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On projectively related Finsler gradient Ricci solitons

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Abstract. In this paper, we study pointwise projectively related Finsler gradient Ricci solitons. We obtain an equation that characterizes the relationship between two pointwise projectively related Finsler gradient Ricci solitons. Further, if two Finsler gradient Ricci solitons $(M, \tilde{F}, dV_{\tilde{F}})$ and (M, F, dV_F) satisfy $\tilde{F}_{;k} = \mu \frac{\partial[\tilde{F}^2]}{\partial y^k}$ and some extra conditions, where ";" denotes the horizontal covariant derivative with respect to F, we characterize their relationships along the geodesics. In particular, if two Finsler gradient Ricci solitons are both complete, then (M, F, dV_F) is expanding or shrinking and $(M, \tilde{F}, dV_{\tilde{F}})$ is shrinking.

Keywords: Finsler metric; Finsler gradient Ricci soliton; projectively related Finsler metrics; S-curvature; weighted Ricci curvature.

1. **Introduction**

The study on gradient Ricci solitons is one of the important topics in Riemannian geometry. A complete Riemannian metric g on a smooth manifold M is called a gradient Ricci soliton if there is a function f so that

$$\mathbf{Ric} + \mathrm{Hess}(f) = \kappa \cdot g,$$

where $\kappa \in \mathbb{R}$. The gradient Ricci solitons are called shrinking if $\kappa > 0$, steady if $\kappa = 0$ and expanding if $\kappa < 0$. The Ricci solitons were first introduced by R. Hamiltons in ([7]) as the self-similar solutions of Ricci flow, which play an

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important role in the proof of Poincaré conjecture by Perelmann. In the past decades, a significant number of results have been obtained in study of gradient Ricci solitons in Riemannian geometry. In particular, if f is a constant, the gradient Ricci soliton becomes an Einstein metric.

It is natural to study and develop the theory of gradient Ricci solitons in Finsler geometry. However, the study of gradient Ricci solitons in Finsler geometry becomes more complicated because of some obstructions. In this paper, we always use (M, F, dV_F) to denote a Finsler manifold (M, F) equipped with a volume measure $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ which we call a Finsler metric measure manifold (or Finsler measure space briefly). Let Y be a C^∞ geodesic field on an open subset $U \subset M$ and $\hat{g} := g_Y$ denote the Riemannian metric induced by Y. Write

$$dV_F = e^{-f}\mathrm{Vol}_{\hat{g}}, \quad \mathrm{Vol}_{\hat{g}} = \sqrt{\det\left(g_{ij}(x, Y_x)\right)}dx^1 \cdots dx^n.$$

It is easy to see that f is given by

$$f(x) = \ln \frac{\sqrt{\det(g_{ij}(x, Y_x))}}{\sigma_F(x)} = \tau(x, Y_x),$$

which is just the distortion of F along Y_x at $x \in M$ ([6, 11]). Let $y := Y_x \in T_xM$ (that is, Y is the geodesic extension of $y \in T_xM$). Then, by the definitions of the S-curvature, we have

$$\begin{aligned} \mathbf{S}(x, y) &= Y[\tau(x, Y)]|_x = df(y), \\ \dot{\mathbf{S}}(x, y) &= Y[\mathbf{S}(x, Y)]|_x = \mathrm{Hess}f(y), \end{aligned}$$

where $\dot{\mathbf{S}}(x, y) := \mathbf{S}_{;m}(x, y)y^m$ and ";" denotes the horizontal covariant derivative with respect to the Chern connection ([10, 11]). Further, the weighted Ricci curvatures are defined as follows ([5, 8])

$$\mathbf{Ric}_N(y) = \mathbf{Ric}(y) + \dot{\mathbf{S}}(x, y) - \frac{\mathbf{S}^2(x, y)}{N - n}, \tag{1.1}$$

$$\mathbf{Ric}_\infty(y) = \mathbf{Ric}(y) + \dot{\mathbf{S}}(x, y) = \mathbf{Ric}(y) + \mathrm{Hess}f(y). \tag{1.2}$$

An n-dimensional Finsler measure space (M, F, dV_F) with volume form dV_F is called a Finsler gradient Ricci soliton if there is a constant $\lambda \in \mathbb{R}$ such that the weighted Ricci curvature $\mathbf{Ric}_\infty$ of (M, F, dV_F) satisfies the following equation ([3, 13, 14])

$$\mathbf{Ric}_\infty = (n - 1)\lambda F^2. \tag{1.3}$$

The Finsler gradient Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Note that a Finsler gradient Ricci soliton is just an Einstein-Finsler metric when $\dot{\mathbf{S}}(x, y) = 0$ for any $y \in T_xM$ and $x \in M$, particularly, when F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \in \mathbb{R}$. For the research on Finsler gradient Ricci soliton, please refer to [1, 2, 13, 14].

In [12], Shen studies pointwise projectively related Einstein-Finsler metrics. He shows that pointwise projectively related Einstein-Finsler metrics satisfy a simple equation along geodesics. In particular, he shows that if two pointwise projectively related Einstein-Finsler metrics are complete with negative Einstein constants, then one is a multiple of another.

In this paper, we mainly study pointwise projectively related Finsler gradient Ricci solitons. We will give an equation that characterizes the relationship between two pointwise projectively related Finsler gradient Ricci solitons. In particular, we have the following theorem.

Theorem 1.1. *Let F and $\tilde{F}$ be Finsler gradient Ricci solitons on an n-dimensional manifold M with*

$$\mathbf{Ric}_\infty = (n-1)\lambda F^2, \quad \widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2,$$

where $\lambda,\ \tilde{\lambda} \in \mathbb{R}$. Suppose that $\tilde{F}_{;k} = \mu \frac{\partial[\tilde{F}^2]}{\partial y^k}$, where ";" denotes the horizontal covariant derivative with respect to F and $\mu \neq 0$ is a constant. Let $m := \tilde{\lambda} - \mu^2$, $b := \frac{n+1}{n-1}$ and $\theta := \sqrt{\lambda m + b^2 c^2 \mu^2}$. Assume that $\tilde{\mathbf{S}} = \mathbf{S}$ and F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \neq 0$. Then $\tilde{F}$ is pointwise projectively related to F and for any unit speed geodesic $\sigma(t)$ of F, the following equalities hold.

(i) *If $\theta \neq 0$ and $m \neq 0$,*

$$\tilde{F}(\dot{\sigma}(t)) = \frac{\theta \tanh\left[\frac{\theta}{\mu} t + \tanh^{-1}\left(\frac{ma^2 + bc\mu}{\theta}\right)\right] - bc\mu}{m}, \tag{1.4}$$

where $a \geq 0$ is a constant.

(ii) *If $\theta \neq 0$ and $m = 0$,*

$$\tilde{F}(\dot{\sigma}(t)) = \left(a_0^2 - \frac{\lambda}{2bc\mu}\right) e^{-2bct} + \frac{\lambda}{2bc\mu}, \tag{1.5}$$

where $a_0 \geq 0$ is a constant.

(iii) *If $\theta = 0$ and $m \neq 0$,*

$$\tilde{F}(\dot{\sigma}(t)) = \frac{\mu}{m}\left(\frac{1}{t + t_0} - cb\right), \tag{1.6}$$

where $t_0 \in \{0, \frac{\mu}{m{a_1}^2 + bc\mu}\}$ and $a_1 \geq 0$ is a constant.

In particular, if F and $\tilde{F}$ are both complete, then F is expanding and $\tilde{F}$ is shrinking (resp. F is shrinking or expanding and $\tilde{F}$ is shrinking). In this case, $\lambda < 0$, $\mu^2 < \tilde{\lambda} < \mu^2\left(1 - \frac{b^2c^2}{\lambda}\right)$ (resp. $0 < \lambda < 2bc\mu a_0^2$ or $2bc\mu a_0^2 < \lambda < 0$ $(a_0 \neq 0)$, $\tilde{\lambda} = \mu^2$).

It should be pointed out that, when F is of constant S-curvature, $\mathbf{S} = (n+1)cF$, the Finsler gradient Ricci soliton F with $\mathbf{Ric}_\infty = (n-1)\lambda F^2$ is actually an Einstein metric with $\mathbf{Ric} = (n-1)\lambda F^2$. Further, if $c = 0$, $\tilde{F}$ and

F are both Einstein-Finsler metrics under the condition that $\tilde{\mathbf{S}} = \mathbf{S}$. Hence, in the following discussions, we always assume that $c \neq 0$.

2. Preliminaries

Let M be an n-dimensional smooth manifold. A Finsler metric on manifold M is a function $F : TM \to [0, \infty)$ satisfying the following properties: (1) F is C^∞ on $TM \backslash \{0\}$; (2) $F(x, \lambda y) = \lambda F(x, y)$ for any $(x, y) \in TM$ and all $\lambda > 0$; (3) F is strongly convex, that is, the matrix $(g_{ij}(x, y)) = \left(\frac{1}{2}(F^2)_{y^i y^j}\right)$ is positive definite for any nonzero $y \in T_x M$. The pair (M, F) is called a Finsler manifold and $g := g_{ij}(x, y)dx^i \otimes dx^j$ is called the fundamental tensor of F. For a non-vanishing vector field V on M, one introduces the weighted Riemannian metric g_V on M given by

$$g_V(y, w) = g_{ij}(x, V_x)y^i w^j$$

for $y, w \in T_x M$. In particular, $g_V(V, V) = F^2(V)$.

Let (M, F) be a Finsler manifold of dimension n. Let $TM_0 := TM \backslash \{0\}$ and $\pi : TM_0 \to M$ be the natural projective map. The pull-back $\pi^* TM$ admits a unique linear connection, which is called the Chern connection. The Chern connection D is determined by the following equations

$$\begin{aligned} &D^V_X Y - D^V_Y X = [X, Y], \\ &Z g_V(X, Y) = g_V(D^V_Z X, Y) + g_V(X, D^V_Z Y) + 2C_V(D^V_Z V, X, Y) \end{aligned}$$

for $V \in TM \backslash \{0\}$ and $X, Y, Z \in TM$, where

$$C_V(X, Y, Z) = C_{ijk}(x, V)X^i Y^j Z^k = \frac{1}{4}\frac{\partial^3 F^2(x, V)}{\partial V^i \partial V^j \partial V^k} X^i Y^j Z^k$$

is the Cartan tensor of F and $D^V_X Y$ is the covariant derivative with respect to the reference vector V.

Given a non-vanishing vector field V on M, the Riemannian curvature $\mathbf{R}^V$ is defined by

$$\mathbf{R}^V(X, Y)Z = D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]} Z$$

for any vector fields X, Y, Z on M. For two linearly independent vectors $V, W \in T_x M \backslash \{0\}$, the flag curvature is defined by

$$\mathcal{K}^V(V, W) = \frac{g_V(\mathbf{R}^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2}.$$

Then the Ricci curvature is defined as

$$\mathbf{Ric}(V) := F(x, V)^2 \sum_{i=1}^{n-1} \mathcal{K}^V(V, e_i), \tag{2.1}$$

where $e_1, \cdots e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of $T_x M$ with respect to g_V.

A curve $\sigma = \sigma(t)$ is a geodesic if and only if, in local coordinates, its coordinates $\big(\sigma^i(t)\big)$ satisfy

$$\ddot{\sigma}^i(t) + 2G^i\left(\sigma(t), \dot{\sigma}(t)\right) = 0, \tag{2.2}$$

where

$$G^i(x,y) = \frac{1}{4} g^{il}(x,y) \left\{ \frac{\partial g_{kl}}{\partial x^j}(x,y) + \frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y) \right\} y^j y^k,$$

which are called the geodesic coefficients of F. F is said to be positively complete (resp. negatively complete), if any geodesic on an open interval (a,b) can be extended to a geodesic on (a,∞) (resp. $(-\infty,b)$). F is said to be complete if it is positively and negatively complete. Besides, two Finsler metrics on a manifold are said to be pointwise projectively related if they have the same geodesics as point sets ([12]).

A vector field Y on an open subset $U \subset M$ is called a geodesic field if every integral curve $\sigma(t)$ of Y in U is a geodesic of F. In local coordinates, a geodesic field $Y = Y^i \frac{\partial}{\partial x^i}$ is characterized by $D_Y^Y Y = 0$, that is,

$$Y^j(x) \frac{\partial Y^i}{\partial x^j}(x) + 2G^i(x, Y_x) = 0.$$

For any non-zero vector $y \in T_xM$, there is an open neighborhood U_x and a geodesic field Y on U_x such that $Y_x = y$. Y is called the geodesic extension of y.

We now consider pointwise projectively related Finsler metrics $-$ those having the same geodesics as set points. Given two Finsler metrics F and $\tilde{F}$ on an n-dimensional manifold M, it is easy to verify that

$$\tilde{G}^i = G^i + \frac{\tilde{F}_{;k} y^k}{2\tilde{F}} y^i + \frac{\tilde{F}}{2} \tilde{g}^{il} \left\{ \frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k - \tilde{F}_{;l} \right\}, \tag{2.3}$$

where $\tilde{F}_{;k}$ denotes the covariant derivative of $\tilde{F}$ on (M,F), that is,

$$\tilde{F}_{;k} := \frac{\partial \tilde{F}}{\partial x^k} - \frac{\partial G^l}{\partial y^k} \frac{\partial \tilde{F}}{\partial y^l}. \tag{2.4}$$

By (2.3), we introduce the following important lemma.

Lemma 2.1. ([9]) *Let (M,F) be a Finsler space. A Finsler metric $\tilde{F}$ is pointwise projective to F if and only if*

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k - \tilde{F}_{;l} = 0.$$

In this case,

$$\tilde{G}^i = G^i + P y^i, \tag{2.5}$$

where P is called the projective factor and determined by

$$P = \frac{\tilde{F}_{;k} y^k}{2\tilde{F}}. \tag{2.6}$$

Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. Assume that $\tilde{F}$ is pointwise projective to F. Then we have ([12])

$$\widetilde{\mathbf{Ric}}(y) = \mathbf{Ric}(y) + (n-1)\Xi(y), \tag{2.7}$$

where

$$\Xi(y) := P^2 - P_{;k} y^k.$$

Let (M, F, dV_F) be an n-dimensional Finsler manifold with a smooth volume measure dV_F. Write the volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$. Define

$$\tau(x, y) := \ln \frac{\sqrt{\det\left(g_{ij}(x, y)\right)}}{\sigma_F(x)}. \tag{2.8}$$

We call τ the distortion of F. It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Set

$$\mathbf{S}(x, y) := \frac{d}{dt}\left[\tau(\sigma(t), \dot{\sigma}(t))\right]|_{t=0}. \tag{2.9}$$

$\mathbf{S}$ is called the S-curvature of F. Further, we have ([6, 10])

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^i \frac{\partial}{\partial x^i}(\ln \sigma_F(x)). \tag{2.10}$$

By (2.5) and (2.10), we can get the following important lemma.

Lemma 2.2. ([4]) *Let F and $\tilde{F}$ be Finsler metrics on n-dimensional manifold M. Suppose that $\tilde{F}$ is pointwise projective equivalent to F. Then the projective factor P is given by*

$$P = \frac{1}{n+1}\left(\tilde{\mathbf{S}} - \mathbf{S}\right) - y^i \frac{\partial}{\partial x^i}\left[\ln f\right], \tag{2.11}$$

where $f = f(x)$ is a scalar function on M determined by $dV_{\tilde{F}} = \left(1/f^{n+1}\right) dV_F$, that is, $f(x) = \left(\frac{\sigma_F}{\tilde{\sigma}_{\tilde{F}}}\right)^{1/(n+1)}$.

3. Projectively Related Finsler Gradient Ricci Solitons

In this section, we first introduce the relations of weighted Ricci curvatures of projectively related Finsler gradient Ricci solitons. Then we will give an equation that projectively related Finsler gradient Ricci solitons satisfy.

Assume that $\tilde{F}$ is pointwise projective to F. Let "|" denote the horizontal covariant derivative with respect to $\tilde{F}$. From (2.5), (2.10) and (2.11), we have

$$
\begin{aligned}
\tilde{\mathbf{S}}_{|j}(x,y) &= \frac{\partial \tilde{\mathbf{S}}}{\partial x^j} - \frac{\partial \tilde{G}^i}{\partial y^j}\frac{\partial \tilde{\mathbf{S}}}{\partial y^i} \\
&= \frac{\partial \mathbf{S}}{\partial x^j} + (n+1)\left(\frac{\partial P}{\partial x^j} + \frac{\partial\left((\ln f)_{;0}\right)}{\partial x^j}\right) \\
&\quad - \left(\frac{\partial G^i}{\partial y^j} + P_{y^j}y^i + P\delta^i_j\right)\left\{\frac{\partial \mathbf{S}}{\partial y^i} + (n+1)\left(\frac{\partial P}{\partial y^i} + \frac{\partial}{\partial x^i}[\ln f]\right)\right\} \\
&= \left(\frac{\partial \mathbf{S}}{\partial x^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial \mathbf{S}}{\partial y^i}\right) + (n+1)\left(\frac{\partial P}{\partial x^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial P}{\partial y^i}\right) \\
&\quad + (n+1)\left(\frac{\partial\left((\ln f)_{;0}\right)}{\partial x^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial\left((\ln f)_{;0}\right)}{\partial y^i}\right) \\
&\quad - (P_{y^j}y^i + P\delta^i_j)\left\{\frac{\partial \mathbf{S}}{\partial y^i} + (n+1)\left(\frac{\partial P}{\partial y^i} + \frac{\partial\left((\ln f)_{;0}\right)}{\partial y^i}\right)\right\} \\
&= \mathbf{S}_{;j}(x,y) + (n+1)\left[P_{;j} + (\ln f)_{;0;j}\right] - \mathbf{S}\frac{\partial P}{\partial y^j} - (n+1)P\frac{\partial P}{\partial y^j} \\
&\quad - (n+1)\frac{\partial P}{\partial y^j}(\ln f)_{;0} - P\frac{\partial \mathbf{S}}{\partial y^j} - (n+1)P\frac{\partial P}{\partial y^j} - (n+1)P\frac{\partial((\ln f)_{;0})}{\partial y^j} \\
&= \mathbf{S}_{;j}(x,y) + (n+1)\left[P_{;j} + (\ln f)_{;0;j}\right] \\
&\quad - \frac{\partial}{\partial y^j}\left[P\left(\mathbf{S} + (n+1)\left(P + (\ln f)_{;0}\right)\right)\right],
\end{aligned}
$$

where

$$
(\ln f)_{;0} := \left(\frac{\partial}{\partial x^i}[\ln f]\right)y^i, \quad (\ln f)_{;0;j} := (\ln f)_{;i;j}y^i.
$$

Further, we have

$$
\begin{aligned}
\dot{\tilde{\mathbf{S}}}(x,y) &= \tilde{\mathbf{S}}_{|j}(x,y)y^j \\
&= \dot{\mathbf{S}}(x,y) + (n+1)\left[P_{;0} + (\ln f)_{;0;0}\right] \\
&\quad -2P\left\{\mathbf{S} + (n+1)\left[P + (\ln f)_{;0}\right]\right\},
\end{aligned} \tag{3.1}
$$

where

$$
P_{;0} := P_{;j}y^j, \quad (\ln f)_{;0;0} := (\ln f)_{;i;j}y^iy^j.
$$

By the definition of $\mathbf{Ric}_\infty$, (2.7) and (3.1), we have

$$
\begin{aligned}
\widetilde{\mathbf{Ric}}_\infty &= \widetilde{\mathbf{Ric}} + \dot{\tilde{\mathbf{S}}} \\
&= \mathbf{Ric} + (n+1)\Xi + \dot{\mathbf{S}} + (n+1)\left[P_{;0} + (\ln f)_{;0;0}\right] \\
&\quad -2P\left\{\mathbf{S} + (n+1)[P + (\ln f)_{;0}]\right\} \\
&= \mathbf{Ric} + \dot{\mathbf{S}} + 2P_{;0} - (n+3)P^2 - 2P\mathbf{S} + (n+1)(\ln f)_{;0;0} \\
&\quad -2(n+1)P(\ln f)_{;0} \\
&= \mathbf{Ric}_\infty + 2P_{;0} - (n+3)P^2 - 2P\left[\mathbf{S} + (n+1)(\ln f)_{;0}\right] \\
&\quad +(n+1)(\ln f)_{;0;0}.
\end{aligned} \tag{3.2}
$$

Then we have the following proposition.

Proposition 3.1. *Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. If $\tilde{F}$ is pointwise projective equivalent to F. Then*

$$\widetilde{\mathbf{Ric}}_\infty = \mathbf{Ric}_\infty + 2P_{;0} - (n+3)P^2 - 2P\left[\mathbf{S} + (n+1)(\ln f)_{;0}\right] + (n+1)(\ln f)_{;0;0}. \tag{3.3}$$

Similarly, from (1.1), (2.11) and (3.1), we have the following result.

Proposition 3.2. *Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. If $\tilde{F}$ is pointwise projective equivalent to F. Then*

$$\begin{aligned}\widetilde{\mathbf{Ric}}_N \quad = \quad & \mathbf{Ric}_N + 2P_{;0} - \frac{(n+1)^2}{N-n}\left((\ln f)_{;0}\right)^2 - \left(\frac{N(n+3)-n+1}{N-n}\right)P^2 \\ & -\left(\frac{2(N+1)}{N-n}\right)\mathbf{S}P - \frac{2(n+1)}{N-n}\left[(N+1)P + \mathbf{S}\right](\ln f)_{;0} \\ & +(n+1)(\ln f)_{;0;0}.\end{aligned} \tag{3.4}$$

By Proposition 3.1, we immediately obtain the following

Theorem 3.3. *Let (M, F, dV_F) and $(M, \tilde{F}, dV_{\tilde{F}})$ be two Finsler measure spaces on M. Assume that $\tilde{F}$ is pointwise projective equivalent to F and $\mathbf{Ric}_\infty = (n-1)\lambda F^2$. Then $\widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2$ if and only if*

$$2P_{;0} - (n+3)P^2 - 2P\left[\mathbf{S} + (n+1)(\ln f)_{;0}\right] + (n+1)(\ln f)_{;0;0} = (n-1)\left(\tilde{\lambda}\tilde{F}^2 - \lambda F^2\right). \tag{3.5}$$

If the projective transformation preserves S-curvature, that is, $\tilde{\mathbf{S}} = \mathbf{S}$, by (2.11), we know that $(\ln f)_{;0} = -P$. Then (3.3) becomes

$$\widetilde{\mathbf{Ric}}_\infty = \mathbf{Ric}_\infty - (n-1)P_{;0} + (n-1)P^2 - 2P\mathbf{S}. \tag{3.6}$$

Further, we have the following

Proposition 3.4. *Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. Suppose that $\tilde{F}$ is pointwise projectively related to F and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant c. If $\mathbf{Ric}_\infty = (n-1)\lambda F^2$, then $\widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2$ if and only if*

$$\tilde{\lambda}\tilde{F}^2 - \lambda F^2 = P^2 - P_{;k}y^k - \frac{2(n+1)c}{n-1}FP. \tag{3.7}$$

By (2.6), we can rewrite (3.7) as follows.

$$\tilde{\lambda}\tilde{F}^2 - \lambda F^2 = \frac{3}{4}\left(\frac{\tilde{F}_{;k}y^k}{\tilde{F}}\right)^2 - \frac{\tilde{F}_{;k;l}y^k y^l}{2\tilde{F}} - \frac{(n+1)cF}{n-1}\frac{\tilde{F}_{;k}y^k}{\tilde{F}}. \tag{3.8}$$

The equation (3.8) is a starting point for our discussions in next section. We should point out that the projective transformations considered in Proposition

3.4 are actually those transformations which change an Einstein-Finsler metric as a Finsler gradient Ricci soliton.

4. An important class of projective equivalences between Finsler gradient Ricci solitons

In this section, we will discuss in detail a special kind of projective equivalence between two Finsler gradient Ricci solitons. Firstly, we have the following fundamental lemma.

Lemma 4.1. ([12]) *Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. Suppose that*

$$\tilde{F}_{;k} = \mu \frac{\partial[\tilde{F}^2]}{\partial y^k}, \tag{4.1}$$

where μ is a constant. Then $\tilde{F}$ is pointwise projective to F. In this case, $P = \mu\widetilde{F}$.

Under the condition (4.1), if $\tilde{\mathbf{S}} = \mathbf{S}$, by (3.6), we get

$$\widetilde{\mathbf{Ric}}_\infty = \mathbf{Ric}_\infty - (n-1)\mu\left[\tilde{F}_{;0} - \mu\tilde{F}^2 + \frac{2\mathbf{S}}{n-1}\tilde{F}\right].$$

Further, we have the following

Proposition 4.2. *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that (4.1) holds for some constant $\mu \neq 0$ and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant c. If $\mathbf{Ric}_\infty = (n-1)\lambda F^2$, then $\widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2$ if and only if*

$$\tilde{\lambda}\tilde{F}^2 - \lambda F^2 = \mu^2\tilde{F}^2 - \mu\tilde{F}_{;k}y^k - \frac{2(n+1)c\mu}{n-1}F\tilde{F}. \tag{4.2}$$

Remark 4.3. *When $\mu = 0$, $\tilde{F}_{;k} = 0$. In this case, F is affinely equivalent to $\tilde{F}$, that is, F and $\tilde{F}$ have the same geodesics as parametrized curves ([6]). Hence, we will not consider the case that $\mu = 0$ in this paper.*

Let $\sigma(t)$ be an arbitrary unit speed geodesic in (M, F) and

$$\tilde{F}(t) := \tilde{F}(\sigma(t), \dot{\sigma}(t)).$$

Observe that $\tilde{F}'(t) = \tilde{F}_{;k}(\sigma(t), \dot{\sigma}(t))\dot{\sigma}^k(t)$. Let

$$g(t) := \sqrt{\tilde{F}(t)}.$$

(4.2) simplifies to

$$\left(\tilde{\lambda} - \mu^2\right)g^4(t) + \frac{2(n+1)c\mu}{n-1}g^2(t) + 2\mu g(t)g'(t) = \lambda. \tag{4.3}$$

The equation (4.3) is solvable.

For simplicity, let

$$m := \tilde{\lambda} - \mu^2, \qquad b := \frac{n+1}{n-1}, \tag{4.4}$$

$$\theta := \sqrt{\lambda\left(\tilde{\lambda} - \mu^2\right) + c^2\mu^2\left(\frac{n+1}{n-1}\right)^2} = \sqrt{\lambda m + b^2c^2\mu^2} \geq 0. \tag{4.5}$$

When $\theta > 0$ and $m \neq 0$, the solution of (4.3) with $g(0) = a \geq 0$ is determined by

$$g(t) = \sqrt{\frac{\theta \tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - bc\mu}{m}}. \tag{4.6}$$

The following fact is notable

$$-1 < \tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] < 1.$$

We can find that if $\theta = 0$ and $m = 0$, then $\mu = 0$ or $c = 0$. In this case, the discussion is trivial, so we omit it. The discussions in Sections 4.1-4.3 are based on the condition that $\theta > 0$, and the situation when $\theta = 0$ will be discussed in Section 4.4.

4.1. $\theta > 0$ and $m < 0$

In this subsection, we study the solution (4.6) when $\theta > 0$ and $m < 0$. In this case, $\tilde{\lambda} < \mu^2$.

Case 1: $\lambda < 0$. In this case, the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - \frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}}\right\}}$$

and $-\infty < \tilde{\lambda} < \mu^2$. As the result, we have

$$\frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}} \in (-1, 1).$$

(i) If $\mu > 0$, then $g(t)$ is defined on $I = (-\infty, \tau)$ and

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{\tau} g^2(t)dt < \infty.$$

(ii) If $\mu < 0$, then $g(t)$ is defined on $I = (-\delta, +\infty)$ and

$$\int_{-\delta}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

Case 2: $\lambda > 0$. In this case, the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2 + bc\mu}{\theta}\right)\right] - \frac{bc\mu}{\sqrt{b^2c^2\mu^2 + \lambda m}}\right\}}$$

and $\mu^2(1 - \frac{b^2c^2}{\lambda}) < \tilde{\lambda} < \mu^2$ and $c\mu > 0$. As the result, we have

$$\frac{bc\mu}{\sqrt{b^2c^2\mu^2 + \lambda m}} \in (1, \infty).$$

Thus $g(t)$ is defined on $I = (-\infty, +\infty)$.

(i) If $\mu > 0$, $c > 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt < \infty.$$

(ii) If $\mu < 0$, $c < 0$, then

$$\int_{-\infty}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

Case 3: $\lambda = 0$. In this case, $c\mu > 0$ and the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2 + bc\mu}{\theta}\right)\right] - 1\right\}}$$

and $\theta = bc\mu > 0$. Thus $g(t)$ is defined on $I = (-\infty, +\infty)$.

(i) If $\mu > 0,\ c > 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt < \infty.$$

(ii) If $\mu < 0,\ c < 0$, then

$$\int_{-\infty}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

From the above arguments, we obtain the following

Proposition 4.4. *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that (4.1) holds for some constant $\mu \neq 0$ and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \neq 0$. If F and $\tilde{F}$ are Finsler gradient Ricci solitons on M with*

$$\mathbf{Ric}_\infty = (n-1)\lambda F^2, \quad \widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2,$$

where $\lambda \in \mathbb{R}$, $\tilde{\lambda} < \mu^2$. Then $\tilde{F}$ is pointwise projectively related to F and along any unit speed geodesic $\sigma(t)$ of F,

$$\tilde{F}(\dot{\sigma}(t)) = \frac{\theta \tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - bc\mu}{m}, \tag{4.7}$$

where $m := \tilde{\lambda} - \mu^2$, $b := \frac{n+1}{n-1}$, $\theta := \sqrt{\lambda m + b^2c^2\mu^2} > 0$.

(i) *If $\lambda < 0$, then F and $\tilde{F}$ are both not complete. In particular, F is positively complete (resp. negatively complete) if and only if $\tilde{F}$ is positively complete (resp. negatively complete).*

(ii) *If $\lambda \geq 0$, then F is complete.*

(iia) *If $\mu > 0$, $c > 0$, then $\tilde{F}$ is negatively complete.*

(iib) *If $\mu < 0$, $c < 0$, then $\tilde{F}$ is positively complete.*

4.2. $\theta > 0$ and $m > 0$

In this subsection, we study the solution (4.6) when $\theta > 0$ and $m > 0$. In this case, $\tilde{\lambda} > \mu^2$.

Case 1: $\lambda < 0$. In this case, the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - \frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}}\right\}}$$

and

$$\mu^2 < \tilde{\lambda} < \mu^2(1 - \frac{b^2c^2}{\lambda}), \qquad c\mu < 0.$$

As the result, we have

$$\frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}} < -1.$$

Thus $g(t)$ is defined on $I = (-\infty, +\infty)$.

(i) If $\mu > 0$, $c < 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

(ii) If $\mu < 0$, $c > 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

Case 2: $\lambda > 0$. In this case, the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - \frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}}\right\}}$$

and $\mu^2 < \tilde{\lambda} < \infty$. As the result, we have

$$\frac{bc\mu}{\sqrt{b^2c^2\mu^2+\lambda m}} \in (-1,1).$$

(i) If $\mu > 0$, then $g(t)$ is defined on $I = (-\delta, +\infty)$ and

$$\int_{-\delta}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt < \infty.$$

(ii) If $\mu < 0$, then $g(t)$ is defined on $I = (-\infty, \tau)$ and

$$\int_{-\infty}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{\tau} g^2(t)dt < \infty.$$

Case 3: $\lambda = 0$. In this case, $c\mu < 0$ and the solution (4.6) can be rewritten as

$$g(t) = \sqrt{\frac{\theta}{m}\left\{\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] + 1\right\}}$$

and $\theta = -bc\mu > 0$. Thus $g(t)$ is defined on $I = (-\infty, +\infty)$.

(i) If $\mu > 0,\ c < 0$, then

$$\int_{-\infty}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

(ii) If $\mu < 0,\ c > 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt < \infty.$$

Proposition 4.5. *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that (4.1) holds for some constant $\mu \neq 0$ and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \neq 0$. If F and $\tilde{F}$ are Finsler gradient Ricci solitons on M with*

$$\mathbf{Ric}_\infty = (n-1)\lambda F^2, \quad \widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2,$$

where $\lambda \in \mathbb{R}$, $\tilde{\lambda} > \mu^2$. Then $\tilde{F}$ is pointwise projectively related to F and along any unit speed geodesic $\sigma(t)$ of F,

$$\tilde{F}(\dot{\sigma}(t)) = \frac{\theta\tanh\left[\frac{\theta}{\mu}t + \tanh^{-1}\left(\frac{ma^2+bc\mu}{\theta}\right)\right] - bc\mu}{m}, \tag{4.8}$$

where $m := \tilde{\lambda} - \mu^2,\ b := \frac{n+1}{n-1},\ \theta := \sqrt{\lambda m + b^2c^2\mu^2} > 0$.

(i) *If $\lambda < 0$, then F and $\tilde{F}$ are both complete.*

(ii) *If $\lambda > 0$, then any geodesic of $\tilde{F}$ has finite length. Hence, $\tilde{F}$ is neither positively complete, nor negatively complete.*

(iii) *If $\lambda = 0$, then F is complete.*

(iiia) *If $\mu > 0$, $c < 0$, then $\tilde{F}$ is positively complete.*

(iiib) *If $\mu < 0$, $c > 0$, then $\tilde{F}$ is negatively complete.*

4.3. $\theta > 0$ and $m = 0$

In this subsection, we will discuss the equation (4.3) with $\theta > 0$ and $m = 0$. In this case, $\tilde{\lambda} = \mu^2 \neq 0$ and $c \neq 0$, and then the equation (4.3) can be simplified to

$$2bc\mu g^2(t) + 2\mu g(t)g'(t) = \lambda. \tag{4.9}$$

The solution of (4.9) with $g(0) = a_0 \geq 0$ is determined by

$$g(t) = \sqrt{\left({a_0}^2 - \frac{\lambda}{2bc\mu}\right) e^{-2bct} + \frac{\lambda}{2bc\mu}}. \tag{4.10}$$

Case 1: $\lambda c\mu > 0$. In this case, $e^{2bct} > 1 - \frac{2bc\mu a_0^2}{\lambda}$.

(i) $\frac{2bc\mu a_0^2}{\lambda} > 1$. In this case, $g(t)$ is defined on $I = (-\infty, +\infty)$ and

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

(ii) $\frac{2bc\mu a_0^2}{\lambda} < 1$.

(iia) If $c > 0$, then $g(t)$ is defined on $I = (-\delta, +\infty)$ and

$$\int_{-\delta}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

(iib) If $c < 0$, then $g(t)$ is defined on $I = (-\infty, \tau)$ and

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{\tau} g^2(t)dt < \infty.$$

Case 2: $\lambda c\mu < 0$. In this case,

$$e^{2bct} < 1 - \frac{2bc\mu a_0^2}{\lambda}.$$

(i) If $c > 0$, then $g(t)$ is defined on $I = (-\infty, \tau)$ and

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{\tau} g^2(t)dt < \infty.$$

(ii) If $c < 0$, then $g(t)$ is defined on $I = (-\delta, +\infty)$ and

$$\int_{-\delta}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

Case 3: $\lambda = 0$. In this case, the solution (4.10) can be rewritten as

$$g(t) = a_0 e^{-bct}.$$

Thus $g(t)$ is defined on $I = (-\infty, +\infty)$.

(i) If $c > 0$, then

$$\int_{-\infty}^{0} g^2(t)dt = \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt < \infty.$$

(ii) If $c < 0$, then

$$\int_{-\infty}^{0} g^2(t)dt < \infty \quad \text{and} \quad \int_{0}^{+\infty} g^2(t)dt = \infty.$$

Proposition 4.6. *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that (4.1) holds for some constant $\mu \neq 0$ and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \neq 0$. If F and $\tilde{F}$ are Finsler gradient Ricci solitons on M with*

$$\mathbf{Ric}_\infty = (n-1)\lambda F^2, \quad \widetilde{\mathbf{Ric}}_\infty = (n-1)\mu^2 \tilde{F}^2,$$

where $\lambda \in \mathbb{R}$. Then $\tilde{F}$ is pointwise projectively related to F and along any unit speed geodesic $\sigma(t)$ of F,

$$\tilde{F}(\dot{\sigma}(t)) = \left({a_0}^2 - \frac{\lambda}{2bc\mu}\right) e^{-2bct} + \frac{\lambda}{2bc\mu}, \tag{4.11}$$

where $b := \frac{n+1}{n-1}$.

(i) $\lambda \neq 0$.
   (ia) *If $\frac{2bc\mu a_0^2}{\lambda} > 1$, then F and $\tilde{F}$ are both complete.*
   (ib) *If $\frac{2bc\mu a_0^2}{\lambda} < 1$, then no geodesic of F and $\tilde{F}$ are defined on $(-\infty, +\infty)$. Further, F is positively complete (resp. negatively complete) if and only if $\tilde{F}$ is positively complete (resp. negatively complete).*

(ii) $\lambda = 0$. *In this case, F is complete, but $\tilde{F}$ is not complete.*
   (iia) *If $c > 0$, then $\tilde{F}$ is negatively complete.*
   (iib) *If $c < 0$, then $\tilde{F}$ is positively complete.*

4.4. $\theta = 0$ and $m \neq 0$

In this subsection, we will discuss the equation (4.3) with $\theta = 0$ and $m \neq 0$. In this case, $\lambda = -\frac{b^2c^2\mu^2}{m}$, and the equation (4.3) can be rewritten as

$$m^2 g^4(t) + 2bcm\mu g^2(t) + 2m\mu g(t)g'(t) = -b^2c^2\mu^2. \tag{4.12}$$

The equation (4.12) is solvable and the solution can be expressed in the following form

$$g(t)=\sqrt{\frac{\mu}{m}\left(\frac{1}{t+t_0}-bc\right)},\qquad t_0\in\mathbb{R}. \tag{4.13}$$

The following discussions are based on the condition that $t_0\neq 0$ or $t_0=0$.

Case 1: $t_0\neq 0$. In this case, the solution (4.13) with $g(0)=a_1\geq 0$ is determined by $t_0=\frac{\mu}{m{a_1}^2+bc\mu}$ and $cm\mu>0$.

(i) If $m\mu<0$ and $c<0$, then $t_0<0$, $g(t)$ is defined on $I=(-\delta,+\infty)$ and

$$\int_{-\delta}^{0} g^2(t)dt<\infty \quad\text{and}\quad \int_{0}^{+\infty} g^2(t)dt=\infty.$$

(ii) If $m\mu>0$ and $c>0$, then $t_0>0$, $g(t)$ is defined on $I=(-\infty,\tau)$ and

$$\int_{-\infty}^{0} g^2(t)dt=\infty \quad\text{and}\quad \int_{0}^{\tau} g^2(t)dt<\infty.$$

Case 2: $t_0=0$. In this case, the solution (4.13) can be written as

$$g(t)=\sqrt{\frac{\mu}{m}\left(\frac{1}{t}-bc\right)}. \tag{4.14}$$

(i) $m\mu>0$.

(ia) If $c>0$, then $g(t)$ is defined on $I=(0,\tau)$ and

$$\int_{0}^{\tau} g^2(t)dt=\infty.$$

(ib) If $c<0$, then $g(t)$ is defined on $I=(-\infty,-\tau)$ or $(0,+\infty)$ and

$$\int_{-\infty}^{-\tau} g^2(t)dt=\infty \quad\text{and}\quad \int_{0}^{+\infty} g^2(t)dt=\infty.$$

(ii) $m\mu<0$.

(iia) If $c<0$, then $g(t)$ is defined on $I=(-\delta,0)$ and

$$\int_{-\delta}^{0} g^2(t)dt=\infty.$$

(iib) If $c>0$, then $g(t)$ is defined on $I=(-\infty,0)$ or $(\delta,+\infty)$ and

$$\int_{-\infty}^{0} g^2(t)dt=\infty \quad\text{and}\quad \int_{\delta}^{+\infty} g^2(t)dt=\infty.$$

Proposition 4.7. *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that (4.1) holds for some constant $\mu \neq 0$ and $\tilde{\mathbf{S}} = \mathbf{S}$. Furthermore, assume that F is of constant S-curvature, $\mathbf{S} = (n+1)cF$ for some constant $c \neq 0$. If F and $\tilde{F}$ are Finsler gradient Ricci solitons on M with*

$$\mathbf{Ric}_\infty = -(n-1)\frac{b^2c^2\mu^2}{m}F^2, \quad \widetilde{\mathbf{Ric}}_\infty = (n-1)\tilde{\lambda}\tilde{F}^2,$$

where $b := \frac{n+1}{n-1}$ and $m := \tilde{\lambda} - \mu^2 \neq 0$. Then $\tilde{F}$ is pointwise projectively related to F and along any unit speed geodesic $\sigma(t)$ of F,

$$\tilde{F}(\dot{\sigma}(t)) = \frac{\mu}{m}\left(\frac{1}{t+t_0} - cb\right), \tag{4.15}$$

where $t_0 \in \{0, \frac{\mu}{m{a_1}^2 + bc\mu}\}$ with $a_1 \geq 0$. In this case, no geodesic of F and $\tilde{F}$ are defined on $(-\infty, +\infty)$. Furthermore, if F is positively complete (resp. negatively complete), then $\tilde{F}$ is positively complete (resp. negatively complete).

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