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Research Paper

SOME SOLITONS ON ANTI-INVARIANT SUBMANIFOLD OF LP-KENMOTSU MANOFOLD ADMITTING ZAMKOVOY CONNECTION

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ABSTRACT

In this paper we prove some curvature properties of anti-invariant submanifold of Lorentzian para-Kenmotsu manifold (briefly, LP-Kenmotsu manifold) with respect to Zamkovoy connection (∇^*). Next, we study Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to Zamkovoy connection. Further, we study η-Einstein soliton on this submanifold with respect to Zamkovoy connection under different curvature conditions. Finally, we give an example of anti-invariant submanifold of 5-dimensional LP-Kenmotsu manifold admitting η-Einstein soliton with respect to ∇^* and verify a relation on the manifold under consideration.

1. Introduction

In 2008, the notion of Zamkovoy canonical connection (briefly, Zamkovoy connection) was introduced by Zamkovoy [30] for a para-contact manifold. And this connection was defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold. Later, Biswas and Baishya [1, 2] studied this connection on generalized pseudo Ricci symmetric Sasakian manifolds and on almost pseudo symmetric Sasakian manifolds. This connection was further studied by Blaga [3] on para-Kenmotsu

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manifolds. In 2020, Mandal and Das [7, 13, 14, 15] studied in detail on various curvature tensors of Sasakian and LP-Sasakian manifolds admitting Zamkovoy connection. In 2021, they discussed LP-Sasakian manifolds equipped with Zamkovoy connection and conharmonic curvature tensor [16]. Recently, they introduced Zamkovoy connection on Lorentzian para-Kenmotsu manifold [17] and studied Ricci soliton on it with respect to this connection. Zamkovoy connection for an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y, \tag{1.1}$$

for all X, $Y \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M.

In 2018, the notion of Lorentzian para-Kenmotsu manifold (LP- Kenmotsu manifold for short) has been introduced by Haseeb and Prasad [9]. Later, Shukla and Dixit [25] studied ϕ-recurrent Lorentzian para-Kenmotsu manifolds and find that such type of manifolds are η-Einstein. Further, Chandra and Lal [6] studied some special results on 3-dimensional Lorentzian para-Kenmotsu manifolds. This manifold is also studied by Sai Prasad, Sunitha Devi [22].

In 1977, anti-invariant submanifolds of Sasakian space forms were introduced by Yano and Kon [28]. Later in 1985, Pandey and Kumar investigated properties of anti-invariant submanifolds of almost para-contact manifolds [20]. Recently, Karmakar and Bhattyacharyya [11] studied anti-invariant submanifolds of some indefinite almost contact and para-contact manifolds. Most recently, Karmakar [10] studied η-Ricci-Yamabe soliton on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection.

Let ϕ be a differential map from a manifold $\widetilde{N}$ into another manifold $\widetilde{M}$ and let the dimensions of $\widetilde{N}$, $\widetilde{M}$ be $\widetilde{n}$, $\widetilde{m}$ $(\widetilde{n} < \widetilde{m})$, respectively. If rank$\phi = \widetilde{n}$, then ϕ is called an immersion of $\widetilde{N}$ into $\widetilde{M}$. If $\phi(p) \neq \phi(q)$ for $p \neq q$, then ϕ is called an imbedding of $\widetilde{N}$ into $\widetilde{M}$. If the manifolds $\widetilde{N}$ and $\widetilde{M}$ satisfy the following two conditions, then $\widetilde{N}$ is called submanifold of $\widetilde{M}$ - (i) $\widetilde{N} \subset \widetilde{M}$, (ii) the inclusion map from $\widetilde{N}$ into $\widetilde{M}$ is an imbedding of $\widetilde{N}$ into $\widetilde{M}$.

A submanifold $\widetilde{N}$ is called anti-invariant if $X \in T_x(\widetilde{N}) \Rightarrow \phi X \in T_x^{\perp}(\widetilde{N})$ for all $X \in \widetilde{N}$, where $T_x(\widetilde{N})$ and $T_x^{\perp}(\widetilde{N})$ are respectively tangent space and normal space at $x \in \widetilde{N}$. Thus in an anti-invariant submanifold $\widetilde{N}$, we have for all $X, Y \in \widetilde{N}$

$$g(X, \phi Y) = 0.$$

The concept of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. Hamilton [8] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S, \tag{1.2}$$

where g is a Riemannian metric, S is Ricci tensor and t is time. The solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where S is Ricci curvature tensor and $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the associated function is called potential function. Ricci soliton was further studied by many researchers. For instance, we see [19, 21, 24, 26] and their references.

Catino and Mazzieri [5] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure (ϕ, ξ, η, g) is said to have an Einstein soliton (g, V, λ) if

$$L_V g + 2S + (2\lambda - r)g = 0, \tag{1.4}$$

holds, where r being the scalar curvature. The Einstein soliton (g, V, λ) is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation

$$\frac{\partial g}{\partial t} = -2S + rg.$$

Again as a generalization of Einstein soliton the η-Einstein soliton on manifold $M\,(\phi, \xi, \eta, g)$ is introduced by A. M. Blaga [4] and it is given by

$$L_V g + 2S + (2\lambda - r)g + 2\beta\eta \otimes \eta = 0, \tag{1.5}$$

where, β is some constant. When $\beta = 0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\beta \neq 0$, the data (g, V, λ, β) is called proper η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [12, 29]. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. An interesting invariant of a concircular transformation is the concircular curvature tensor $(\mathcal{W})$, which was defined in [27, 29] as

$$\mathcal{W}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}\left[g(Y, Z)X - g(X, Z)Y\right], \tag{1.6}$$

for all X, Y, $Z \in \chi(M)$, set of all vector fields of the manifold M, where R is the Riemannian curvature tensor and r is the scalar curvature.

Definition 1.1. A Riemannian manifold M is called an η-Einstein manifold if its Ricci curvature tensor is of the form

$$S(Y, Z) = k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z),$$

for all Y, $Z \in \chi(M)$, where k_1, k_2 are scalars.

This paper is structured as follows:

First two sections of the paper have been kept for introduction and preliminaries. In **Section-3**, we give expression for Zamkovoy connection on anti-invariant submanifold of LP-Kenmotsu manifold. In **Section-4**, we study Einstein soliton with respect to Zamkovoy connection on anti-invariant submanifold of LP-Kenmotsu manifold. **Section-5** concerns with η-Einstein soliton with respect to Zamkovoy connection on anti-invariant submanifold of LP-Kenmotsu manifold. **Section-6** contains η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi .)_{R^*} .S^* = 0$. **Section-7** deals with η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi .)_{\mathcal{W}^*} .S^* = 0$. In **Section-8**, we discuss η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi .)_{S^*} .\mathcal{W}^* = 0$. Finally **Section-9,** contains an example of anti-invariant submanifold of 5-dimensional LP-Kenmotsu manifold admitting η-Einstein soliton with respect to Zamkovoy connection.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional Lorentzian almost para-contact manifold with structure (ϕ, ξ, η, g), where η is a 1-form, ξ is the structure vector field, ϕ is a $(1,1)$-tensor field and g is a Lorentzian metric satisfying

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X, Y on $\overline{M}$. A Lorentzian almost para-contact manifold is said to be Lorentzian para-contact manifold if η becomes a contact form. In a Lorentzian para-contact manifold the following relations also hold [18, 23]:

$$\phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.4}$$

$$g(X, \phi Y) = g(\phi X, Y). \tag{2.5}$$

The manifold $\overline{M}$ is called a Lorentzian para-Kenmotsu manifold if

$$(\nabla_X \varphi) Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.6}$$

for all smooth vector fields X, Y on $\overline{M}$.

In a Lorentzian para-Kenmotsu manifold the following relations also hold [9, 17]:

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.7}$$

$$(\nabla_X \eta) Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.8}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.12}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.13}$$

$$S(\xi, \xi) = -(n-1), \tag{2.14}$$

$$Q\xi = (n-1)\xi, \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.16}$$

for all smooth vector fields X, Y, Z on $\overline{M}$.

3. Zamkovoy connection on anti-invariant submanifold of LP-Kenmotsu manifold

Expression of Zamkovoy connection on an n-dimensional LP-Kenmotsu manifold $\overline{M}$ [17] is

$$\nabla^*_X Y = \nabla_X Y - g(X,Y)\xi + \eta(Y)X + \eta(X)\phi Y. \tag{3.1}$$

Setting $Y=\xi$ in (3.1) we obtain

$$\nabla^*_X \xi = -2\left[X + \eta(X)\xi\right]. \tag{3.2}$$

The Riemannian curvature tensor R^* with respect to Zamkovoy connection [17] on $\overline{M}$ is given by

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z + 3g(Y,Z)X - 3g(X,Z)Y \\ &\quad +2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi \\ &\quad +2g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Z)\eta(Y)\xi \\ &\quad +2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y \\ &\quad -2\eta(Y)\eta(Z)\phi X + 2\eta(X)\eta(Z)\phi Y. \end{aligned} \tag{3.3}$$

For an anti-invariant submanifold M of $\overline{M}$ the Riemannian curvature tensor with respect to Zamkovoy connection is given by

$$\begin{aligned} R^*(X,Y)Z &= R(X,Y)Z + 3g(Y,Z)X - 3g(X,Z)Y \\ &\quad +2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi \\ &\quad +2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y \\ &\quad -2\eta(Y)\eta(Z)\phi X + 2\eta(X)\eta(Z)\phi Y. \end{aligned} \tag{3.4}$$

Writing the equation (3.4) by the cyclic permutations of X, Y and Z and using the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we have

$$R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0. \tag{3.5}$$

Therefore, the Riemannian curvature tensor with respect to Zamkovoy connection on M satisfies the 1st Bianchi identity.

Taking inner product of (3.4) with a vector field U, we get

$$\begin{aligned} R^*(X,Y,Z,U) &= R(X,Y,Z,U) + 3g(Y,Z)g(X,U) - 3g(X,Z)g(Y,U) \\ &\quad +2g(Y,Z)\eta(X)\eta(U) - 2g(X,Z)\eta(Y)\eta(U) \\ &\quad +2g(X,U)\eta(Y)\eta(Z) - 2\eta(X)\eta(Z)g(Y,U), \end{aligned} \tag{3.6}$$

where $R^*(X,Y,Z,U) = g(R^*(X,Y)Z,U)$ and $X,\ Y,\ Z,\ U \in \chi(M)$.

Contracting (3.6) over X and U, we get

$$S^*(Y,Z) = S(Y,Z) + (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{3.7}$$

where S^* is the Ricci curvature tensor with respect to Zamkovoy connection.

Proposition 3.1. *The Riemannian curvature tensor with respect to Zamkovoy connection on an anti-invariant submanifold of LP-Kenmotsu manifold satisfies the 1st Bianchi identity.*

Proposition 3.2. *Ricci tensor with respect to Zamkovoy connection of an anti-invariant submanifold of LP-Kenmotsu manifold is symmetric and it is given by (3.7).*

Lemma 3.3. *Let M be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting Zamkovoy connetion, then*

$$R^{*}(X,Y)\xi = 2\left[\eta(Y)X-\eta(X)Y+\eta(Y)\phi X-\eta(X)\phi Y\right], \tag{3.8}$$

$$R^{*}(\xi,Y)Z = 2\left[g(Y,Z)\xi-\eta(Z)Y-\eta(Z)\phi Y\right], \tag{3.9}$$

$$R^{*}(\xi,Y)\xi = 2\left[\eta(Y)\xi+Y+\phi Y\right], \tag{3.10}$$

$$S^{*}(\xi,Z) = S^{*}(Z,\xi)=2(n-1)\eta(Z), \tag{3.11}$$

$$Q^{*}Y = QY+(3n-5)Y+2(n-2)\eta(Y)\xi, \tag{3.12}$$

$$Q^{*}\xi = 2(n-1)\xi, \tag{3.13}$$

$$r^{*} = r+(n-1)(3n-4), \tag{3.14}$$

for all X, Y, $Z\in\chi(M)$, where R^{*}, Q^{*} and r^{*} denote Riemannian curvature tensor, Ricci operator and scalar curvature of M with respect to ∇^{*}, respectively.

Theorem 3.4. *If an n-dimensional anti-invariant submanifold M of an LP-Kenmotsu manifold is Ricci flat with respect to Zamkovoy connection, then M is η-Einstein manifold.*

Proof. Let M be an n-dimensional anti-invariant submanifold of an LP-Kenmotsu manifold, which is Ricci flat with respect to Zamkovoy connection i.e., $S^{*}(Y,Z)=0$, for all Y, $Z\in\chi(M)$. Then from (3.7), we have

$$S(Y,Z)=-(3n-5)g(Y,Z)-2(n-2)\eta(Y)\eta(Z),$$

which implies that M is an η-Einstein manifold. □

Concircular curvature tensor of M with respect to Zamkovoy connection is given by

$$\mathcal{W}^{*}(X,Y)Z = R^{*}(X,Y)Z - \frac{r^{*}}{n(n-1)}\left[g(Y,Z)X-g(X,Z)Y\right], \tag{3.15}$$

for all $X,Y,Z\in\chi(M)$, where R^{*}, $\mathcal{W}^{*}$ and r^{*} are Riemannian curvature tensor, concircular curvature tensor and scalar curvature tensor of M with respect to ∇^{*}, respectively.

Lemma 3.5. *Let M be an n-dimensional anti-invariant submanifold of LP-Kenmotsu manifold admitting Zamkovoy connetion, then*

$$\eta(\mathcal{W}^{*}(X,Y)Z) = \left[\frac{r+(n-1)(3n-4)}{n(n-1)}\right]\left[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\right], \tag{3.16}$$

$$\eta(\mathcal{W}^{*}(X,Y)\xi)=0,\eta(\mathcal{W}^{*}(X,\xi)\xi)=0,\eta(\mathcal{W}^{*}(\xi,Y)\xi)=0, \tag{3.17}$$

$$\mathcal{W}^{*}(X,Y)\xi = \left[\frac{r+(n-1)(n-4)}{n(n-1)}\right]\left[\eta(X)Y-\eta(Y)X\right] +2\left[\eta(Y)\phi X-\eta(X)\phi Y\right], \tag{3.18}$$

$$\mathcal{W}^{*}(\xi,X)Y=-\left[\frac{r+(n-1)(n-4)}{n(n-1)}\right]\left[g(X,Y)\xi-\eta(Y)X\right], \tag{3.19}$$

for all $X,\ Y,\ Z\in\chi\left(M\right).$

4. Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to Zamkovoy connection

Theorem 4.1. *An Einstein soliton* (g, V, λ) *on an anti-invariant submanifold of LP-Kenmotsu manifold is invariant under Zamkovoy connection if relation holds*

$$\begin{aligned} 0 &= 2g(X,Y)\eta(V)-g(X,V)\eta(Y)-g(Y,V)\eta(X) \\ &\quad -(n-2)(3n-7)g(X,Y)+4(n-2)\eta(X)\eta(Y). \end{aligned} \tag{4.1}$$

Proof. The equation (1.4) with respect to Zamkovoy connection on an anti-invariant submanifold M of LP-Kenmotsu manifold may be written as

$$(L_V^*g)(X,Y)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y)=0, \tag{4.2}$$

where L_V^*g denote Lie derivative of g with respect to ∇^* along the vector field V and S^* is the Ricci curvature tensor of M with respect to ∇^*.

After expanding (4.2) and using (3.1) and (3.7) we have

$$\begin{aligned} &(L_V^*g)(X,Y)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y) \\ =\ & g(\nabla_X^*V,Y)+g(X,\nabla_Y^*V)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y) \\ =\ & (L_Vg)(X,Y)+2S(X,Y)+(2\lambda-r)g(X,Y) \\ & +2g(X,Y)\eta(V)-g(X,V)\eta(Y)-g(Y,V)\eta(X) \\ & -(n-2)(3n-7)g(X,Y)+4(n-2)\eta(X)\eta(Y), \end{aligned} \tag{4.3}$$

which shows that the Einstein soliton (g, V, λ) is invariant on M under Zamkovoy connection, if (4.1) holds. □

Theorem 4.2. *Let* M *be an anti-invariant submanifold of LP-Kenmotsu manifold admitting an Einstein soliton* (g, V, λ) *with respect to* ∇^**. If the non-zero potential vector field* V *be collinear with the structure vector field of* M*, then the soliton is*

1. *expanding if* $r>-(3n-8)(n-1)$,
2. *steady if* $r=-(3n-8)(n-1)$,
3. *shrinking if* $r<-(3n-8)(n-1)$.

Proof. Setting $V=\xi$ in (4.2) and using (3.2) we get

$$\begin{aligned} 0 &= (L_\xi^*g)(X,Y)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y) \\ &= g(\nabla_X^*\xi,Y)+g(X,\nabla_Y^*\xi)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y) \\ &= \left[-4-(n-2)(3n-7)+2\lambda-r\right]g(X,Y) \\ &\quad +2S(X,Y)+4(n-3)\eta(X)\eta(Y). \end{aligned} \tag{4.4}$$

Putting $X=Y=\xi$ and using (2.1), (2.14) in (4.4) we get

$$\lambda=\frac{1}{2}\left[r+(3n-8)(n-1)\right],$$

which proves the theorem. □

5. η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold with respect to Zamkovoy connection

Theorem 5.1. *If an n-dimensional anti-invariant submanifold M of an LP-Kenmotsu manifold admits η-Einstein soliton (g,ξ,λ,β) with respect to Zamkovoy connection, then the soliton scalars are given by the following equations*

$$\begin{aligned}\lambda &= \frac{r}{2}\left[\frac{n-2}{n-1}\right]+\frac{1}{2}(3n^2-10n+12),\\ \beta &= -\frac{1}{2(n-1)}\left[r-(n-1)(n+4)\right].\end{aligned}$$

Proof. The equation (1.5) with respect to Zamkovoy connection on an anti-invariant submanifold M of LP-Kenmotsu manifold may be written as

$$(L^*_V g)(X,Y)+2S^*(X,Y)+(2\lambda-r^*)g(X,Y)+2\beta\eta(X)\eta(Y)=0. \tag{5.1}$$

Applying $V=\xi$ in (5.1) we get

$$\begin{aligned}0 &= g(\nabla^*_X\xi,Y)+g(X,\nabla^*_Y\xi)+2S^*(X,Y)\\ &\quad +(2\lambda-r^*)g(X,Y)+2\beta\eta(X)\eta(Y).\end{aligned} \tag{5.2}$$

Using (3.2) in (5.2) we obtain

$$0=2S^*(X,Y)+(2\lambda-r^*-4)g(X,Y)+2(\beta-2)\eta(X)\eta(Y). \tag{5.3}$$

Using (3.7) in (5.3) we get

$$\begin{aligned}0 &= 2S(X,Y)+\left[2\lambda-(r+4)-(n-2)(3n-7)\right]g(X,Y)\\ &\quad +2(\beta+2n-6)\eta(X)\eta(Y).\end{aligned} \tag{5.4}$$

Setting $X=Y=\xi$ in (5.4) we have

$$\lambda=\beta+\frac{1}{2}\left[r+(3n-8)(n-1)\right]. \tag{5.5}$$

Taking an orthonormal frame field and contracting (5.4) over X and Y we obtain

$$\beta=\lambda n-\frac{r}{2}(n-2)-\frac{1}{2}(n-1)(3n^2-10n+12). \tag{5.6}$$

Comparing the value of β from (5.5) and (5.6) we get

$$\lambda=\frac{r}{2}\left[\frac{n-2}{n-1}\right]+\frac{1}{2}(3n^2-10n+12). \tag{5.7}$$

Putting the value of λ from (5.7) in (5.5) we get

$$\beta=-\frac{1}{2(n-1)}\left[r-(n-1)(n+4)\right].$$

□

Corollary 5.2. *If an n-dimensional anti-invariant submanifold M of an LP-Kenmotsu manifold contains η-Einstein soliton (g,ξ,λ,β) with respect to ∇^* then M is η-Einstein manifold*

Proof. From equation (5.4) we have

$$S(X,Y) = -\left[\frac{2\lambda-(r+4)-(n-2)(3n-7)}{2}\right]g(X,Y) -(\beta+2n-6)\eta(X)\eta(Y),$$

which shows that M is η-Einstein manifold. □

Theorem 5.3. *Let M be an anti-invariant submanifold of an LP-Kenmotsu manifold admitting η-Einstein soliton (g,ξ,λ,β) with respect to ∇^*. If the structure vector field ξ of M be parallel i.e., $\nabla_X\xi=0$, then M is an η-Einstein manifold.*

Proof. If ξ is parallel, then from (3.1) we have

$$\nabla^*_X\xi = -X-\eta(X)\xi. \tag{5.8}$$

After expanding the Lie derivative and setting $V=\xi$ in (5.1) we get

$$0 = g(\nabla^*_X\xi,Y)+g(X,\nabla^*_Y\xi)+2S^*(X,Y) +(2\lambda-r^*)g(X,Y)+2\beta\eta(X)\eta(Y). \tag{5.9}$$

Using (3.7), (3.14) and (5.8) in (5.9) we get

$$S(X,Y) = -\frac{1}{2}\left[2\lambda-r+(3n-7)(n-2)\right]g(X,Y)-(\beta+2n-5)\eta(X)\eta(Y),$$

which shows that M is η-Einstein. □

Theorem 5.4. *If M be an anti-invariant submanifold of an LP-Kenmotsu manifold admitting η-Einstein soliton (g,V,λ,β) with respect to ∇^* such that $V\in D$, then scalar curvature of M is given by*

$$r = 2(\lambda-\beta)-(n-1)(3n-8),$$

where D is a distribution on M defined by $D=\ker\eta$.

Proof. Here $V\in D$ and hence

$$\eta(V)=0. \tag{5.10}$$

Taking covariant derivative of (5.10) with respect to ξ and using $(\nabla_\xi\eta)V=0$, we get

$$\eta(\nabla_\xi V)=0. \tag{5.11}$$

In view of (3.1) and (5.11) we have

$$\eta\left(\nabla^*_\xi V\right)=0. \tag{5.12}$$

After expanding the Lie derivative of (5.1) we get

$$0 = g(\nabla^*_X V,Y)+g(X,\nabla^*_Y V)+2S^*(X,Y) +(2\lambda-r^*)g(X,Y)+2\beta\eta(X)\eta(Y). \tag{5.13}$$

Setting $X=Y=\xi$ in (5.13) and using (3.11), (5.12), we obtain

$$0 = 2\lambda-r-(n-1)(3n-8)-2\beta.$$

This gives the theorem. □

6. η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi .)_{R^*} .S^* = 0$

Theorem 6.1. *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional anti-invariant submanifold of an LP-Kenmotsu manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to ∇^*. If M satisfies $(\xi .)_{R^*} .S^* = 0$, then the soliton constants are given by*

$$\beta = 2, \lambda = \frac{1}{2}\left[r + (3n-8)(n-1) + 4\right].$$

Proof. If M contains an η-Einstein soliton (g, ξ, λ, β) with respect to ∇^*, then (5.2) gives

$$S^*(X,Y) = \left[2 - \lambda + \frac{r^*}{2}\right] g(X,Y) - (\beta - 2)\eta(X)\eta(Y). \tag{6.1}$$

The condition that must be satisfied by S^* is

$$S^*(R^*(\xi, X)Y, Z) + S^*(Y, R^*(\xi, X)Z) = 0, \tag{6.2}$$

for all $X, Y, Z \in \chi(M)$.

Using (3.9) and replacing the expression of S^* from (6.1) in (6.2) we get

$$\begin{aligned} 0 &= (\beta - 2)\left[g(X,Y)\eta(Z) + \eta(Y)\eta(Y)\eta(Z)\right] \\ &\quad +(\beta - 2)\left[g(X,Z)\eta(Y) + \eta(Y)\eta(Y)\eta(Z)\right]. \end{aligned} \tag{6.3}$$

For $Z = \xi$, we have

$$(\beta - 2)g(\phi X, \phi Y) = 0,$$

for all $X, Y \in \chi(M)$, which gives

$$\beta = 2.$$

From (5.5) and (6.3) it follows that

$$\beta = 2, \lambda = \frac{1}{2}\left[r + (3n-8)(n-1) + 4\right].$$

□

Corollary 6.2. *The η-Einstein soliton (g, ξ, λ, β) on an n-dimensional anti-invariant submanifold M of an LP-Kenmotsu manifold satisfying $(\xi .)_{R^*} .S^* = 0$ is shrinking, steady or expanding according as*

$$\begin{aligned} r &< -\left[(3n-8)(n-1) + 4\right], \\ r &= -\left[(3n-8)(n-1) + 4\right], \\ r &> -\left[(3n-8)(n-1) + 4\right]. \end{aligned}$$

Corollary 6.3. *There is no Einstein soliton on M satisfying $(\xi .)_{R^*} .S^* = 0$ with potential vector field ξ.*

7. η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi .)_{\mathcal{W}^*} .S^* = 0$

Theorem 7.1. *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional anti-invariant submanifold of an LP-Kenmotsu manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to ∇^*. If M satisfies $(\xi .)_{\mathcal{W}^*} .S^* = 0$, then the scalar curvature of M is given by*

$$r = -2(n-1)(n-2),$$

provided $\beta \neq 2$.

Proof. The condition that must be satisfied by S^* is

$$0 = S^*(\mathcal{W}^*(\xi, X)Y, Z) + S^*(Y, \mathcal{W}^*(\xi, X)Z), \tag{7.1}$$

for all $X,\, Y,\, Z \in \chi(M)$.

Replacing the expression of S^* from (6.1) in (7.1) we obtain

$$\begin{aligned} 0 &= (\beta-2)\left[1-\frac{r^*}{n(n-1)}\right][g(X,Y)\eta(Z)+\eta(Y)\eta(Y)\eta(Z)] \\ &\quad +(\beta-2)\left[1-\frac{r^*}{n(n-1)}\right][g(X,Z)\eta(Y)+\eta(Y)\eta(Y)\eta(Z)]. \end{aligned} \tag{7.2}$$

Setting $Z = \xi$ in (7.2) we get

$$0 = (\beta-2)\left[1-\frac{r^*}{n(n-1)}\right] g(\phi X, \phi Y). \tag{7.3}$$

Using (3.14) in (7.3) we get

$$r = -2(n-1)(n-2),$$

if

$$\beta \neq 2,$$

which gives the theorem. □

8. η-Einstein soliton on anti-invariant submanifold of LP-Kenmotsu manifold satisfying $(\xi.)_{S^*}.\mathcal{W}^* = 0$

Theorem 8.1. *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional anti-invariant submanifold of an LP-Kenmotsu manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to ∇^*. If M satisfies $(\xi.)_{S^*}.\mathcal{W}^* = 0$, then the soliton constants are given by*

$$\begin{aligned} \lambda &= \frac{r+(n-1)(3n-4)+4}{2} + \frac{2(n-1)[r+(n-1)(3n-4)]}{r+(n-1)(n-4)}, \\ \beta &= 2n + \frac{2(n-1)[r+(n-1)(3n-4)]}{r+(n-1)(n-4)}. \end{aligned}$$

Proof. The condition that must be satisfied by S^* is

$$\begin{aligned} 0 &= S^*(X, \mathcal{W}^*(Y,Z)V)\xi - S^*(\xi, \mathcal{W}^*(Y,Z)V)X \\ &\quad + S(X,Y)\mathcal{W}^*(\xi,Z)V - S^*(\xi,Y)\mathcal{W}^*(X,Z)V \\ &\quad + S^*(X,Z)\mathcal{W}^*(Y,\xi)V - S^*(\xi,Z)\mathcal{W}^*(Y,X)V \\ &\quad + S^*(X,V)\mathcal{W}^*(Y,Z)\xi - S^*(\xi,V)\mathcal{W}^*(Y,Z)X, \end{aligned} \tag{8.1}$$

for all $X,\, Y,\, Z,\, V \in \chi(M)$. Taking inner product with ξ the relation (8.1) becomes

$$\begin{aligned} 0 &= -S^*(X, \mathcal{W}^*(Y,Z)V) - S^*(\xi, \mathcal{W}^*(Y,Z)V)\eta(X) \\ &\quad + S^*(X,Y)\eta(\mathcal{W}^*(\xi,Z)V) - S^*(\xi,Y)\eta(\mathcal{W}^*(X,Z)V) \\ &\quad + S^*(X,Z)\eta(\mathcal{W}^*(Y,\xi)V) - S^*(\xi,Z)\eta(\mathcal{W}^*(Y,X)V) \\ &\quad + S^*(X,V)\eta(\mathcal{W}^*(Y,Z)\xi) - S^*(\xi,V)\eta(\mathcal{W}^*(Y,Z)X). \end{aligned} \tag{8.2}$$

Setting $V=\xi$ and using (3.16), (3.17), (3.18), (3.19) we get

$$\begin{aligned} 0 &= S^*(X,\mathcal{W}^*(Y,Z)\xi)+S^*(\xi,\mathcal{W}^*(Y,Z)\xi)\eta(X) \\ &\quad +S^*(\xi,\xi)\eta(\mathcal{W}^*(Y,Z)X). \end{aligned} \tag{8.3}$$

Replacing the expression of S^* from (6.1) in (8.3) we obtain

$$\begin{aligned} 0 &= \left[2-\lambda+\frac{r^*}{2}\right]\left[2-\frac{r^*}{n(n-1)}\right][g(X,Y)\eta(Z)-g(X,Z)\eta(Y)] \\ &\quad +\frac{2r^*}{n}[g(X,Y)\eta(Z)-g(X,Z)\eta(Y)]. \end{aligned} \tag{8.4}$$

Setting $Z=\xi$ in (8.4) we get

$$0=\left[2-\lambda+\frac{r^*}{2}\right]\left[2-\frac{r^*}{n(n-1)}\right]g(\phi X,\phi Y)+\frac{2r^*}{n}g(\phi X,\phi Y), \tag{8.5}$$

Using (3.14) in (8.5) we obtain

$$\lambda=\frac{r+(n-1)(3n-4)+4}{2}+\frac{2(n-1)[r+(n-1)(3n-4)]}{r+(n-1)(n-4)}.$$

Putting the value of λ in (5.5) we get

$$\beta=2n+\frac{2(n-1)[r+(n-1)(3n-4)]}{r+(n-1)(n-4)}.$$

This gives the theorem. □

9. Example of anti-invariant submanifold of 5-dimensional LP-Kenmotsu manifold admitting η-Einstein soliton with respect to Zamkovoy connection

We consider a 5-dimensional manifold

$$M=\left\{(x,y,z,u,v)\in R^5\right\},$$

where (x,y,z,u,v) are the standard co-ordinates in R^5.
We choose the linearly independent vector fields

$$E_1=x\frac{\partial}{\partial x},E_2=x\frac{\partial}{\partial y},E_3=x\frac{\partial}{\partial z},E_4=x\frac{\partial}{\partial u},E_5=x\frac{\partial}{\partial v}.$$

Let g be the Riemannian metric defined by $g(E_i,E_j)=0,\ if\ i\neq j\ for\ i,j=1,2,3,4,5$, and $g(E_1,E_1)=-1$, $g(E_2,E_2)=1$, $g(E_3,E_3)=1$, $g(E_4,E_4)=1$, $g(E_5,E_5)=1$.

Let η be the 1-form defined by $\eta(X)=g(X,E_1)$, for any $X\in\chi\left(M^5\right)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi E_1=0,\phi E_2=-E_3,\phi E_3=-E_2,\phi E_4=-E_5,\phi E_5=-E_4. \tag{9.1}$$

Let $X,\ Y,\ Z\in\chi\left(M^5\right)$ be given by

$$\begin{aligned} X &= x_1E_1+x_2E_2+x_3E_3+x_4E_4+x_5E_5, \\ Y &= y_1E_1+y_2E_2+y_3E_3+y_4E_4+y_5E_5, \\ Z &= z_1E_1+z_2E_2+z_3E_3+z_4E_4+z_5E_5. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X,Y) &= x_1y_1+x_2y_2+x_3y_3+x_4y_4+x_5y_5,\\ \eta(X) &= -x_1,\\ g(\phi X,\phi Y) &= x_2y_2+x_3y_3+x_4y_4+x_5y_5. \end{aligned}$$

Using the linearity of g and ϕ, $\eta(E_1)=-1$, $\phi^2 X = X+\eta(X)E_1$ and $g(\phi X,\phi Y) = g(X,Y)+\eta(X)\eta(Y)$ for all $X,Y\in\chi(M)$.

We have

$$\begin{aligned} [E_1,E_2] &= E_2, [E_1,E_3]=E_3, [E_1,E_4]=E_4, [E_1,E_5]=E_5,\\ [E_2,E_1] &= -E_2, [E_3,E_1]=-E_3, [E_4,E_1]=-E_4, [E_5,E_1]=-E_5,\\ [E_i,E_j] &= 0 \text{ for all others } i \text{ and } j. \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{aligned} \nabla_{E_1}E_1 &= 0, \nabla_{E_1}E_2=0, \nabla_{E_1}E_3=0, \nabla_{E_1}E_4=0, \nabla_{E_1}E_5=0,\\ \nabla_{E_2}E_1 &= -E_2, \nabla_{E_2}E_2=-E_1, \nabla_{E_2}E_3=0,\ \nabla_{E_2}E_4=0, \nabla_{E_2}E_5=0,\\ \nabla_{E_3}E_1 &= -E_3,\ \nabla_{E_3}E_2=0\ , \nabla_{E_3}E_3=-E_1\ , \nabla_{E_3}E_4=0\ , \nabla_{E_3}E_5=0,\\ \nabla_{E_4}E_1 &= -E_4, \nabla_{E_4}E_2=0, \nabla_{E_4}E_3=0, \nabla_{E_4}E_4=-E_1, \nabla_{E_4}E_5=0,\\ \nabla_{E_5}E_1 &= -E_5, \nabla_{E_5}E_2=0, \nabla_{E_5}E_3=0, \nabla_{E_5}E_4=0,\ \nabla_{E_5}E_5=-E_1. \end{aligned}$$

From the above results we see that the structure (ϕ,ξ,η,g) satisfies

$$(\nabla_X\phi)Y = -g(\phi X,Y)\xi - \eta(Y)\phi X,$$

for all $X,\ Y\in\chi\left(M^5\right)$, where $\eta(\xi)=\eta(E_1)=-1$. Hence $M^5(\phi,\xi,\eta,g)$ is a LP-Kenmotsu manifold.

Let $M^*(\phi,\xi,\eta,g)$ be an anti-invariant submanifold of $M^5(\phi,\xi,\eta,g)$. Then the non-zero components of Riemannian curvature of M^* with respect to Levi-Civita connection ∇ are given by

$R(E_1,E_2)E_1=E_2, R(E_1,E_2)E_2=-E_1, R(E_1,E_3)E_1=E_3,$
$R(E_1,E_3)E_3=-E_1, R(E_1,E_4)E_1=E_4, R(E_1,E_4)E_4=-E_1,$
$R(E_1,E_5)E_1=E_5, R(E_1,E_5)E_5=-E_1, R(E_2,E_1)E_2=E_1,$
$R(E_2,E_1)E_1=-E_2, R(E_2,E_3)E_2=E_3, R(E_2,E_3)E_3=-E_2,$
$R(E_2,E_4)E_2=E_4, R(E_2,E_4)E_4=-E_2, R(E_2,E_5)E_2=E_5,$
$R(E_2,E_5)E_5=-E_2, R(E_3,E_1)E_3=E_1, R(E_3,E_1)E_1=-E_3,$
$R(E_3,E_2)E_3=E_2, R(E_3,E_2)E_2=-E_3, R(E_3,E_4)E_3=E_4,$
$R(E_3,E_4)E_4=-E_3, R(E_3,E_5)E_3=E_5, R(E_3,E_5)E_5=-E_3,$
$R(E_4,E_1)E_4=E_1, R(E_4,E_1)E_1=-E_4, R(E_4,E_2)E_4=E_2,$
$R(E_4,E_2)E_2=-E_4, R(E_4,E_3)E_4=E_3, R(E_4,E_3)E_3=-E_4,$
$R(E_4,E_5)E_4=E_5, R(E_4,E_5)E_5=-E_4, R(E_5,E_1)E_5=E_1,$
$R(E_5,E_1)E_1=-E_5, R(E_5,E_2)E_5=E_2, R(E_5,E_2)E_2=-E_5,$
$R(E_5,E_3)E_5=E_3, R(E_5,E_3)E_3=-E_5, R(E_5,E_4)E_5=E_4.$

By the help of (3.1), we obtain

$$\begin{aligned}
\nabla^*_{E_1} E_1 &= 0,\ \nabla^*_{E_1} E_2 = E_3, \nabla^*_{E_1} E_3 = E_2, \nabla^*_{E_1} E_4 = E_5, \nabla^*_{E_1} E_5 = E_4,\\
\nabla^*_{E_2} E_1 &= -2E_2, \nabla^*_{E_2} E_2 = -2E_1, \nabla^*_{E_2} E_3 = 0,\ \nabla^*_{E_2} E_4 = 0, \nabla^*_{E_2} E_5 = 0,\\
\nabla^*_{E_3} E_1 &= -2E_3, \nabla^*_{E_3} E_2 = 0, \nabla^*_{E_3} E_3 = -2E_1, \nabla^*_{E_3} E_4 = 0, \nabla^*_{E_3} E_5 = 0,\\
\nabla^*_{E_4} E_1 &= -2E_4,\ \nabla^*_{E_4} E_2 = 0, \nabla^*_{E_4} E_3 = 0, \nabla^*_{E_4} E_4 = -2E_1, \nabla^*_{E_4} E_5 = 0,\\
\nabla^*_{E_5} E_1 &= -2E_5, \nabla^*_{E_5} E_2 = 0, \nabla^*_{E_5} E_3 = 0, \nabla^*_{E_5} E_4 = 0,\ \nabla^*_{E_5} E_5 = -2E_1.
\end{aligned}$$

Some of the non-zero components of Riemannian curvature tensor of M^* with respect to Zamkovoy connection are given by

$$\begin{aligned}
R^*(E_1, E_3) E_1 &= 2(E_2 - E_3), R^*(E_2, E_3) E_2 = -4E_3,\\
R^*(E_4, E_3) E_4 &= -4E_3, R^*(E_5, E_3) E_5 = -4E_3,\\
R^*(E_3, E_1) E_1 &= 2(E_2 - E_3), R^*(E_3, E_2) E_2 = 4E_3,\\
R^*(E_3, E_4) E_4 &= 4E_3, R^*(E_3, E_5) E_5 = 4E_4.
\end{aligned}$$

Using the above curvature tensors the Ricci curvature tensors of M^* with respect to ∇ and ∇^* are

$$\begin{aligned}
S(E_1, E_1) &= -4, S(E_2, E_2) = S(E_3, E_3) = -2,\\
S(E_4, E_4) &= S(E_5, E_5) = -2,\\
S^*(E_1, E_1) &= -8, S^*(E_2, E_2) = S^*(E_4, E_4) = 14,\\
S^*(E_5, E_5) &= S^*(E_3, E_3) = 14.
\end{aligned}$$

Therefore, the scalar curvature tensor of M^* with respect to Levi-Civita connection is $r = -12$ and scalar curvature tensor with respect to Zamkovoy connection is $r^* = 32$.

Setting $V = X = Y = E_1$ in (5.1) we have

$$\begin{aligned}
0 &= \left(L^*_{E_1} g\right)(E_1, E_1) + 2S^*(E_1, E_1) + (2\lambda - r^*) g(E_1, E_1) + 2\beta\eta(E_1)\eta(E_1),\\
&= g\left(\nabla^*_{E_1} E_1, E_1\right) + g\left(E_1, \nabla^*_{E_1} E_1\right)\\
&\quad +2S^*(E_1, E_1) + (2\lambda - r^*) g(E_1, E_1) + 2\beta\eta(E_1)\eta(E_1),\\
&= 0 + 0 + 2(-8) + (2\lambda - 32)(-1) + 2\beta,\\
&= \beta - \lambda + 8,
\end{aligned}$$

which gives

$$\begin{aligned}
\lambda &= \beta + 8,\\
&= \lambda + \frac{1}{2}\left[-12 + 28\right],\\
&= \lambda + \frac{1}{2}\left[-12 + (3 \times 5 - 8)(5 - 1)\right],\\
&= \lambda + \frac{1}{2}\left[r + (3n - 8)(n - 1)\right],
\end{aligned}$$

which shows that λ and β satisfies relation (5.5).

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