\left(p^3 - 3p^2(3q+1) + p(-9q^2+18q+2) + q(q^2-3q+2)\right) \\
&- \frac{3}{2}\left(3p^4 - 12p^3q - 20p^3 - 30p^2q^2 + 84p^2q\right) \\
&- \frac{3}{2}\left(36p^2 - 12pq^3 + 84pq^2 - 120pq - 16p + 3q^4 - 20q^3 + 36q^2 - 16q\right)
\end{aligned}$$

Similarly, when p, q are of same parity with $q < p$

$$\begin{aligned} L'_5 = L'_6 &= -4(8pq - 8q^2) - 8\left(p^3 - 3p^2(3q+1) + p(-9q^2 + 18q + 2) + q(q^2 - 3q + 2)\right) \\ &- \frac{3}{2}\left(3p^4 - 12p^3q - 20p^3 - 30p^2q^2 + 84p^2q + 36p^2 - 12pq^3\right) \\ &- \frac{3}{2}\left(84pq^2 - 120pq - 16p + 3q^4 - 20q^3 + 36q^2 - 16q\right) \end{aligned}$$

When p, q are of same parity with $p = q$

$$\begin{aligned} L'_5 = L'_6 &= -8\left(p^3 - 3p^2(3q+1) + p(-9q^2 + 18q + 2) + q(q^2 - 3q + 2)\right) \\ &- \frac{3}{2}\left(3p^4 - 12p^3q - 20p^3 - 30p^2q^2 + 84p^2q + 36p^2 - 12pq^3\right) \\ &- \frac{3}{2}\left(84pq^2 - 120pq - 16p + 3q^4 - 20q^3 + 36q^2 - 16q\right) \end{aligned}$$

When p, q are of different parity with $p < q$

$$\begin{aligned} L_5 = L_6 &= \sum_{e=uv\in H1} (d(u) + d(v))|n_u(e|G) - n_v(e|G)| \\ &= 2\left(\sum_{l=1}^{\frac{p+q-1}{2}} 10(8pq - 8l^2)\right) + 2\left(\sum_{l=1}^{\frac{p+q-1}{2}} 6(2l-2)(8pq - 8l^2)\right) - 2(8pq - 8p^2) \\ &= -3p^4 + 4p^3(3q+2) + 6p^2(5q^2 - 12q + 1) - 2(8pq - 8p^2) \\ &- \frac{20}{3}\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) \\ &+ 4p\left(3q^3 - 18q^2 + 21q - 2\right) - 3q^4 + 8q^3 + 6q^2 - 8q - 3 \end{aligned}$$

When p, q are of different parity with $q < p$

$$\begin{aligned} L_5 = L_6 &= -3p^4 + 4p^3(3q+2) + 6p^2\left(5q^2 - 12q + 1\right) - \frac{20}{3}\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) \\ &- 2(8pq - 8q^2) + 4p\left(3q^3 - 18q^2 + 21q - 2\right) - 3q^4 + 8q^3 + 6q^2 - 8q - 3 \end{aligned}$$

When p, q are of different parity with $p < q$

$$\begin{aligned} L'_5 = L'_6 &= \sum_{e=uv\in H1} (d(u)d(v))|n_u(e|G) - n_v(e|G)| \\ &= 2\left(\sum_{l=1}^{\frac{p+q-1}{2}} 12(8pq - 8l^2)\right) + 2\left(\sum_{l=1}^{\frac{p+q-1}{2}} 9(2l-2)(8pq - 8l^2)\right) - 4(8pq - 8p^2) \\ &= -4(8pq - 8p^2) - 8\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) \\ &- \frac{3}{2}\left(3p^4 - 4p^3(3q+2) - 6p^2(5q^2 - 12q + 1)\right) \\ &- \frac{3}{2}\left(p(-12q^3 + 72q^2 - 84q + 8) + 3q^4 - 8q^3 - 6q^2 + 8q + 3\right) \end{aligned}$$

When p, q are of different parity with $q < p$

$$\begin{aligned} L'_5 = L'_6 &= -4(8pq - 8q^2) - 8\left(p^3 - 9p^2q - 9pq^2 + 12pq - p + q^3 - q\right) \\ &- \frac{3}{2}\left(3p^4 - 4p^3(3q+2) - 6p^2(5q^2 - 12q + 1)\right) \\ &- \frac{3}{2}\left(p(-12q^3 + 72q^2 - 84q + 8) + 3q^4 - 8q^3 - 6q^2 + 8q + 3\right) \end{aligned}$$

Now, $Mo_A(T^1UC_4C_8[p,q]) = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$ and $Mo_M(T^1UC_4C_8[p,q]) = L'_1 + L'_2 + L'_3 + L'_4 + L'_5 + L'_6$. Therefore, combining all the terms together, we get the required results □

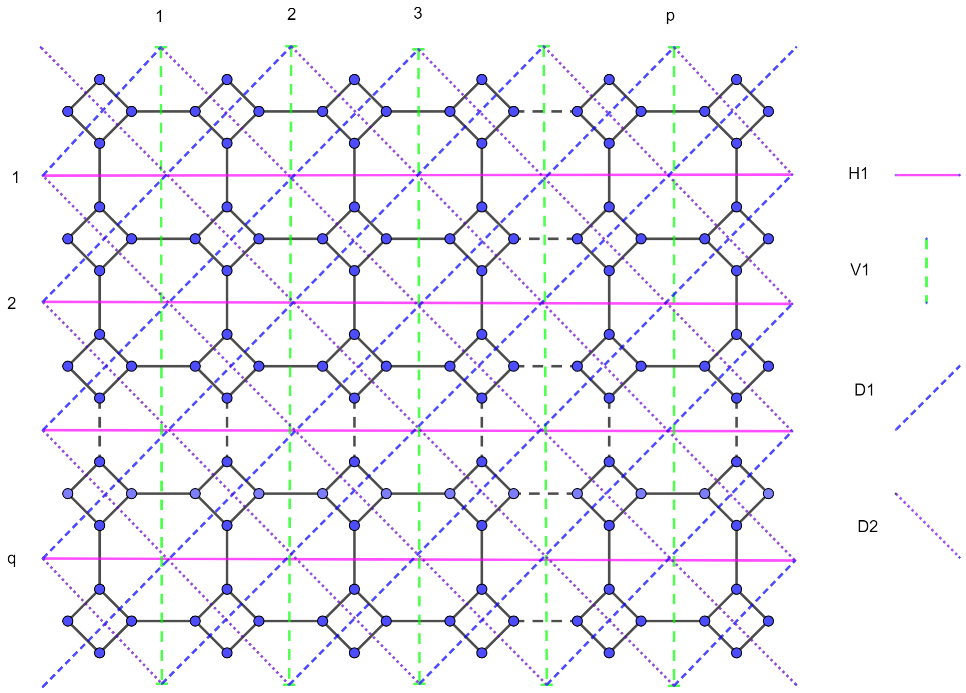

FIGURE 2. All the different cuts of the components $T^2UC_4C_8[p,q]$

Theorem 2.2. *For the graph $T^2UC_4C_8[p,q]$, the weighted versions of Mostar indices are as follows*

(a.) *When p, q are even,*

$$\begin{aligned} Mo_A(T^2UC_4C_8[p,q]) &= 12p^2(q+1)^2 + 12q^2(p+1)^2 \\ &+ \frac{20}{3}\left(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)\right) \\ &- 3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2) \\ &+ 4p(3q^3+3q^2-6q-8) - q(3q^3-20q^2+12q+32) - 4I' \end{aligned}$$

$$\text{where } I' = \begin{cases} (4(p+1)(q+1) - 4(p+1)^2) & \text{if } p < q \\ (4(p+1)(q+1) - 4(q+1)^2) & \text{if } q < p \\ 0 & \text{if } p = q \end{cases}$$

(b.) *When p, q are odd,*

$$\begin{aligned} Mo_A(T^2UC_4C_8[p,q]) &= 12(p^2-1)(q+1)^2 + 12(q^2-1)(p+1)^2 \\ &+ \frac{20}{3}\left(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)\right) \\ &- 3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2) \\ &+ 4p(3q^3+3q^2-6q-8) - q(3q^3-20q^2+12q+32) - 4I' \end{aligned}$$

(c.) *When p is even and q is odd,*

$$\begin{aligned} Mo_A(T^2UC_4C_8[p,q]) &= 12p^2(q+1)^2 + 12(q^2-1)(p+1)^2 \\ &+ \frac{20}{3}\left(-p^3 + p^2(9q+6) + p(9q^2+24q+13) - q^3 + 6q^2 + 13q + 6\right) \\ &- 3p^4 + 4p^3(3q+2) + 6p^2(5q^2+4q+1) \\ &+ 4p(3q^3+6q^2-3q-2) - 3q^4 + 8q^3 + 6q^2 - 8q - 3 - 4I' \end{aligned}$$

(d.) *When p is odd and q is even,*

$$\begin{aligned} Mo_A(T^2UC_4C_8[p,q]) &= 12(p^2-1)(q+1)^2 + 12q^2(p+1)^2 \\ &+ \frac{20}{3}\left(-p^3 + p^2(9q+6) + p(9q^2+24q+13) - q^3 + 6q^2 + 13q + 6\right) \\ &- 3p^4 + 4p^3(3q+2) + 6p^2(5q^2+4q+1) \\ &+ 4p(3q^3+6q^2-3q-2) - 3q^4 + 8q^3 + 6q^2 - 8q - 3 - 4I' \end{aligned}$$

(e.) *When p, q are even,*

$$\begin{aligned} Mo_M(T^2UC_4C_8[p,q]) &= 18p^2(q+1)^2 + 18q^2(p+1)^2 \\ &+ 8\left(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)\right) \\ &+ \frac{3}{2}\left(-3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2)\right) \\ &+ \frac{3}{2}\left(4p(3q^3+3q^2-6q-8) - q(3q^3-20q^2+12q+32)\right) - 8I' \end{aligned}$$

(f.) *When p, q are odd,*

$$\begin{aligned} Mo_M(T^2UC_4C_8[p,q]) &= 18(p^2-1)(q+1)^2 + 18(q^2-1)(p+1)^2 \\ &+ 8\left(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)\right) \\ &+ \frac{3}{2}\left(-3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2)\right) \\ &+ \frac{3}{2}\left(4p(3q^3+3q^2-6q-8) - q(3q^3-20q^2+12q+32)\right) - 8I' \end{aligned}$$

(g.) *When p is even and q is odd,*

$$\begin{aligned} Mo_M(T^2UC_4C_8[p,q]) &= 18p^2(q+1)^2 + 18(q^2-1)(p+1)^2 \\ &+ 8\left(-p^3 + p^2(9q+6) + p(9q^2+24q+13) - q^3 + 6q^2 + 13q + 6\right) \\ &+ \frac{3}{2}\left(-3p^4 + 4p^3(3q+2) + 6p^2(5q^2+4q+1)\right) \\ &+ \frac{3}{2}\left(4p(3q^3+6q^2-3q-2) - 3q^4 + 8q^3 + 6q^2 - 8q - 3\right) - -8I' \end{aligned}$$

(h.) *When p is odd and q is even,*

$$\begin{aligned} Mo_M(T^2UC_4C_8[p,q]) &= 18(p^2-1)(q+1)^2 + 18q^2(p+1)^2 \\ &+ 8\left(-p^3 + p^2(9q+6) + p(9q^2+24q+13) - q^3 + 6q^2 + 13q + 6\right) \\ &+ \frac{3}{2}\left(-3p^4 + 4p^3(3q+2) + 6p^2(5q^2+4q+1)\right) \\ &+ \frac{3}{2}\left(4p(3q^3+6q^2-3q-2) - 3q^4 + 8q^3 + 6q^2 - 8q - 3\right) - -8I' \end{aligned}$$

Proof. We use a variant of cut method to determine the explicit expressions of the indices. We partition the edges of the graph into four different sets according to the different cuts on the graph. We consider one horizontal cut $H1$, one vertical cut $V1$ and two diagonal cuts $D1$ and $D2$ respectively. Let S_1, S_2, S_2 and S_4 denote the sum corresponding to the cuts $H1, V1, D1$ and $D2$ respectively for the additively weighted Mostar index. Similarly, let S'_1, S'_2, S'_3 and S'_4 denote the sum corresponding to the cuts $H1, V1, D1$ and $D2$ respectively for the multiplicatively weighted Mostar index. The following tables gives the summary of the contribution of the edges in different cuts.

Table 1: Details regarding the contribution of the edges to the additively/multiplicative weighted Mostar index for the edges in the horizontal cut $H1$.

Vertical cut $V1$						
No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u) + d(v)$	No. of edges
q	$p+1$	$4l(p+1)$	$4(p+1)((q+1)-l)$	$\|4(p+1)((q+1)-2l)\|$	6	$p+1$
No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u)d(v)$	No. of edges
q	$p+1$	$4l(p+1)$	$4(p+1)((q+1)-l)$	$\|4(p+1)((q+1)-2l)\|$	9	$p+1$

Table 2: Details regarding the contribution of the edges to the additively/multiplicative weighted Mostar index for the edges in the vertical cut $V1$.

Horizontal cut $H1$						
No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u) + d(v)$	No. of edges
p	$q+1$	$4l(q+1)$	$4(q+1)((p+1)-l)$	$\|4(q+1)((p+1)-2l)\|$	6	$q+1$
No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u)d(v)$	No. of edges
p	$q+1$	$4l(q+1)$	$4(q+1)((p+1)-l)$	$\|4(q+1)((p+1)-2l)\|$	9	$q+1$

Therefore, the contribution of the edges corresponding to the edges in $H1$ and $V1$ cut is as follows. When p is even

$$\begin{aligned} S_1 &= \sum_{e=uv\in V1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| \\ &= 12(q+1)\left(\sum_{l=1}^{\frac{p}{2}}(4(p+1)(q+1) - 8l(q+1))\right) \\ &= 12p^2(q+1)^2 \end{aligned}$$

$$\begin{aligned} S'_1 &= \sum_{e=uv\in V1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| \\ &= 18(q+1)\left(\sum_{l=1}^{\frac{p}{2}}(4(p+1)(q+1) - 8l(q+1))\right) \\ &= 18p^2(q+1)^2 \end{aligned}$$

When p is odd

$$S_1 = \sum_{e=uv\in V1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)|$$
$$= 12(q+1)\left(\sum_{l=1}^{\frac{p-1}{2}}(4(p+1)(q+1) - 8l(q+1))\right)$$
$$= 12(p^2-1)(q+1)^2$$

$$S_1' = \sum_{e=uv\in V1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)|$$
$$= 18(q+1)\left(\sum_{l=1}^{\frac{p-1}{2}}(4(p+1)(q+1) - 8l(q+1))\right)$$
$$= 18(p^2-1)(q+1)^2$$

From the previous expressions, we can compute the contribution of horizontal cut by replacing p and q by q and p respectively. Therefore, When q is even

$$S_2 = 12q^2(p+1)^2 \text{ and } S_2' = 18q^2(p+1)^2$$

When q is odd,

$$S_2 = 12(q^2-1)(p+1)^2 \text{ and } S_2' = 18(q^2-1)(p+1)^2$$

Now, in the case of diagonal cuts, $D1$ and $D2$, due to symmetry the contributions are same. When $p < q$, note that the $p+1-$th diagonal cut has one edge with $d(u)+d(v) = 4, (d(u)d(v) = 4)$ and one edge with $d(u)+d(v) = 5, (d(u)d(v) = 6)$ and $2p$ edges with $d(u)+d(v) = 6, (d(u)d(v) = 9)$. Every other diagonal cut has two edges with $d(u)+d(v) = 5, (d(u)d(v) = 6)$ and $2l-2$ edges with $d(u)+d(v) = 6, (d(u)d(v) = 9)$(similarly for $q < p$). When $p = q$ the middle diagonal cut contains exactly two edges with degree $d(u)+d(v) = 4, (d(u)d(v) = 4)$.

Table 3: Details regarding the contribution of the edges to the additively/multiplicatively weighted Mostar index for the edges in the diagonal cuts $D1$ and $D2$.

No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u)+d(v)$	No. of edges
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	5	1 or 2
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	4	1 or 2
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	6	$2l-2$

Table 4: Details regarding the contribution of the edges to the multiplicatively weighted Mostar index for the edges in the vertical cut $V2$.

No. of cuts	No. of edges in each cut	$n_u(e\|G)$	$n_v(e\|G)$	$\|n_u(e\|G) - n_v(e\|G)\|$	$d(u)d(v)$	No. of edges
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	6	1 or 2
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	4	1 or 2
$p+q+1$	$2l$	$2l^2$	$4(p+1)(q+1)-2l^2$	$\|4(p+1)(q+1)-4l^2\|$	9	$2l-2$

Due to symmetry of the structure, the total number of diagonal cuts of $D1$ and $D2$ must be same. Hence the total contribution of both the diagonal cuts must be same. Thus, the contribution when p, q are of same parity with $p < q$ is

$$\begin{aligned}
S_3 = S_4 &= \sum_{e=uv\in D_1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| \\
&= 20\left(\sum_{l=1}^{\frac{p+q}{2}} (4(p+1)(q+1) - 4l^2)\right) + 12\left(\sum_{l=1}^{\frac{p+q}{2}} (2l-2)(4(p+1)(q+1) - 4l^2)\right) \\
&- 2(4(p+1)(q+1) - 4(p+1)^2) \\
&= 10/3(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) + 1/2(-3p^4 + 4p^3(3q+5) \\
&+ 6p^2(5q^2+2q-2) + 4p(3q^3+3q^2-6q-8) \\
&- q(3q^3 - 20q^2 + 12q + 32)) - 2(4(p+1)(q+1) - 4(p+1)^2)
\end{aligned}$$

when p, q are of same parity with $q < p$ is

$$\begin{aligned}
S_3 = S_4 &= 10/3(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) + 1/2(-3p^4 + 4p^3(3q+5) \\
&+ 6p^2(5q^2+2q-2) + 4p(3q^3+3q^2-6q-8) \\
&- q(3q^3 - 20q^2 + 12q + 32)) - 2(4(p+1)(q+1) - 4(q+1)^2)
\end{aligned}$$

when p, q are of same parity with $p = q$ is

$$\begin{aligned}
S_3 = S_4 &= 20\left(\sum_{l=1}^{\frac{p+q}{2}} (4(p+1)(q+1) - 4l^2)\right) + 12\left(\sum_{l=1}^{\frac{p+q}{2}} (2l-2)(4(p+1)(q+1) - 4l^2)\right) \\
&= 10/3(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) + 1/2(-3p^4 + 4p^3(3q+5) \\
&+ 6p^2(5q^2+2q-2) + 4p(3q^3+3q^2-6q-8) - q(3q^3 - 20q^2 + 12q + 32))
\end{aligned}$$

when p, q are of same parity with $p < q$

$$\begin{aligned}
S'_3 = S'_4 &= \sum_{e=uv\in D_1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)| \\
&= 24\left(\sum_{l=1}^{\frac{p+q}{2}} (4(p+1)(q+1) - 4l^2)\right) + 18\left(\sum_{l=1}^{\frac{p+q}{2}} (2l-2)(4(p+1)(q+1) - 4l^2)\right) \\
&- 4(4(p+1)(q+1) - 4(p+1)^2) \\
&= 4(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) + 3/4(-3p^4 + 4p^3(3q+5) \\
&+ 6p^2(5q^2+2q-2) + 4p(3q^3+3q^2-6q-8) \\
&- q(3q^3 - 20q^2 + 12q + 32)) - 4(4(p+1)(q+1) - 4(p+1)^2)
\end{aligned}$$

when p, q are of same parity with $q < p$

$$\begin{aligned}
S'_3 = S'_4 &= 4(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) \\
&+ \frac{3}{4}\left((-3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2)\right) \\
&+ \frac{3}{4}\left(4p(3q^3+3q^2-6q-8) - q(3q^3 - 20q^2 + 12q + 32)\right) - 4(4(p+1)(q+1) - 4(q+1)^2)
\end{aligned}$$

when p, q are of same parity with $p = q$

$$\begin{aligned}
S'_3 = S'_4 &= 4(-p^3 + 9p^2(q+1) + p(9q^2+18q+10) + q(-q^2+9q+10)) \\
&+ \frac{3}{4}\left(-3p^4 + 4p^3(3q+5) + 6p^2(5q^2+2q-2)\right) \\
&+ \frac{3}{4}\left(4p(3q^3+3q^2-6q-8) - q(3q^3 - 20q^2 + 12q + 32)\right)
\end{aligned}$$

when p, q are of different parity with $p < q$,

$$S_3 = S_4 = \sum_{e=uv\in D1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)|$$
$$= 20\left(\sum_{l=1}^{\frac{p+q+1}{2}} (4(p+1)(q+1)-4l^2)\right) + 12\left(\sum_{l=1}^{\frac{p+q+1}{2}} (2l-2)(4(p+1)(q+1)-4l^2)\right)$$
$$-2(4(p+1)(q+1)-4(p+1)^2)$$
$$= 10/3(-p^3+p^2(9q+6)+p(9q^2+24q+13)-q^3+6q^2+13q+6)$$
$$+1/2(-3p^4+4p^3(3q+2)+6p^2(5q^2+4q+1)$$
$$+4p(3q^3+6q^2-3q-2)-3q^4+8q^3+6q^2-8q-3)-2(4(p+1)(q+1)-4(p+1)^2)$$

when p, q are of different parity with $q < p$,

$$S_3 = S_4 = 10/3(-p^3+p^2(9q+6)+p(9q^2+24q+13)-q^3+6q^2+13q+6)$$
$$+1/2(-3p^4+4p^3(3q+2)+6p^2(5q^2+4q+1)+4p(3q^3+6q^2-3q-2)$$
$$-3q^4+8q^3+6q^2-8q-3)-2(4(p+1)(q+1)-4(q+1)^2)$$

when p, q are of different parity with $p < q$,

$$S'_3 = S'_4 = \sum_{e=uv\in D_1} (d(u)+d(v))|n_u(e|G) - n_v(e|G)|$$
$$= 24\left(\sum_{l=1}^{\frac{p+q+1}{2}} (4(p+1)(q+1)-4l^2)\right) + 18\left(\sum_{l=1}^{\frac{p+q+1}{2}} (2l-2)(4(p+1)(q+1)-4l^2)\right)$$
$$-4(4(p+1)(q+1)-4(p+1)^2)$$
$$= 4(-p^3+p^2(9q+6)+p(9q^2+24q+13)-q^3+6q^2+13q+6)+3/4(-3p^4+4p^3(3q+2)$$
$$+6p^2(5q^2+4q+1)+4p(3q^3+6q^2-3q-2)-3q^4+8q^3+6q^2-8q-3)$$
$$-4(4(p+1)(q+1)-4(p+1)^2)$$

when p, q are of different parity with $q < p$,

$$S'_3 = S'_4 = 4(-p^3+p^2(9q+6)+p(9q^2+24q+13)-q^3+6q^2+13q+6)+3/4(-3p^4+4p^3(3q+2)$$
$$+6p^2(5q^2+4q+1)+4p(3q^3+6q^2-3q-2)-3q^4+8q^3+6q^2-8q-3)$$
$$-4(4(p+1)(q+1)-4(q+1)^2)$$

Now, $Mo_A(T^2UC_4C_8[p,q]) = S_1+S_2+S_3+S_4$ and $Mo_M(T^2UC_4C_8[p,q]) = S'_1+S'_2+S'_3+S'_4$, by considering the different possibilities we get the result. □

3. Conclusion

In this paper, we computed the explicit expressions of the weighted Mostar type bond additive indices of certain class of carbon nanostructures using a variant of cut method. There are several other carbon nanostructures in which several bond additive indices have not been computed. Which is a problem which needs further research.

4. Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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