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A NOTE ON OPERATIONS OF INTUITIONISTIC FUZZY SOFT SETS

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Abstract. In this article, it an achievement for our part to be had of notions of a few operations on intuitionistic fuzzy soft sets and generalization of De Morgan's laws with respect to some operations in intuitionistic fuzzy soft set theory . It is being validated by suitable example. Thereafter properties like associative, distributive relative to different operations on intuitionistic fuzzy soft set are established and justified by suitable example . It is shown that restricted symmetric difference operation does not satisfy associative and restricted intersection does not distribute over restricted symmetric difference in intuitionistic fuzzy soft set theory.

Key Words: Intuitionistic fuzzy soft set, Restricted (extended) union, intersection of intuitionistic fuzzy soft sets, Restricted (extended) difference, Restricted symmetric difference of intuitionistic fuzzy soft sets.
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1. Introduction

Traditional mathematical tools are not appropriate to model uncertainties that occur in various ways in our real world problems. Theory of probability, fuzzy sets [9], intuitionistic fuzzy sets [1], soft sets [4], rough sets etc. were proposed by Zadeh(in 1965), Atanassov(in 1986), Molodtsov(in 1999), Pawlak(in 1982) respectively, are present mathematical tools for describing uncertainties. Each of these theories has its

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own inherent difficulties. Maji et al. [5] proposed the concept of intuitionistic fuzzy soft set, a hybrid structure of fuzzy set and soft set, to deal with uncertainties in decision making problem in more appropriate way. Thereafter it has gained more popularity to model uncertainty, vagueness in economics, engineering, environmental science, social science etc. In this paper, we use this hybrid concept to appraise a few new results on different operations on intuitionistic fuzzy soft set. In [8], authors have illustrated De Morgan's laws in soft set theory and given interrelation among many operations like restricted (extended) union, intersection, restricted difference and restricted symmetric difference. In [5, 6], authors introduced many operations like subset, complement, intersection, union, AND, OR operation and dot etc. with some properties. In [3], authors introduce the concept of relation on intuitionistic fuzzy soft set.
In this paper , in section 3, we prove De Morgan's laws in intuitionistic fuzzy soft set theory relative to some operations like restricted (extended) union, restricted (extended) intersection, restricted difference etc. and a few of them will be validated by suitable examples . Also we establish a generalization of De Morgan's laws. In section 4, we prove associative, distributive properties of operations on intuitionistic fuzzy soft set theory and substantiate a few of them by suitable examples and lastly, we prove that restricted symmetric difference operation satisfies associative property and restricted intersection operation satisfies both distributive laws over restricted symmetric difference operation in soft theory . Here we consider an example so that these do not hold in intuitionistic fuzzy soft set theory. Proofs of certain results in the sequel are routine. However, we include them for the sake of completeness.

2. Preliminaries

In this section, we remind some basic definitions in soft set theory, intuitionistic fuzzy set theory and also in intuitionistic fuzzy soft set theory which will be needed in the sequel. Let U be the initial universe set, E be the set of parameters with respect to U and $P(U)$ stands for the power set of U.

Definition 2.1 ([4]). Let $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.2 ([2]). The relative complement of a soft set (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$, where $F^r : A \to P(U)$ is a mapping given by $F^r(e) = U \setminus F(e)$, for all $e \in A$.

Definition 2.3 ([7]). A soft set (F, A) over U is said to be a null soft set denoted by Φ_A, if $\forall e \in A, F(e) = \emptyset$ (null set).

Definition 2.4 ([7]). A soft set (F, A) over U is said to be an absolute soft set denoted by $\mathcal{U}_A$, if $\forall e \in A, F(e) = U$.

Definition 2.5 ([2]). The restricted intersection of two soft sets (F, A) and (G, B) over common universe U, denoted by $(F, A) \Cap (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$.

Definition 2.6 ([2]). The extended intersection of two soft sets (F, A) and (G, B) over common universe U, denoted by $(F, A) \sqcap_{\mathcal{E}} (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall\, e \in C$,

$$
\begin{aligned}
H(e) &= F(e), && \text{if } e \in A \setminus B,\\
&= G(e), && \text{if } e \in B \setminus A,\\
&= F(e) \cap G(e), && \text{if } e \in A \cap B.
\end{aligned}
$$

Definition 2.7 ([2]). The restricted union of two soft sets (F, A) and (G, B) over common universe U, denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$, is defined as the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$.

Definition 2.8 ([7]). The extended union of two soft sets (F, A) and (G, B) over common universe U, denoted by $(F, A) \mathbin{\widetilde{\cup}} (G, B)$, is defined as the soft set (H, C), where $C = A \cup B$, and $\forall\, e \in C$,

$$
\begin{aligned}
H(e) &= F(e), && \text{if } e \in A \setminus B,\\
&= G(e), && \text{if } e \in B \setminus A,\\
&= F(e) \cup G(e), && \text{if } e \in A \cap B.
\end{aligned}
$$

Definition 2.9 ([2]). Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. The restricted difference of (F, A) and (G, B) is denoted by $(F, A) \sim_{\mathcal{R}} (G, B)$ and defined as $(F, A) \sim_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \setminus G(e)$.

Definition 2.10 ([8]). The restricted symmetric difference of two soft sets (F, A) and (G, B) over a common universe U is defined by
$(F, A) \mathbin{\widetilde{\triangle}} (G, B) = ((F, A) \sim_{\mathcal{R}} (G, B)) \cup_{\mathcal{R}} ((G, B) \sim_{\mathcal{R}} (F, A))$.

Note that we use the notations $\Cup$, $\sqcup_{\mathcal{E}}$, $(F,A)^c$ instead of $\cup_{\mathcal{R}}$, $\widetilde{\cup}$, $(F,A)^r$ as in [2] throughout this paper. Also we use this new notation for IFSS theory throughout this paper.

Definition 2.11 ([1])**.** An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set U is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in U \},$$

where the functions $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ denote the degree of membership and degree of nonmembership of the element $x \in U$ to A, respectively and satisfy $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in U$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in U \}$.

Definition 2.12 ([1])**.** If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are any two IFS in X, then
(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ $\forall x \in X$;
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$;
(3) $A^c = (\nu_A, \mu_A)$;
(4) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$, where for all $x \in X$,
$(\mu_A \cap \mu_B)(x) = \mu_A(x) \wedge \mu_B(x)$ and $(\nu_A \cup \nu_B)(x) = \nu_A(x) \vee \nu_B(x)$;
(5) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$, where for all $x \in X$,
$(\mu_A \cup \mu_B)(x) = \mu_A(x) \vee \mu_B(x)$ and $(\nu_A \cap \nu_B)(x) = \nu_A(x) \wedge \nu_B(x)$.

Now we recollect some of the operations and redefine some others on IFSS theory, which are stated above on soft set theory.

Definition 2.13 ([5])**.** Let $A \subseteq E$ and I^U denotes the collection of all intuitionistic fuzzy subsets of U. A pair $(\mathcal{F}, A)$ is called an intuitionistic fuzzy soft set (briefly, IFSS) over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : A \to I^U$.

Definition 2.14 ([5])**.** Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over U. $(\mathcal{F}, A)$ is called an intuitionistic fuzzy soft subset of $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \widetilde{\subseteq} (\mathcal{G}, B)$, if $(i)\, A \subseteq B$, (ii) for all $e \in A$, $\mathcal{F}(e) \subseteq \mathcal{G}(e)$.

Definition 2.15 ([5])**.** The restricted intersection of two IFSS $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \Cap (\mathcal{G}, B)$, is defined as the IFSS $(\mathcal{H}, C)$, where $C = A \cap B$ and for all $e \in C$, $\mathcal{H}(e) = \mathcal{F}(e) \cap \mathcal{G}(e)$.

Definition 2.16. The extended intersection of two IFSS $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B)$, is defined as the IFSS $(\mathcal{H}, C)$, where $C = A \cup B$, and $\forall\, e \in C$,

$$\mathcal{H}(e) = \mathcal{F}(e), \quad \text{if } e \in A \setminus B,$$
$$= \mathcal{G}(e), \quad \text{if } e \in B \setminus A,$$
$$= \mathcal{F}(e) \cap \mathcal{G}(e), \quad \text{if } e \in A \cap B.$$

Definition 2.17. The restricted union of two IFSS $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \Cup (\mathcal{G}, B)$, is defined as the IFSS $(\mathcal{H}, C)$, where $C = A \cap B$ and for all $e \in C$, $\mathcal{H}(e) = \mathcal{F}(e) \cup \mathcal{G}(e)$.

Definition 2.18 ([5])**.** The extended union of two IFSS $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B)$, is defined as the IFSS $(\mathcal{H}, C)$, where $C = A \cup B$, and $\forall\, e \in C$,

$$\mathcal{H}(e) = \mathcal{F}(e), \quad \text{if } e \in A \setminus B,$$
$$= \mathcal{G}(e), \quad \text{if } e \in B \setminus A,$$
$$= \mathcal{F}(e) \cup \mathcal{G}(e), \quad \text{if } e \in A \cap B.$$

Definition 2.19. The complement of an IFSS $(\mathcal{F}, A)$ is denoted by $(\mathcal{F}, A)^c$ and defined by $(\mathcal{F}, A)^c = (\mathcal{F}^c, A)$, where $\mathcal{F}^c : A \to I^U$ is a mapping given by $\mathcal{F}^c(e) = [\mathcal{F}(e)]^c$ for all $e \in A$.

Definition 2.20. An IFSS $(\mathcal{F}, A)$ over U is said to be a null intuitionistic fuzzy soft set, denoted by Φ_A, if for all $e \in A$, $\mathcal{F}(e)$ is the null intuitionistic fuzzy set i.e. $\mathcal{F}(e) = (\widetilde{0}, \widetilde{1})$, where $\widetilde{0}(x) = 0$, $\widetilde{1}(x) = 1$ for all $x \in U$.

Definition 2.21. An IFSS $(\mathcal{F}, A)$ over U is said to be an absolute intuitionistic fuzzy soft set, denoted by $\mathcal{U}_A$, if for all $e \in A$, $\mathcal{F}(e)$ is the absolute intuitionistic fuzzy set i.e. $\mathcal{F}(e)(x) = (\widetilde{1}, \widetilde{0})$, where $\widetilde{1}(x) = 1, \widetilde{0}(x) = 0$ for all $x \in U$.

Definition 2.22 ([5])**.** Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U. Then "$(\mathcal{F}, A)$ AND $(\mathcal{G}, B)$" denoted by $(\mathcal{F}, A) \wedge (\mathcal{G}, B)$ is defined as $(\mathcal{F}, A) \wedge (\mathcal{G}, B) = (\mathcal{H}, A \times B)$, where $\mathcal{H}(x, y) = \mathcal{F}(x) \cap \mathcal{G}(y)$ for all $(x, y) \in A \times B$.

Definition 2.23 ([5])**.** Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U. Then "$(\mathcal{F}, A)$ OR $(\mathcal{G}, B)$" denoted by $(\mathcal{F}, A) \vee (\mathcal{G}, B)$ is defined as $(\mathcal{F}, A) \vee (\mathcal{G}, B) = (\mathcal{H}, A \times B)$, where $\mathcal{H}(x, y) = \mathcal{F}(x) \cup \mathcal{G}(y)$ for all $(x, y) \in A \times B$.

Definition 2.24. Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U such that $A \cap B \neq \phi$. The restricted difference of $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ is denoted by $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)$ and defined as $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B) = (\mathcal{H}, C)$, where $C = A \cap B$ and $\forall\, e \in C$, $\mathcal{H}(e) = \mathcal{F}(e) \cap \mathcal{G}^c(e)$.

Definition 2.25. The restricted symmetric difference of two IFSS $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ over a common universe U is defined by
$(\mathcal{F}, A)\widetilde{\triangle}(\mathcal{G}, B) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)) \Cup ((\mathcal{G}, B) \sim_{\mathcal{R}} (\mathcal{F}, A))$.

Now we define Extended difference operation on IFSS

Definition 2.26. Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U. The extended difference of $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ is denoted by $(\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{G}, B)$ and is defined as $(\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{G}, B) = (\mathcal{H}, C)$, where $C = A \cup B$ and for all $e \in C$,
$\mathcal{H}(e) = \mathcal{F}(e)$, if $e \in A \setminus B$,
$= \Phi_e$, if $e \in B \setminus A$, where Φ_e is null intuitionistic fuzzy set,
$= \mathcal{F}(e) \cap \mathcal{G}^c(e)$, if $e \in A \cap B$.

3. De Morgan's laws in intuitionistic fuzzy soft set theory

In this section, at first we recall De Morgan's laws in IFS theory given by [1]. Then we state some properties relating to operations mentioned below and illustrate De Morgan's laws in IFSS theory for restricted (extended) union, restricted (extended) intersection and complement. Thereafter we illustrate De Morgan's laws in IFSS theory for AND-operation, OR-operation and complement with suitable examples.

Proposition 3.1. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two IFS in X. Then we have the following:*
(1) $(A \cup B)^c = A^c \cap B^c$,
(2) $(A \cap B)^c = A^c \cup B^c$.

Proposition 3.2. *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U such that $A \cap B \neq \phi$. Then we have the following:*
(1) $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c \mathrel{\widetilde{\subseteq}} (\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \Cap (\mathcal{G}, B))^c \mathrel{\widetilde{\supseteq}} (\mathcal{F}, A)^c \Cap (\mathcal{G}, B)^c$.

Proof. (1) Suppose that $(\mathcal{F}, A) \Cup (\mathcal{G}, B) = (\mathcal{H}, C)$, where $C = A \cap B$. Therefore $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c = (\mathcal{H}, C)^c = (\mathcal{H}^c, C)$.
Now $(\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \Cup (\mathcal{G}^c, B) = (\mathcal{K}, C)$, where $\mathcal{K}(e) = \mathcal{F}^c(e) \cup \mathcal{G}^c(e)$ for all $e \in C = A \cap B$.
Let $e \in C$. Then by Definition 2.19,
$\mathcal{H}^c(e) = [\mathcal{H}(e)]^c = [\mathcal{F}(e) \cup \mathcal{G}(e)]^c = \mathcal{F}^c(e) \cap \mathcal{G}^c(e)$ (by Proposition 3.1)
$\subseteq \mathcal{F}^c(e) \cup \mathcal{G}^c(e) = \mathcal{K}(e)$. Therefore $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c \mathrel{\widetilde{\subseteq}} (\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c$.
(2) Proof is similar as in (1). □

Proposition 3.3. *Let* $(\mathcal{F}, A)$ *and* $(\mathcal{G}, B)$ *be two IFSS over a common universe* U*. Then we have the following:*
(1) $((\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B))^c \,\widetilde{\subseteq}\, (\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B))^c \,\widetilde{\supseteq}\, (\mathcal{F}, A)^c \sqcap_{\mathcal{E}} (\mathcal{G}, B)^c$.

Proof. Using similar technique as in Proposition 3.2 and using Definition 2.16, 2.18, it can be proved. □

Proposition 3.4. *Let* $(\mathcal{F}, A)$ *and* $(\mathcal{G}, B)$ *be two IFSS over a common universe* U*. Then we have the following:*
(1) $((\mathcal{F}, A) \vee (\mathcal{G}, B))^c \,\widetilde{\subseteq}\, (\mathcal{F}, A)^c \vee (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \wedge (\mathcal{G}, B))^c \,\widetilde{\supseteq}\, (\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c$.

Proof. We prove part (2) of this theorem. Similarly, part (1) can be proved. Suppose that $(\mathcal{F}, A) \wedge (\mathcal{G}, B) = (\mathcal{H}, A \times B)$. Therefore $((\mathcal{F}, A) \wedge (\mathcal{G}, B))^c = (\mathcal{H}, A \times B)^c = (\mathcal{H}^c, A \times B)$. Now $(\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \wedge (\mathcal{G}^c, B) = (\mathcal{K}, A \times B)$, where $\mathcal{K}(x, y) = \mathcal{F}^c(x) \cap \mathcal{G}^c(y)$, $\forall\, (x, y) \in A \times B$. Let $(x, y) \in A \times B$. Then by Definition 2.19, $\mathcal{H}^c(x, y) = [\mathcal{H}(x, y)]^c = [\mathcal{F}(x) \cap \mathcal{G}(y)]^c = \mathcal{F}^c(x) \cup \mathcal{G}^c(y)$ (by Proposition 3.1)
$\supseteq \mathcal{F}^c(x) \cap \mathcal{G}^c(y) = \mathcal{K}(x, y)$. So, $((\mathcal{F}, A) \wedge (\mathcal{G}, B))^c \,\widetilde{\supseteq}\, (\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c$. □

Now we give an example in support of Proposition 3.2.

Example 3.5. *Let* U *be the initial universe,* E *be the set of parameters and* A, B *be two subsets of* E *such that* $U = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$. *Also let* $(\mathcal{F}, A)$, $(\mathcal{G}, B)$ *be two IFSS over the common universe* U, *where* $\mathcal{F} : A \to I^U$ *is defined by:*
$\mathcal{F}(e_1) = \{(h_1, 0.2, 0.3), (h_2, 0.4, 0.3), (h_3, 0.8, 0.1), (h_4, 0.5, 0.5)\}$,
$\mathcal{F}(e_2) = \{(h_1, 0.7, 0.2), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$.
And $\mathcal{G} : B \to I^U$ *is defined by:*
$\mathcal{G}(e_2) = \{(h_1, 0.1, 0.1), (h_2, 0.3, 0.2), (h_3, 0.9, 0), (h_4, 0.7, 0.2)\}$,
$\mathcal{G}(e_3) = \{(h_1, 0.3, 0.2), (h_2, 0.1, 0.2), (h_3, 0.7, 0.2), (h_4, 0.7, 0.1)\}$.
Let $(\mathcal{F}, A) \Cup (\mathcal{G}, B) = (\mathcal{L}, A \cap B)$, *where* $A \cap B = \{e_2\}$ *and* $\mathcal{L}(e_2) = \mathcal{F}(e_2) \cup \mathcal{G}(e_2) = \{(h_1, 0.7, 0.1), (h_2, 0.3, 0.2), (h_3, 0.9, 0), (h_4, 0.7, 0.2)\}$.
Hence $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c = (\mathcal{L}^c, A \cap B)$,
where $\mathcal{L}^c(e_2) = \{(h_1, 0.1, 0.7), (h_2, 0.2, 0.3), (h_3, 0, 0.9), (h_4, 0.2, 0.7)\}$.
Now let $(\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c = (\mathcal{M}, A \cap B)$, *where* $A \cap B = \{e_2\}$ *and* $\mathcal{M}(e_2) = \mathcal{F}^c(e_2) \cup \mathcal{G}^c(e_2) = \{(h_1, 0.2, 0.1), (h_2, 0.2, 0.1), (h_3, 0.5, 0.4), (h_4, 0.3, 0.3)\}$. *This shows that* $\mathcal{L}^c(e_2) \subseteq \mathcal{M}(e_2)$.
Therefore $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c \,\widetilde{\subseteq}\, (\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c$.

Theorem 3.6. *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U such that $A \cap B \neq \phi$. Then we have the following:*
(1) $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \Cap (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \Cap (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \Cup (\mathcal{G}, B)^c$.

Proof. (1) Suppose that $(\mathcal{F}, A) \Cup (\mathcal{G}, B) = (\mathcal{H}, C)$, where $C = A \cap B$. Therefore $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c = (\mathcal{H}, C)^c = (\mathcal{H}^c, C)$. Now $(\mathcal{F}, A)^c \Cap (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \Cap (\mathcal{G}^c, B) = (\mathcal{K}, C)$, where $\mathcal{K}(e) = \mathcal{F}^c(e) \cap \mathcal{G}^c(e)$ for all $e \in C = A \cap B$. Let $e \in C$. Then by Definition 2.19,
$\mathcal{H}^c(e) = [\mathcal{H}(e)]^c = [\mathcal{F}(e) \cup \mathcal{G}(e)]^c = \mathcal{F}^c(e) \cap \mathcal{G}^c(e)$ (by Proposition 3.1) $= \mathcal{K}(e)$. Therefore $((\mathcal{F}, A) \Cup (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \Cap (\mathcal{G}, B)^c$.
(2) By similar technique as in (1), it can be proved. □

Theorem 3.7. *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U. Then we have the following:*
(1) $((\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \sqcap_{\mathcal{E}} (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c$.

Proof. We prove part (2) of this theorem. By similar technique, part (1) can be proved. Suppose that $(\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B) = (\mathcal{H}, C)$, where $C = A \cup B$. Therefore $((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B))^c = (\mathcal{H}, C)^c = (\mathcal{H}^c, C)$. Now $(\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \sqcup_{\mathcal{E}} (\mathcal{G}^c, B) = (\mathcal{K}, C)$, where $C = A \cup B$, and $\forall\, e \in C$,

$$\begin{aligned}\mathcal{K}(e) &= \mathcal{F}^c(e), && \text{if } e \in A \setminus B,\\ &= \mathcal{G}^c(e), && \text{if } e \in B \setminus A,\\ &= \mathcal{F}^c(e) \cup \mathcal{G}^c(e), && \text{if } e \in A \cap B.\end{aligned}$$

Then using Definition 2.19, 2.16, we have
for $e \in A \setminus B$, $\mathcal{H}^c(e) = [\mathcal{H}(e)]^c = [\mathcal{F}(e)]^c = \mathcal{F}^c(e) = \mathcal{K}(e)$;
for $e \in B \setminus A$, $\mathcal{H}^c(e) = [\mathcal{H}(e)]^c = [\mathcal{G}(e)]^c = \mathcal{G}^c(e) = \mathcal{K}(e)$; and
for $e \in A \cap B$, $\mathcal{H}^c(e) = [\mathcal{H}(e)]^c = [\mathcal{F}(e) \cap \mathcal{G}(e)]^c = \mathcal{F}^c(e) \cup \mathcal{G}^c(e) = \mathcal{K}(e)$.
Therefore $((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c$. □

Theorem 3.8 ([5])**.** *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two IFSS over a common universe U. Then we have the following:*
(1) $((\mathcal{F}, A) \vee (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c$,
(2) $((\mathcal{F}, A) \wedge (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \vee (\mathcal{G}, B)^c$.

Theorem 3.9. *Let $(\mathcal{F}, A)$, $(\mathcal{G}, B)$ and $(\mathcal{H}, C)$ be three IFSS over a common universe U. Then we have the following:*
(1) $(\mathcal{F}, A) \sim_{\mathcal{R}} ((\mathcal{G}, B) \Cap (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)) \Cup ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{H}, C))$;
(2) $(\mathcal{F}, A) \sim_{\mathcal{R}} ((\mathcal{G}, B) \Cup (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)) \Cap ((\mathcal{F}, A) \sim_{\mathcal{R}}$

$(\mathcal{H}, C))$;
(3) $(\mathcal{F}, A) \sim_{\mathcal{R}} ((\mathcal{G}, B) \sqcap_{\mathcal{E}} (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)) \sqcup_{\mathcal{E}} ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{H}, C))$;
(4) $(\mathcal{F}, A) \sim_{\mathcal{R}} ((\mathcal{G}, B) \sqcup_{\mathcal{E}} (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B)) \sqcap_{\mathcal{E}} ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{H}, C))$;
(5) $(\mathcal{F}, A) \sim_{\mathcal{E}} ((\mathcal{G}, B) \Cap (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{G}, B)) \Cup ((\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{H}, C))$;
(6) $(\mathcal{F}, A) \sim_{\mathcal{E}} ((\mathcal{G}, B) \Cup (\mathcal{H}, C)) = ((\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{G}, B)) \Cap ((\mathcal{F}, A) \sim_{\mathcal{E}} (\mathcal{H}, C))$;

Proof. (1) Suppose that $(\mathcal{G}, B) \Cap (\mathcal{H}, C) = (\mathcal{S}, B \cap C)$, where $\mathcal{S}(x) = \mathcal{G}(x) \cap \mathcal{H}(x)$ for all $x \in B \cap C \neq \phi$. and let $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{S}, B \cap C) = (\mathcal{T}, A \cap (B \cap C))$, where for all $x \in A \cap (B \cap C)$, $\mathcal{T}(x) = \mathcal{F}(x) \cap \mathcal{S}^c(x)$
$= \mathcal{F}(x) \cap [\mathcal{G}(x) \cap \mathcal{H}(x)]^c$
$= \mathcal{F}(x) \cap [\mathcal{G}^c(x) \cup \mathcal{H}^c(x)]$, using Proposition 3.1 and Definition 2.19
$= [\mathcal{F}(x) \cap \mathcal{G}^c(x)] \cup [\mathcal{F}(x) \cap \mathcal{H}^c(x)]$, since distributive law holds in IFS.
Again assume that $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B) = (\mathcal{I}, A \cap B)$, $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{H}, C) = (\mathcal{J}, A \cap C)$, where for all $x \in A \cap B$, $\mathcal{I}(x) = \mathcal{F}(x) \cap \mathcal{G}^c(x)$ and for all $x \in A \cap C$, $\mathcal{J}(x) = \mathcal{F}(x) \cap \mathcal{H}^c(x)$. Now let $(\mathcal{I}, A \cap B) \Cup (\mathcal{J}, A \cap C) = (\mathcal{K}, (A \cap B) \cap (A \cap C)) = (\mathcal{K}, A \cap B \cap C)$, where for all $x \in A \cap B \cap C$, $\mathcal{K}(x) = \mathcal{I}(x) \cup \mathcal{J}(x) = [\mathcal{F}(x) \cap \mathcal{G}^c(x)] \cup [\mathcal{F}(x) \cap \mathcal{H}^c(x)]$. Therefore we conclude that $\mathcal{T}(x) = \mathcal{K}(x)$ for all $x \in A \cap B \cap C$, which proves the desired part.
(2) To proof this we can use similar technique as in (1).
(3) Suppose that $(\mathcal{G}, B) \sqcap_{\mathcal{E}} (\mathcal{H}, C) = (\mathcal{S}, B \cup C)$, where for all $x \in B \cup C$,

$$\begin{aligned} \mathcal{S}(x) &= \mathcal{G}(x), && \text{if } x \in B \setminus C, \\ &= \mathcal{H}(x), && \text{if } x \in C \setminus B, \\ &= \mathcal{G}(x) \cap \mathcal{H}(x), && \text{if } x \in B \cap C. \end{aligned}$$

and let $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{S}, B \cup C) = (\mathcal{T}, A \cap (B \cup C))$, where for all $x \in A \cap (B \cup C)$, $\mathcal{T}(x) = \mathcal{F}(x) \cap \mathcal{S}^c(x)$, i.e.

$$\begin{aligned} \mathcal{T}(x) &= \mathcal{F}(x) \cap \mathcal{G}^c(x), && \text{if } x \in A \cap (B \setminus C), \\ &= \mathcal{F}(x) \cap \mathcal{H}^c(x), && \text{if } x \in A \cap (C \setminus B), \\ &= \mathcal{F}(x) \cap [\mathcal{G}(x) \cap \mathcal{H}(x)]^c, && \text{if } x \in A \cap B \cap C. \end{aligned}$$

Again let $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{G}, B) = (\mathcal{I}, A \cap B)$ and $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{H}, C) = (\mathcal{J}, A \cap C)$, where for all $x \in A \cap B$, $\mathcal{I}(x) = \mathcal{F}(x) \cap \mathcal{G}^c(x)$ and for all $x \in A \cap C$, $\mathcal{J}(x) = \mathcal{F}(x) \cap \mathcal{H}^c(x)$.
Now assume that $(\mathcal{I}, A \cap B) \sqcup_{\mathcal{E}} (\mathcal{J}, A \cap C) = (\mathcal{K}, (A \cap B) \cup (A \cap C))$, where for all $x \in (A \cap B) \cup (A \cap C) = A \cap (B \cup C)$, we have three cases:
(i) if $x \in (A \cap B) \setminus (A \cap C) = A \cap (B \setminus C)$, $\mathcal{K}(x) = \mathcal{I}(x) = \mathcal{F}(x) \cap \mathcal{G}^c(x)$;
(ii) if $x \in (A \cap C) \setminus (A \cap B) = A \cap (C \setminus B)$, $\mathcal{K}(x) = \mathcal{J}(x) = \mathcal{F}(x) \cap \mathcal{H}^c(x)$;

(iii) if $x \in (A \cap B) \cap (A \cap C) = A \cap B \cap C$, $\mathcal{K}(x) = \mathcal{I}(x) \cup \mathcal{J}(x) = [\mathcal{F}(x) \cap \mathcal{G}^c(x)] \cup [\mathcal{F}(x) \cap \mathcal{H}^c(x)] = \mathcal{F}(x) \cap [\mathcal{G}^c(x) \cup \mathcal{H}^c(x)] = \mathcal{F}(x) \cap [\mathcal{G}(x) \cap \mathcal{H}(x)]^c$. Therefore we conclude that $\mathcal{T}(x) = \mathcal{K}(x)$ for all $x \in A \cap (B \cup C)$, which completes the proof.

(4) Proof is similar as in (3).

(5)-(6) can be proved similarly as (1), (2) and using Definition 2.26. □

Note 3.10. *Theorems 3.6, 3.7, 3.8 are the De Morgan's laws in IFSS and the Theorem 3.9 is the generalisation of De Morgan's laws in IFSS.*

Example 3.11. *Now we illustrate part (2) of the Theorem 3.7. As a continuation of Example 3.5, let* $(\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B) = (\mathcal{L}, A \cup B)$. *Hence by Definition 2.16,* $\mathcal{L}(e_1) = \mathcal{F}(e_1)$, $\mathcal{L}(e_2) = \mathcal{F}(e_2) \cap \mathcal{G}(e_2) = \{(h_1, 0.1, 0.2), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$, $\mathcal{L}(e_3) = \mathcal{G}(e_3)$. *Now let* $(\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \sqcup_{\mathcal{E}} (\mathcal{G}^c, B) = (\mathcal{K}, A \cup B)$. *Hence by Definition 2.18,* $\mathcal{K}(e_1) = \mathcal{F}^c(e_1)$, $\mathcal{K}(e_2) = \mathcal{F}^c(e_2) \cup \mathcal{G}^c(e_2) = \{(h_1, 0.2, 0.1), (h_2, 0.2, 0.1), (h_3, 0.5, 0.4), (h_4, 0.3, 0.3)\}$, $\mathcal{K}(e_3) = \mathcal{G}^c(e_3)$. *This shows that* $\mathcal{L}^c(e_1) = \mathcal{K}(e_1)$, $\mathcal{L}^c(e_2) = \mathcal{K}(e_2)$, $\mathcal{L}^c(e_3) = \mathcal{K}(e_3)$. *Therefore*
$((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \sqcup_{\mathcal{E}} (\mathcal{G}, B)^c$ *is verified.*

Now we illustrate part (1) of Theorem 3.8. Let $(\mathcal{F}, A) \vee (\mathcal{G}, B) = (\mathcal{M}, A \times B)$, *then* $((\mathcal{F}, A) \vee (\mathcal{G}, B))^c = (\mathcal{M}^c, A \times B)$. *Note that* $A \times B = \{(e_1, e_2), (e_1, e_3), (e_2, e_2), (e_2, e_3)\}$. *Then by Definition 2.23,* $\mathcal{M}(e_1, e_2) = \mathcal{F}(e_1) \cup \mathcal{G}(e_2) = \{(h_1, 0.2, 0.1), (h_2, 0.4, 0.2), (h_3, 0.9, 0), (h_4, 0.7, 0.2)\}$.
Similarly,
$\mathcal{M}(e_1, e_3) = \mathcal{F}(e_1) \cup \mathcal{G}(e_3)$
$= \{(h_1, 0.3, 0.2), (h_2, 0.4, 0.2), (h_3, 0.8, 0.1), (h_4, 0.7, 0.1)\}$,
$\mathcal{M}(e_2, e_2) = \mathcal{F}(e_2) \cup \mathcal{G}(e_2)$
$= \{(h_1, 0.7, 0.1), (h_2, 0.3, 0.2), (h_3, 0.9, 0), (h_4, 0.7, 0.2)\}$,
$\mathcal{M}(e_2, e_3) = \mathcal{F}(e_2) \cup \mathcal{G}(e_3)$
$= \{(h_1, 0.7, 0.2), (h_2, 0.1, 0.2), (h_3, 0.7, 0.2), (h_4, 0.7, 0.1)\}$.
Now let $(\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c = (\mathcal{F}^c, A) \wedge (\mathcal{G}^c, B) = (\mathcal{N}, A \times B)$.
Then by Definition 2.22,
$\mathcal{N}(e_1, e_2) = \mathcal{F}^c(e_1) \cap \mathcal{G}^c(e_2)$
$= \{(h_1, 0.1, 0.2), (h_2, 0.2, 0.4), (h_3, 0, 0.9), (h_4, 0.2, 0.7)\}$.
Similarly,
$\mathcal{N}(e_1, e_3) = \{(h_1, 0.2, 0.3), (h_2, 0.2, 0.4), (h_3, 0.1, 0.8), (h_4, 0.1, 0.7)\}$,
$\mathcal{N}(e_2, e_2) = \{(h_1, 0.1, 0.7), (h_2, 0.2, 0.3), (h_3, 0, 0.9), (h_4, 0.2, 0.7)\}$,
$\mathcal{N}(e_2, e_3) = \{(h_1, 0.2, 0.7), (h_2, 0.2, 0.1), (h_3, 0.2, 0.7), (h_4, 0.1, 0.7)\}$.
This shows that $\mathcal{M}^c(e_1, e_2) = \mathcal{N}(e_1, e_2)$, $\mathcal{M}^c(e_1, e_3) = \mathcal{N}(e_1, e_3)$,
$\mathcal{M}^c(e_2, e_2) = \mathcal{N}(e_2, e_2)$, $\mathcal{M}^c(e_2, e_3) = \mathcal{N}(e_2, e_3)$.
Therefore $((\mathcal{F}, A) \vee (\mathcal{G}, B))^c = (\mathcal{F}, A)^c \wedge (\mathcal{G}, B)^c$ *is verified.*

4. Properties of operations on IFSS and their interrelations

In this section, at first we recall some properties like associative, distributive property in IFS theory given by [1]. Then we illustrate those on IFSS theory using different operations on IFSS. We have been motivated from properties of operation on soft set theory [8]. Throughout this section, let $(\mathcal{F}, A)$, $(\mathcal{G}, B)$, $(\mathcal{H}, C)$ be any three IFSS over the same universe U, where A, B, C are subsets of the parameter set E.

Proposition 4.1. *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$, $C = (\mu_C, \nu_C)$ be any three IFS in X. Then we have the following:*
(1) $A \cap (B \cap C) = (A \cap B) \cap C$,
(2) $A \cup (B \cup C) = (A \cup B) \cup C$,
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(4) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$,
(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
(6) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Theorem 4.2. *Associative properties:*
(1) $(\mathcal{F}, A) \Cap ((\mathcal{G}, B) \Cap (\mathcal{H}, C)) = ((\mathcal{F}, A) \Cap (\mathcal{G}, B)) \Cap (\mathcal{H}, C)$;
(2) $(\mathcal{F}, A) \sqcap_{\mathcal{E}} ((\mathcal{G}, B) \sqcap_{\mathcal{E}} (\mathcal{H}, C)) = ((\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, B)) \sqcap_{\mathcal{E}} (\mathcal{H}, C)$;
(3) $(\mathcal{F}, A) \Cup ((\mathcal{G}, B) \Cup (\mathcal{H}, C)) = ((\mathcal{F}, A) \Cup (\mathcal{G}, B)) \Cup (\mathcal{H}, C)$;
(4) $(\mathcal{F}, A) \sqcup_{\mathcal{E}} ((\mathcal{G}, B) \sqcup_{\mathcal{E}} (\mathcal{H}, C)) = ((\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B)) \sqcup_{\mathcal{E}} (\mathcal{H}, C)$;
(5) $(\mathcal{F}, A) \wedge ((\mathcal{G}, B) \wedge (\mathcal{H}, C)) = ((\mathcal{F}, A) \wedge (\mathcal{G}, B)) \wedge (\mathcal{H}, C)$;
(6) $(\mathcal{F}, A) \vee ((\mathcal{G}, B) \vee (\mathcal{H}, C)) = ((\mathcal{F}, A) \vee (\mathcal{G}, B)) \vee (\mathcal{H}, C)$.

Proof. (1) Let $(\mathcal{G}, B) \Cap (\mathcal{H}, C) = (\mathcal{L}, B \cap C)$,
where $\mathcal{L}(e) = \mathcal{G}(e) \cap \mathcal{H}(e)$ for all $e \in B \cap C \neq \phi$.
Then $(\mathcal{F}, A) \Cap ((\mathcal{G}, B) \Cap (\mathcal{H}, C)) = (\mathcal{F}, A) \Cap (\mathcal{L}, B \cap C) = (\mathcal{M}, A \cap B \cap C)$,
where $\mathcal{M}(e) = \mathcal{F}(e) \cap \mathcal{L}(e) = \mathcal{F}(e) \cap (\mathcal{G}(e) \cap \mathcal{H}(e))$, $\forall\, e \in A \cap B \cap C \neq \phi$.
Similarly, we have $((\mathcal{F}, A) \Cap (\mathcal{G}, B)) \Cap (\mathcal{H}, C) = (\mathcal{N}, A \cap B \cap C)$,
where $\mathcal{N}(e) = (\mathcal{F}(e) \cap \mathcal{G}(e)) \cap \mathcal{H}(e)$ for all $e \in A \cap B \cap C \neq \phi$.
Now using Proposition 4.1, we have $\mathcal{M}(e) = \mathcal{N}(e)$, $\forall\, e \in A \cap B \cap C \neq \phi$.
Therefore $(\mathcal{F}, A) \Cap ((\mathcal{G}, B) \Cap (\mathcal{H}, C)) = ((\mathcal{F}, A) \Cap (\mathcal{G}, B)) \Cap (\mathcal{H}, C)$.
(2) We omit this proof, it is similar as in (4).
(3) To prove this, we can use similar technique as in (1).
(4) Let $(\mathcal{G}, B) \sqcup_{\mathcal{E}} (\mathcal{H}, C) = (\mathcal{L}, B \cup C)$, where for all $e \in B \cup C$,

$$\begin{aligned} \mathcal{L}(e) &= \mathcal{G}(e), && \text{if } e \in B \setminus C, \\ &= \mathcal{H}(e), && \text{if } e \in C \setminus B, \\ &= \mathcal{G}(e) \cup \mathcal{H}(e), && \text{if } e \in B \cap C. \end{aligned}$$

Then $(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{L}, B \cup C) = (\mathcal{M}, A \cup B \cup C)$, where for all $e \in A \cup B \cup C$,
$\mathcal{M}(e) = \mathcal{F}(e)$, if $e \in A \setminus (B \cup C)$,
$= \mathcal{L}(e)$, if $e \in (B \cup C) \setminus A = \{B \setminus (A \cup C)\} \cup \{C \setminus (A \cup B)\} \cup \{(B \cap C) \setminus A\}$,
$= \mathcal{F}(e) \cup \mathcal{L}(e)$, if $e \in A \cap (B \cup C) = \{A \cap B \cap C\} \cup \{A \cap (B \setminus C)\} \cup \{A \cap (C \setminus B)\}$.

Now, let $(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B) = (\mathcal{K}, A \cup B)$, where for all $e \in A \cup B$,
$\mathcal{K}(e) = \mathcal{F}(e)$, if $e \in A \setminus B$,
$= \mathcal{G}(e)$, if $e \in B \setminus A$,
$= \mathcal{F}(e) \cup \mathcal{G}(e)$, if $e \in A \cap B$.

Then $(\mathcal{K}, A \cup B) \sqcup_{\mathcal{E}} (\mathcal{H}, C) = (\mathcal{N}, A \cup B \cup C)$, where for all $e \in A \cup B \cup C$,
$\mathcal{N}(e) = \mathcal{K}(e)$, if $e \in (A \cup B) \setminus C = \{A \setminus (B \cup C)\} \cup \{B \setminus (A \cup C)\} \cup \{(A \cap B) \setminus C\}$,
$= \mathcal{H}(e)$, if $e \in C \setminus (A \cup B)$,
$= \mathcal{K}(e) \cup \mathcal{H}(e)$, if $e \in (A \cup B) \cap C = \{A \cap B \cap C\} \cup \{(A \setminus B) \cap C\} \cup \{(B \setminus A) \cap C\}$.

Now we decompose $(A \cup B \cup C)$ as a union of pairwise disjoint sets,
$(A \cup B \cup C) = \{A \setminus (B \cup C)\} \cup \{B \setminus (A \cup C)\} \cup \{C \setminus (A \cup B)\} \cup \{(A \cap B) \setminus C\} \cup \{(B \cap C) \setminus A\} \cup \{(C \cap A) \setminus B\} \cup \{A \cap B \cap C\}$.
Now $e \in (A \cup B \cup C)$ means e belongs to one of the disjoint sets.
(i) If $e \in A \setminus (B \cup C)$, then $\mathcal{M}(e) = \mathcal{F}(e) = \mathcal{N}(e)$;
(ii) If $e \in B \setminus (A \cup C)$, then $\mathcal{M}(e) = \mathcal{G}(e) = \mathcal{N}(e)$;
(iii) If $e \in C \setminus (A \cup B)$, then $\mathcal{M}(e) = \mathcal{H}(e) = \mathcal{N}(e)$;
(iv) If $e \in (A \cap B) \setminus C = A \cap (B \setminus C)$, then $\mathcal{M}(e) = \mathcal{F}(e) \cup \mathcal{G}(e) = \mathcal{N}(e)$;
(v) If $(B \cap C) \setminus A = (B \setminus A) \cap C$, then $\mathcal{M}(e) = \mathcal{G}(e) \cup \mathcal{H}(e) = \mathcal{N}(e)$;
(vi) If $(C \cap A) \setminus B = A \cap (C \setminus B) = (A \setminus B) \cap C$, then $\mathcal{M}(e) = \mathcal{H}(e) \cup \mathcal{F}(e) = \mathcal{N}(e)$;
(vii) If $e \in (A \cap B \cap C)$, then $\mathcal{M}(e) = \mathcal{F}(e) \cup \mathcal{G}(e) \cup \mathcal{H}(e) = \mathcal{N}(e)$.
So, for all $e \in (A \cup B \cup C)$ we have $\mathcal{M}(e) = \mathcal{N}(e)$.
Therefore $(\mathcal{F}, A) \sqcup_{\mathcal{E}} ((\mathcal{G}, B) \sqcup_{\mathcal{E}} (\mathcal{H}, C)) = ((\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, B)) \sqcup_{\mathcal{E}} (\mathcal{H}, C)$.
(5) Suppose that $(\mathcal{G}, B) \wedge (\mathcal{H}, C) = (\mathcal{I}, B \times C)$, where $\mathcal{I}(y, z) = \mathcal{G}(y) \cap \mathcal{H}(z)$ for all $(y, z) \in B \times C$ and let $(\mathcal{F}, A) \wedge (\mathcal{I}, B \times C) = (\mathcal{J}, A \times B \times C)$, where $\mathcal{J}(x, y, z) = \mathcal{F}(x) \cap \mathcal{I}(y, z) = \mathcal{F}(x) \cap (\mathcal{G}(y) \cap \mathcal{H}(z))$ for all $(x, y, z) \in A \times B \times C$.
Now, let $((\mathcal{F}, A) \wedge (\mathcal{G}, B)) \wedge (\mathcal{H}, C) = (\mathcal{K}, A \times B \times C)$. Then similarly we can show that $\mathcal{K}(x, y, z) = (\mathcal{F}(x) \cap \mathcal{G}(y)) \cap \mathcal{H}(z)$ for all $(x, y, z) \in A \times B \times C$. By using Proposition 4.1, we have $\mathcal{J}(x, y, z) = \mathcal{K}(x, y, z)$ for all $(x, y, z) \in A \times B \times C$.

Therefore $(\mathcal{F},A)\wedge((\mathcal{G},B)\wedge(\mathcal{H},C))=((\mathcal{F},A)\wedge(\mathcal{G},B))\wedge(\mathcal{H},C)$.
(6) Proof is similar as in (5). □

Theorem 4.3. *Left distributive properties:*
(1) $(\mathcal{F},A)\Cap((\mathcal{G},B)\Cup(\mathcal{H},C))=((\mathcal{F},A)\Cap(\mathcal{G},B))\Cup((\mathcal{F},A)\Cap(\mathcal{H},C))$;
(2) $(\mathcal{F},A)\Cap((\mathcal{G},B)\sqcup_{\mathcal{E}}(\mathcal{H},C))=((\mathcal{F},A)\Cap(\mathcal{G},B))\sqcup_{\mathcal{E}}((\mathcal{F},A)\Cap(\mathcal{H},C))$;
(3) $(\mathcal{F},A)\Cup((\mathcal{G},B)\Cap(\mathcal{H},C))=((\mathcal{F},A)\Cup(\mathcal{G},B))\Cap((\mathcal{F},A)\Cup(\mathcal{H},C))$;
(4) $(\mathcal{F},A)\sqcup_{\mathcal{E}}((\mathcal{G},B)\Cap(\mathcal{H},C))=((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{G},B))\Cap((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{H},C))$;
(5) $(\mathcal{F},A)\sqcap_{\mathcal{E}}((\mathcal{G},B)\Cup(\mathcal{H},C))=((\mathcal{F},A)\sqcap_{\mathcal{E}}(\mathcal{G},B))\Cup((\mathcal{F},A)\sqcap_{\mathcal{E}}(\mathcal{H},C))$;
(6) $(\mathcal{F},A)\Cup((\mathcal{G},B)\sqcap_{\mathcal{E}}(\mathcal{H},C))=((\mathcal{F},A)\Cup(\mathcal{G},B))\sqcap_{\mathcal{E}}((\mathcal{F},A)\Cup(\mathcal{H},C))$.

Proof. (1) It can be proved as a particular case of (2). So, we omit the proof.
(2) At first suppose that $(\mathcal{G},B)\sqcup_{\mathcal{E}}(\mathcal{H},C)=(\mathcal{P},B\cup C)$, where for all $e\in B\cup C$,

$$\begin{aligned}\mathcal{P}(e)&=\mathcal{G}(e), && \text{if } e\in B\setminus C,\\ &=\mathcal{H}(e), && \text{if } e\in C\setminus B,\\ &=\mathcal{G}(e)\cup\mathcal{H}(e), && \text{if } e\in B\cap C.\end{aligned}$$

And let $(\mathcal{F},A)\Cap(\mathcal{P},B\cup C)=(\mathcal{Q},A\cap(B\cup C))$ where for all $e\in A\cap(B\cup C)$, $\mathcal{Q}(e)=\mathcal{F}(e)\cap\mathcal{P}(e)$ i.e.,

$$\begin{aligned}\mathcal{Q}(e)&=\mathcal{F}(e)\cap\mathcal{G}(e), && \text{if } e\in A\cap(B\setminus C),\\ &=\mathcal{F}(e)\cap\mathcal{H}(e), && \text{if } e\in A\cap(C\setminus B),\\ &=\mathcal{F}(e)\cap[\mathcal{G}(e)\cup\mathcal{H}(e)], && \text{if } e\in A\cap B\cap C.\end{aligned}$$

Now we consider the right hand side of the equality and let $(\mathcal{F},A)\Cap(\mathcal{G},B)=(\mathcal{L},A\cap B)$, $(\mathcal{F},A)\Cap(\mathcal{H},C)=(\mathcal{M},A\cap C)$, where
$\mathcal{L}(e)=\mathcal{F}(e)\cap\mathcal{G}(e)$ for all $e\in A\cap B$,
$\mathcal{M}(e)=\mathcal{F}(e)\cap\mathcal{H}(e)$ for all $e\in A\cap C$.
Again let $(\mathcal{L},A\cap B)\sqcup_{\mathcal{E}}(\mathcal{M},A\cap C)=(\mathcal{T},(A\cap B)\cup(A\cap C))$, where for all $e\in(A\cap B)\cup(A\cap C)$,

$$\begin{aligned}\mathcal{T}(e)&=\mathcal{L}(e), && \text{if } e\in(A\cap B)\setminus(A\cap C),\\ &=\mathcal{M}(e), && \text{if } e\in(A\cap C)\setminus(A\cap B),\\ &=\mathcal{L}(e)\cup\mathcal{M}(e), && \text{if } e\in A\cap B\cap C.\end{aligned}$$

We note that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ and decompose $A\cap(B\cup C)$ as an union of pairwise disjoint sets, $A\cap(B\cup C)=[A\cap(B\setminus C)]\cup[A\cap(C\setminus B)]\cup[A\cap B\cap C]$.
Now $e\in A\cap(B\cup C)$ means e belongs to one of the disjoint sets.
(i) If $e\in A\cap(B\setminus C)=(A\cap B)\setminus(A\cap C)$, then $\mathcal{T}(e)=\mathcal{L}(e)=\mathcal{F}(e)\cap\mathcal{G}(e)=\mathcal{Q}(e)$;
(ii) If $e\in A\cap(C\setminus B)=(A\cap C)\setminus(A\cap B))$, then $\mathcal{T}(e)=\mathcal{M}(e)=$

$\mathcal{F}(e)\cap\mathcal{H}(e)=\mathcal{Q}(e)$;
(iii) If $e\in A\cap B\cap C$, then $\mathcal{T}(e)=\mathcal{L}(e)\cup\mathcal{M}(e)=[\mathcal{F}(e)\cap\mathcal{G}(e)]\cup[\mathcal{F}(e)\cap\mathcal{H}(e)]=\mathcal{F}(e)\cap[\mathcal{G}(e)\cup\mathcal{H}(e)]=\mathcal{Q}(e)$, using Proposition 4.1.
Therefore $(\mathcal{Q},A\cap(B\cup C))=(\mathcal{T},(A\cap B)\cup(A\cap C))$. This implies that $(\mathcal{F},A)\Cap((\mathcal{G},B)\sqcup_{\mathcal{E}}(\mathcal{H},C))=((\mathcal{F},A)\Cap(\mathcal{G},B))\sqcup_{\mathcal{E}}((\mathcal{F},A)\Cap(\mathcal{H},C))$.
(3) It can be proved as a particular case of (4). So, we omit the proof.
(4) At first suppose that $(\mathcal{G},B)\Cap(\mathcal{H},C)=(\mathcal{P},B\cap C)$, where for all $e\in B\cap C$, $\mathcal{P}(e)=\mathcal{G}(e)\cap\mathcal{H}(e)$. And let $(\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{P},B\cap C)=(\mathcal{Q},A\cup(B\cap C)$, where for all $e\in A\cup(B\cap C)$,

$$\begin{aligned}\mathcal{Q}(e)&=\mathcal{F}(e), && \text{if } e\in A\setminus(B\cap C),\\ &=\mathcal{P}(e)=\mathcal{G}(e)\cap\mathcal{H}(e), && \text{if } e\in(B\cap C)\setminus A,\\ &=\mathcal{F}(e)\cup\mathcal{P}(e)=\mathcal{F}(e)\cup[\mathcal{G}(e)\cap\mathcal{H}(e)], && \text{if } e\in A\cap B\cap C.\end{aligned}$$

Now consider the right hand side of the equality and let $(\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{G},B)=(\mathcal{L},A\cup B)$, $(\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{H},C)=(\mathcal{M},A\cup C)$, where for all $e\in A\cup B$,

$$\begin{aligned}\mathcal{L}(e)&=\mathcal{F}(e), && \text{if } e\in A\setminus B,\\ &=\mathcal{G}(e), && \text{if } e\in B\setminus A,\\ &=\mathcal{F}(e)\cup\mathcal{G}(e), && \text{if } e\in A\cap B.\end{aligned}$$

And for all $e\in A\cup C$,

$$\begin{aligned}\mathcal{M}(e)&=\mathcal{F}(e), && \text{if } e\in A\setminus C,\\ &=\mathcal{H}(e), && \text{if } e\in C\setminus A,\\ &=\mathcal{F}(e)\cup\mathcal{H}(e), && \text{if } e\in A\cap C.\end{aligned}$$

Again let, $(\mathcal{L},A\cup B)\Cap(\mathcal{M},A\cup C)=(\mathcal{T},(A\cup B)\cap(A\cup C))$, where for all $e\in(A\cup B)\cap(A\cup C)=A\cup(B\cap C)$, $\mathcal{T}(e)=\mathcal{L}(e)\cap\mathcal{M}(e)$.
Now we decompose $A\cup(B\cap C)$ as an union of pairwise disjoint sets, $A\cup(B\cap C)=[A\setminus(B\cap C)]\cup[(B\cap C)\setminus A]\cup[A\cap B\cap C]$.
So, $e\in A\cup(B\cap C)$ means e belongs to one of the disjoint sets.
(i) If $e\in A\setminus(B\cap C)=(A\setminus B)\cup(A\setminus C)$, then $\mathcal{T}(e)=\mathcal{L}(e)\cap\mathcal{M}(e)=\mathcal{F}(e)\cap\mathcal{F}(e)=\mathcal{F}(e)=\mathcal{Q}(e)$;
(ii) If $(B\cap C)\setminus A=(B\setminus A)\cap(C\setminus A)$, then $\mathcal{T}(e)=\mathcal{L}(e)\cap\mathcal{M}(e)=\mathcal{G}(e)\cap\mathcal{H}(e)=\mathcal{Q}(e)$;
(iii) If $e\in A\cap B\cap C$, then $\mathcal{T}(e)=\mathcal{L}(e)\cap\mathcal{M}(e)=[\mathcal{F}(e)\cup\mathcal{G}(e)]\cap[\mathcal{F}(e)\cup\mathcal{H}(e)]$ (by Proposition 4.1) $=\mathcal{F}(e)\cup[\mathcal{G}(e)\cap\mathcal{H}(e)]=\mathcal{Q}(e)$.
This shows that $(\mathcal{Q},A\cup(B\cap C))=(\mathcal{T},(A\cup B)\cap(A\cup C))$. Therefore $(\mathcal{F},A)\sqcup_{\mathcal{E}}((\mathcal{G},B)\Cap(\mathcal{H},C))=((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{G},B))\Cap((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{H},C))$.
(5) Here we can use similar technique as in (4). So, we omit the proof.
(6) Here we can use similar technique as in (2). So, we omit the proof.
□

Note 4.4. *In general $\Cap$ is not left distributive over $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{E}}$. Then we state the following Theorem.*

Theorem 4.5.
(1) $(\mathcal{F},A)\Cap((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))\subseteq((\mathcal{F},A)\Cap(\mathcal{G},B))\sim_{\mathcal{R}}((\mathcal{F},A)\Cap(\mathcal{H},C))$;
(2) $(\mathcal{F},A)\Cap((\mathcal{G},B)\sim_{\mathcal{E}}(\mathcal{H},C))\subseteq((\mathcal{F},A)\Cap(\mathcal{G},B))\sim_{\mathcal{E}}((\mathcal{F},A)\Cap(\mathcal{H},C))$.

Proof. (1) It can be proved as a particular case of (2). So, we omit this part.
(2) Suppose that $(\mathcal{G},B)\sim_{\mathcal{E}}(\mathcal{H},C)=(\mathcal{L},B\cup C)$, where

$$\begin{aligned}\mathcal{L}(e)&=\mathcal{G}(e), && \text{if } e\in B\setminus C,\\ &=\Phi_e, && \text{if } e\in C\setminus B,\\ &=\mathcal{G}(e)\cap\mathcal{H}^c(e), && \text{if } e\in B\cap C.\end{aligned}$$

Now, let $(\mathcal{F},A)\Cap(\mathcal{L},B\cup C)=(\mathcal{W},A\cap(B\cup C))$, where $\mathcal{W}(e)=\mathcal{F}(e)\cap\mathcal{L}(e)$ for all $e\in A\cap(B\cup C)$ i.e.

$$\begin{aligned}\mathcal{W}(e)&=\mathcal{F}(e)\cap\mathcal{G}(e), && \text{if } e\in A\cap(B\setminus C),\\ &=F(x)\cap\Phi_e=\Phi_e, && \text{if } e\in A\cap(C\setminus B),\\ &=\mathcal{F}(e)\cap[\mathcal{G}(e)\cap\mathcal{H}^c(e)], && \text{if } e\in A\cap(B\cap C).\end{aligned}$$

Now suppose $(\mathcal{F},A)\Cap(\mathcal{G},B)=(\mathcal{M},A\cap B)$ and $(\mathcal{F},A)\Cap(\mathcal{H},C)=(\mathcal{N},A\cap C)$, where $\mathcal{M}(e)=\mathcal{F}(e)\cap\mathcal{G}(e)$ for all $e\in A\cap B$ and $\mathcal{N}(e)=\mathcal{F}(e)\cap\mathcal{H}(e)$ for all $e\in A\cap C$. Again, let $(\mathcal{M},A\cap B)\sim_{\mathcal{E}}(\mathcal{N},A\cap C)=(\mathcal{K},(A\cap B)\cup(A\cap C))$, where

$$\begin{aligned}\mathcal{K}(e)&=\mathcal{M}(e), && \text{if } e\in(A\cap B)\setminus(A\cap C),\\ &=\Phi_e, && \text{if } e\in(A\cap C)\setminus(A\cap B),\\ &=\mathcal{M}(e)\cap\mathcal{N}^c(e), && \text{if } e\in(A\cap B)\cap(A\cap C)=A\cap B\cap C.\end{aligned}$$

Hence, for all $e\in(A\cap B)\cup(A\cap C)=A\cap(B\cup C)$,
(i) If $e\in(A\cap B)\setminus(A\cap C)=A\cap(B\setminus C)$, $\mathcal{K}(e)=\mathcal{M}(e)=\mathcal{F}(e)\cap\mathcal{G}(e)=\mathcal{W}(e)$;
(ii) If $e\in(A\cap C)\setminus(A\cap B)=A\cap(C\setminus B)$, $\mathcal{K}(e)=\Phi_e=\mathcal{W}(e)$;
(iii) If $e\in A\cap B\cap C$, $\mathcal{K}(e)=\mathcal{M}(e)\cap\mathcal{N}^c(e)=[\mathcal{F}(e)\cap\mathcal{G}(e)]\cap[\mathcal{F}(e)\cap\mathcal{H}(e)]^c$ $=[\mathcal{F}(e)\cap\mathcal{G}(e)]\cap[\mathcal{F}^c(e)\cup\mathcal{H}^c(e)]=[\mathcal{F}(e)\cap\mathcal{G}(e)\cap\mathcal{F}^c(e)]\cup[\mathcal{F}(e)\cap\mathcal{G}(e)\cap\mathcal{H}^c(e)]\supseteq\mathcal{F}(e)\cap\mathcal{G}(e)\cap\mathcal{H}^c(e)=\mathcal{W}(e)$.
Therefore $(\mathcal{K},(A\cap B)\cup(A\cap C))\supseteq(\mathcal{W},A\cap(B\cup C))$.
Hence $(\mathcal{F},A)\Cap((\mathcal{G},B)\sim_{\mathcal{E}}(\mathcal{H},C))\subseteq((\mathcal{F},A)\Cap(\mathcal{G},B))\sim_{\mathcal{E}}((\mathcal{F},A)\Cap(\mathcal{H},C))$. □

Theorem 4.6. *Right distributive properties:*
(1) $((\mathcal{F},A)\Cup(\mathcal{G},B))\Cap(\mathcal{H},C)=((\mathcal{F},A)\Cap(\mathcal{H},C))\Cup((\mathcal{G},B)\Cap(\mathcal{H},C))$;
(2) $((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{G},B))\Cap(\mathcal{H},C)=((\mathcal{F},A)\Cap(\mathcal{H},C))\sqcup_{\mathcal{E}}((\mathcal{G},B)\Cap(\mathcal{H},C))$;
(3) $((\mathcal{F},A)\Cap(\mathcal{G},B))\Cup(\mathcal{H},C)=((\mathcal{F},A)\Cup(\mathcal{H},C))\Cap((\mathcal{G},B)\Cup(\mathcal{H},C))$;

(4) $((\mathcal{F},A)\Cap(\mathcal{G},B))\sqcup_{\mathcal{E}}(\mathcal{H},C)=((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{H},C))\Cap((\mathcal{G},B)\sqcup_{\mathcal{E}}(\mathcal{H},C))$;
(5) $((\mathcal{F},A)\Cup(\mathcal{G},B))\sqcap_{\mathcal{E}}(\mathcal{H},C)=((\mathcal{F},A)\sqcap_{\mathcal{E}}(\mathcal{H},C))\Cup((\mathcal{G},B)\sqcap_{\mathcal{E}}(\mathcal{H},C))$;
(6) $((\mathcal{F},A)\sqcap_{\mathcal{E}}(\mathcal{G},B))\Cup(\mathcal{H},C)=((\mathcal{F},A)\Cup(\mathcal{H},C))\sqcap_{\mathcal{E}}((\mathcal{G},B)\Cup(\mathcal{H},C))$;
(7) $((\mathcal{F},A)\Cup(\mathcal{G},B))\sim_{\mathcal{R}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C))\Cup((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))$;
(8) $((\mathcal{F},A)\sqcup_{\mathcal{E}}(\mathcal{G},B))\sim_{\mathcal{R}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C))\sqcup_{\mathcal{E}}((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))$;
(9) $((\mathcal{F},A)\Cap(\mathcal{G},B))\sim_{\mathcal{R}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C))\Cap((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))$;
(10) $((\mathcal{F},A)\sqcap_{\mathcal{E}}(\mathcal{G},B))\sim_{\mathcal{R}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C))\sqcap_{\mathcal{E}}((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))$;
(11) $((\mathcal{F},A)\Cup(\mathcal{G},B))\sim_{\mathcal{E}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{E}}(\mathcal{H},C))\Cup((\mathcal{G},B)\sim_{\mathcal{E}}(\mathcal{H},C))$;
(12) $((\mathcal{F},A)\Cap(\mathcal{G},B))\sim_{\mathcal{E}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{E}}(\mathcal{H},C))\Cap((\mathcal{G},B)\sim_{\mathcal{E}}(\mathcal{H},C))$;

Proof. (1)-(6) By using similar technique as in Theorem 4.3, the proofs can be established.
(7) Let $(\mathcal{F},A)\Cup(\mathcal{G},B)=(\mathcal{L}_1,A\cap B)$, where $\mathcal{L}_1(e)=\mathcal{F}(e)\cup\mathcal{G}(e)$ for all $e\in A\cap B$. Now let $(\mathcal{L}_1,A\cap B)\sim_{\mathcal{R}}(\mathcal{H},C)=(\mathcal{L}_2,A\cap B\cap C)$, where $\mathcal{L}_2(e)=\mathcal{L}_1(e)\cap\mathcal{H}^c(e)=[\mathcal{F}(e)\cup\mathcal{G}(e)]\cap\mathcal{H}^c(e)$ for all $e\in A\cap B\cap C$.
Now we consider the right hand side of the equality and let $(\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C)=(\mathcal{M}_1,A\cap C)$, $(\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C)=(\mathcal{M}_2,B\cap C)$, where $\mathcal{M}_1(e)=\mathcal{F}(e)\cap\mathcal{H}^c(e)$ for all $e\in A\cap C$ and $\mathcal{M}_2(e)=\mathcal{G}(e)\cap\mathcal{H}^c(e)$ for all $e\in B\cap C$. Now let $(\mathcal{M}_1,A\cap C)\Cup(\mathcal{M}_2,B\cap C)=(\mathcal{M}_3,A\cap B\cap C)$, where $\mathcal{M}_3(e)=\mathcal{M}_1(e)\cup\mathcal{M}_2(e)=[\mathcal{F}(e)\cap\mathcal{H}^c(e)]\cup[\mathcal{G}(e)\cap\mathcal{H}^c(e)]=[\mathcal{F}(e)\cup\mathcal{G}(e)]\cap\mathcal{H}^c(e)$ for all $e\in A\cap B\cap C$. This shows that $(\mathcal{L}_2,A\cap B\cap C)=(\mathcal{M}_3,A\cap B\cap C)$. Therefore $((\mathcal{F},A)\Cup(\mathcal{G},B))\sim_{\mathcal{R}}(\mathcal{H},C)=((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{H},C))\Cup((\mathcal{G},B)\sim_{\mathcal{R}}(\mathcal{H},C))$.
(8)-(10) The proofs can be established similar to (7) and using similar technique as in Theorem 3.9. So we omit the proofs.
(11)-(12) It can be proved similar to (7), (9) and using Definition 2.26. □

Note 4.7. *In general $\Cap$ is not right distributive over $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{E}}$. Then we state the following Theorem.*

Theorem 4.8.
(1) $((\mathcal{F},A)\sim_{\mathcal{R}}(\mathcal{G},B))\Cap(\mathcal{H},C)\subseteq((\mathcal{F},A)\Cap(\mathcal{H},C))\sim_{\mathcal{R}}((\mathcal{G},B)\Cap(\mathcal{H},C))$;
(2) $((\mathcal{F},A)\sim_{\mathcal{E}}(\mathcal{G},B))\Cap(\mathcal{H},C)\subseteq((\mathcal{F},A)\Cap(\mathcal{H},C))\sim_{\mathcal{E}}((\mathcal{G},B)\Cap(\mathcal{H},C))$.

Proof. It can be easily proved by using similar technique as in Theorem 4.5. So we omit the proofs. □

Now we state some basic properties of operation on IFSS theory, which was proposed on soft set theory by some authors in [7, 8].

Theorem 4.9. *Properties of the union operations:*
(1) $(\mathcal{F}, A) \Cup \mathcal{U}_A = \mathcal{U}_A$;
(2) $(\mathcal{F}, A) \sqcup_{\mathcal{E}} \mathcal{U}_A = \mathcal{U}_A$;
(3) $(\mathcal{F}, A) \Cup \Phi_A = (\mathcal{F}, A)$;
(4) $(\mathcal{F}, A) \sqcup_{\mathcal{E}} \Phi_A = (\mathcal{F}, A)$;
(5) $(\mathcal{F}, A) \Cup (\mathcal{G}, A) = \Phi_A \Leftrightarrow (\mathcal{F}, A) = \Phi_A$ *and* $(\mathcal{G}, A) = \Phi_A$;
(6) $(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, A) = \Phi_A \Leftrightarrow (\mathcal{F}, A) = \Phi_A$ *and* $(\mathcal{G}, A) = \Phi_A$;
(7) $(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{G}, A) = (\mathcal{F}, A) \Cup (\mathcal{G}, A)$.

Proof. (5) $(\mathcal{F}, A) \Cup (\mathcal{G}, A) = \Phi_A \Leftrightarrow \forall e \in A, \mathcal{F}(e) \cup \mathcal{G}(e) = (\widetilde{0}, \widetilde{1})$, (by Definition 2.17) $\Leftrightarrow \forall e \in A, \mathcal{F}(e) = (\widetilde{0}, \widetilde{1})$ and $\mathcal{G}(e) = (\widetilde{0}, \widetilde{1})$, (by Definition 2.12, 2.20)
$\Leftrightarrow (\mathcal{F}, A) = \Phi_A$ and $(\mathcal{G}, A) = \Phi_A$.
All other parts are easy to prove using Definitions 2.20, 2.21, 2.17, 2.18. So we omit the proofs. □

Theorem 4.10. *Properties of the intersection operations:*
(1) $(\mathcal{F}, A) \Cap \mathcal{U}_A = (\mathcal{F}, A)$;
(2) $(\mathcal{F}, A) \sqcap_{\mathcal{E}} \mathcal{U}_A = (\mathcal{F}, A)$;
(3) $(\mathcal{F}, A) \Cap \Phi_A = \Phi_A$;
(4) $(\mathcal{F}, A) \sqcap_{\mathcal{E}} \Phi_A = \Phi_A$;
(5) $(\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{G}, A) = (\mathcal{F}, A) \Cap (\mathcal{G}, A)$.

Proof. All the parts can be easily proved using Definitions 2.20, 2.21, 2.15, 2.16. So we omit the proofs. □

Theorem 4.11. *Properties of complement, restricted difference, restricted symmetric difference operations:*
(1) $(\mathcal{U}_A)^c = \Phi_A, (\Phi_A)^c = \mathcal{U}_A$;
(2) $(\mathcal{F}, A) \sim_{\mathcal{R}} \Phi_A = (\mathcal{F}, A)$;
(3) $\Phi_A \sim_{\mathcal{R}} (\mathcal{F}, A) = \Phi_A$;
(4) $\mathcal{U}_A \sim_{\mathcal{R}} (\mathcal{F}, A) = (\mathcal{F}, A)^c$;
(5) $(\mathcal{F}, A) \widetilde{\triangle} \Phi_A = (\mathcal{F}, A)$;
(6) $(\mathcal{F}, A) \widetilde{\triangle} (\mathcal{F}, A) = (\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A)$;
(7) $(\mathcal{F}, A) \widetilde{\triangle} (\mathcal{G}, B) = (\mathcal{G}, B) \widetilde{\triangle} (\mathcal{F}, A)$.

Proof. (1) It can be proved easily using Definitions 2.20, 2.21, 2.19. So we omit this proof.
(2) Let $(\mathcal{F}, A) \sim_{\mathcal{R}} \Phi_A = (\mathcal{H}, A)$. Then for all $e \in A$, $\mathcal{H}(e) = \mathcal{F}(e) \cap (\widetilde{0}, \widetilde{1})^c = \mathcal{F}(e) \cap (\widetilde{1}, \widetilde{0}) = \mathcal{F}(e)$. Hence $(\mathcal{F}, A) \sim_{\mathcal{R}} \Phi_A = (\mathcal{F}, A)$.
(3) Similar proof as in (2).
(4) Let $\mathcal{U}_A \sim_{\mathcal{R}} (\mathcal{F}, A) = (\mathcal{H}, A)$. Then for all $e \in A$, $\mathcal{H}(e) = (\widetilde{1}, \widetilde{0}) \cap \mathcal{F}^c(e) = \mathcal{F}^c(e)$. Hence $\mathcal{U}_A \sim_{\mathcal{R}} (\mathcal{F}, A) = (\mathcal{F}, A)^c$.
(5) By Definition 2.25, $(\mathcal{F}, A) \widetilde{\triangle} \Phi_A = ((\mathcal{F}, A) \sim_{\mathcal{R}} \Phi_A) \Cup (\Phi_A \sim_{\mathcal{R}} (\mathcal{F}, A)) = (\mathcal{F}, A) \Cup \Phi_A = (\mathcal{F}, A)$.
(6) $(\mathcal{F}, A) \widetilde{\triangle} (\mathcal{F}, A) = ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A)) \Cup ((\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A)) = (\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A)$.
(7) It is easy to prove using Definition 2.25. □

Theorem 4.12. *Let (F, A), (G, B), (H, C) be any three soft sets over U. Then we have the following:*
(1) $(F, A) \sqcup_{\mathcal{E}} (F, A)^c = (F, A) \Cup (F, A)^c = \mathcal{U}_A$ [8];
(2) $(F, A) \sqcap_{\mathcal{E}} (F, A)^c = (F, A) \Cap (F, A)^c = \Phi_A$ [8];
(3) $(F, A) \sim_{\mathcal{R}} (F, A) = (F, A) \widetilde{\triangle} (F, A) = \Phi_A$ [8];
(4) $(F, A) \widetilde{\triangle} ((G, B) \widetilde{\triangle} (H, C)) = ((F, A) \widetilde{\triangle} (G, B)) \widetilde{\triangle} (H, C)$;
(5) $(F, A) \Cap ((G, B) \widetilde{\triangle} (H, C)) = ((F, A) \Cap (G, B)) \widetilde{\triangle} ((F, A) \Cap (H, C))$;
(6) $((F, A) \widetilde{\triangle} (G, B)) \Cap (H, C) = ((F, A) \Cap (H, C)) \widetilde{\triangle} (G, B))(\Cap (H, C))$.

Proof. (4) At first we consider the left hand side of the equality and let $(G, B) \widetilde{\triangle} (H, C) = (L_1, B \cap C)$. By Definition 2.10, $(L_1, B \cap C) = ((G, B) \sim_{\mathcal{R}} (H, C)) \Cup ((H, C) \sim_{\mathcal{R}} (G, B))$. Using Definition 2.9, $L_1(e) = [G(e) \setminus H(e)] \cup [H(e) \setminus G(e)]$, for all $e \in B \cap C$.
Now let, $(F, A) \widetilde{\triangle} (L_1, B \cap C) = (L_2, A \cap B \cap C)$. Therefore $(L_2, A \cap B \cap C) = ((F, A) \sim_{\mathcal{R}} (L_1, B \cap C)) \Cup ((L_1, B \cap C) \sim_{\mathcal{R}} (F, A))$.
Hence for all $e \in A \cap B \cap C$, $L_2(e) = [F(e) \setminus L_1(e)] \cup [L_1(e) \setminus F(e)] = [F(e) \setminus \{(G(e) \setminus H(e)) \cup (H(e) \setminus G(e))\}] \cup [\{(G(e) \setminus H(e)) \cup (H(e) \setminus G(e))\} \setminus F(e)] = [F(e) \setminus \{(G(e) \cap H^c(e)) \cup (H(e) \cap G^c(e))\}] \cup [\{(G(e) \cap H^c(e)) \cup (H(e) \cap G^c(e))\} \setminus F(e)]$
$= [F(e) \cap \{(G(e) \cap H^c(e))^c \cap (H(e) \cap G^c(e))^c\}] \cup [\{(G(e) \cap H^c(e)) \cup (H(e) \cap G^c(e))\} \cap F^c(e)] = [F(e) \cap \{(G^c(e) \cup H(e)) \cap (H^c(e) \cup G(e))\}] \cup [\{(G(e) \cap H^c(e)) \cup (H(e) \cap G^c(e))\} \cap F^c(e)] = [F(e) \cap G^c(e) \cap H^c(e)] \cup [F(e) \cap G^c(e) \cap G(e)] \cup [F(e) \cap H(e) \cap H^c(e)] \cup [F(e) \cap H(e) \cap G(e)] \cup [G(e) \cap H^c(e) \cap F^c(e)] \cup [H(e) \cap G^c(e) \cap F^c(e)] = [F(e) \cap G^c(e) \cap H^c(e)] \cup [F(e) \cap H(e) \cap G(e)] \cup [G(e) \cap H^c(e) \cap F^c(e)] \cup [H(e) \cap G^c(e) \cap F^c(e)]$, (since for any crisp subset A of U, $A \cap A^c = \phi$).

Now we consider the right hand side of the equality and let $(F,A)\,\widetilde{\triangle}\,(G,B) = (M_1, A\cap B)$. By Definition 2.10, $(M_1, A\cap B) = ((F,A) \sim_{\mathcal{R}} (G,B)) \Cup ((G,B) \sim_{\mathcal{R}} (F,A))$. Again using Definition 2.9, $M_1(e) = [F(e)\setminus G(e)] \cup [G(e)\setminus F(e)]$, for all $e \in A\cap B$. Now let, $(M_1, A\cap B)\,\widetilde{\triangle}\,(H,C) = (M_2, A\cap B\cap C)$. Therefore $(M_2, A\cap B\cap C) = ((M_1, A\cap B) \sim_{\mathcal{R}} (H,C)) \Cup ((H,C) \sim_{\mathcal{R}} (M_1, A\cap B))$. Hence $\forall\, e \in A\cap B\cap C$, $M_2(e) = [M_1(e)\setminus H(e)] \cup [H(e)\setminus M_1(e)] = [\{(F(e)\setminus G(e)) \cup (G(e)\setminus F(e))\} \setminus H(e)]$
$\cup\, [H(e) \setminus \{(F(e)\setminus G(e)) \cup (G(e)\setminus F(e))\}]$
$= [\{(F(e)\cap G^c(e)) \cup (G(e)\cap F^c(e))\} \cap H^c(e)] \cup [H(e) \cap \{(F(e)\cap G^c(e)) \cup (G(e)\cap F^c(e))\}^c]$
$= [(F(e)\cap G^c(e)\cap H^c(e)] \cup [G(e)\cap F^c(e)\cap H^c(e)] \cup [H(e)\cap \{(F^c(e) \cup G(e)) \cap (G^c(e)\cup F(e))\}]$
$= [(F(e)\cap G^c(e)\cap H^c(e)] \cup [G(e)\cap F^c(e)\cap H^c(e)] \cup [H(e)\cap F^c(e)\cap G^c(e)] \cup [H(e)\cap F^c(e)\cap F(e)] \cup [H(e)\cap G(e)\cap G^c(e)] \cup [H(e)\cap G(e)\cap F(e)]$
$= [(F(e)\cap G^c(e)\cap H^c(e)] \cup [G(e)\cap F^c(e)\cap H^c(e)] \cup [H(e)\cap F^c(e)\cap G^c(e)] \cup [H(e)\cap G(e)\cap F(e)]$.
It clearly shows that $(L_2, A\cap B\cap C) = (M_2, A\cap B\cap C)$.
Therefore $(F,A)\,\widetilde{\triangle}\,((G,B)\,\widetilde{\triangle}\,(H,C)) = ((F,A)\,\widetilde{\triangle}\,(G,B))\,\widetilde{\triangle}\,(H,C)$.
(5) At first we consider the left hand side of the equality and let $(G,B)\,\widetilde{\triangle}\,(H,C) = (L_1, B\cap C)$. By Definition 2.10, $(L_1, B\cap C) = ((G,B) \sim_{\mathcal{R}} (H,C)) \Cup ((H,C) \sim_{\mathcal{R}} (G,B))$. Using Definition 2.9, $L_1(e) = [G(e)\setminus H(e)] \cup [H(e)\setminus G(e)] = [G(e)\cap H^c(e)] \cup [H(e)\cap G^c(e)]$, for all $e \in B\cap C$. Now let, $(F,A) \Cap (L_1, B\cap C) = (L_2, A\cap B\cap C)$, where $L_2(e) = F(e) \cap \{[G(e)\cap H^c(e)] \cup [H(e)\cap G^c(e)]\} = [F(e)\cap G(e)\cap H^c(e)] \cup [F(e)\cap H(e)\cap G^c(e)]$, for all $e \in A\cap B\cap C$.
Now we consider the right hand side of the equality and let $(F,A) \Cap (G,B) = (M_1, A\cap B)$ and $(F,A) \Cap (H,C) = (M_2, A\cap C)$, where $M_1(e) = F(e)\cap G(e)$, for all $e \in A\cap B$ and $M_2(e) = F(e)\cap H(e)$, for all $e \in A\cap C$. Now let, $(M_1, A\cap B)\,\widetilde{\triangle}\,(M_2, A\cap C) = (M_3, A\cap B\cap C)$. Then $(M_3, A\cap B\cap C) = [(M_1, A\cap B) \sim_{\mathcal{R}} (M_2, A\cap C)] \Cup [(M_2, A\cap C) \sim_{\mathcal{R}} (M_1, A\cap B)]$. Hence for all $e \in A\cap B\cap C$, $M_3(e) =$
$[M_1(e)\setminus M_2(e)] \cup [M_2(e)\setminus M_1(e)] = [M_1(e)\cap M_2^c(e)] \cup [M_2(e)\cap M_1^c(e)]$
$= [\{F(e)\cap G(e)\} \cap \{F(e)\cap H(e)\}^c] \cup [\{F(e)\cap H(e)\} \cap \{F(e)\cap G(e)\}^c]$
$= [\{F(e)\cap G(e)\} \cap \{F^c(e)\cup H^c(e)\}] \cup [\{F(e)\cap H(e)\} \cap \{F^c(e)\cup G^c(e)\}]$
$= [\{F(e)\cap G(e)\cap F^c(e)] \cup [F(e)\cap G(e)\cap H^c(e)] \cup [F(e)\cap H(e)\cap F^c(e)] \cup [F(e)\cap H(e)\cap G^c(e)] = [F(e)\cap G(e)\cap H^c(e)] \cup [F(e)\cap H(e)\cap G^c(e)]$.
This shows that $(L_2, A\cap B\cap C) = (M_3, A\cap B\cap C)$. Therefore $(F,A) \Cap$

$((G,B)\,\widetilde{\triangle}\,(H,C)) = ((F,A) \Cap (G,B))\,\widetilde{\triangle}\,((F,A) \Cap (H,C))$.
(6) Proof is similar as (5). □

Note 4.13. *Properties of the operations in Theorem 4.12 are not true on IFSS. In support of this conclusion, we give an example below.*

Example 4.14. *As a continuation of Example 3.5 and 3.11, let* $(\mathcal{H}, C)$ *be another IFSS over the common universe* U, *where* $C = \{e_2, e_4\}$ *and* $\mathcal{H} : C \to I^U$ *is defined by:* $\mathcal{H}(e_2) = \{(h_1, 0.3, 0.7), (h_2, 0.1, 0.1), (h_3, 0.5, 0.4), (h_4, 0.9, 0.1)\}$; $\mathcal{H}(e_4) = \{(h_1, 0.5, 0.3), (h_2, 0.4, 0.5), (h_3, 0.3, 0.2), (h_4, 0.6, 0.2)\}$.
(1) Let $(\mathcal{F}, A) \Cup (\mathcal{F}, A)^c = (\mathcal{L}, A)$. *Then by Definition 2.17,*
$\mathcal{L}(e_1) = \mathcal{F}(e_1) \cup \mathcal{F}^c(e_1)$
$= \{(h_1, 0.3, 0.2), (h_2, 0.4, 0.3), (h_3, 0.8, 0.1), (h_4, 0.5, 0.5)\}$,
$\mathcal{L}(e_2) = \mathcal{F}(e_2) \cup \mathcal{F}^c(e_2)$
$= \{(h_1, 0.7, 0.2), (h_2, 0.2, 0.1), (h_3, 0.5, 0.4), (h_4, 0.3, 0.3)\}$.
This shows that $(\mathcal{F}, A) \Cup (\mathcal{F}, A)^c \neq \mathcal{U}_A$. *Hence by Theorem 4.9,*
$(\mathcal{F}, A) \sqcup_{\mathcal{E}} (\mathcal{F}, A)^c = (\mathcal{F}, A) \Cup (\mathcal{F}, A)^c \neq \mathcal{U}_A$.
(2) It is easy to check that $(\mathcal{F}, A) \sqcap_{\mathcal{E}} (\mathcal{F}, A)^c = (\mathcal{F}, A) \Cap (\mathcal{F}, A)^c \neq \Phi_A$.
(3) Let $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A) = (\mathcal{M}, A)$. *Then* $\mathcal{M}(e_1) = \mathcal{F}(e_1) \cap \mathcal{F}^c(e_1) = \{(h_1, 0.2, 0.3), (h_2, 0.3, 0.4), (h_3, 0.1, 0.8), (h_4, 0.5, 0.5)\}$, $\mathcal{M}(e_2) = \mathcal{F}(e_2) \cap \mathcal{F}^c(e_2) = \{(h_1, 0.2, 0.7), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$.
This shows that $(\mathcal{F}, A) \sim_{\mathcal{R}} (\mathcal{F}, A) \neq \Phi_A$. *Hence by Theorem 4.11,* $(\mathcal{F}, A)\,\widetilde{\triangle}\,(\mathcal{F}, A) \neq \Phi_A$.
(4) Now we shall establish that
$(\mathcal{F}, A)\,\widetilde{\triangle}\,((\mathcal{G}, B)\,\widetilde{\triangle}\,(\mathcal{H}, C)) \neq ((\mathcal{F}, A)\,\widetilde{\triangle}\,(\mathcal{G}, B))\,\widetilde{\triangle}\,(\mathcal{H}, C)$, *in general.*
Let $(\mathcal{G}, B)\,\widetilde{\triangle}\,(\mathcal{H}, C) = (\mathcal{L}_1, B \cap C)$, *where* $B \cap C = \{e_2\}$ *and by Definition 2.25,* $(\mathcal{L}_1, B \cap C) = ((\mathcal{G}, B) \sim_{\mathcal{R}} (\mathcal{H}, C)) \Cup ((\mathcal{H}, C) \sim_{\mathcal{R}} (\mathcal{G}, B))$.
Then $\mathcal{L}_1(e_2) = [\mathcal{G}(e_2) \cap \mathcal{H}^c(e_2)] \cup [\mathcal{H}(e_2) \cap \mathcal{G}^c(e_2)]$.
Now
$\mathcal{G}(e_2) \cap \mathcal{H}^c(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.1, 0.9)\}$;
$\mathcal{H}(e_2) \cap \mathcal{G}^c(e_2) = \{(h_1, 0.1, 0.7), (h_2, 0.1, 0.3), (h_3, 0, 0.9), (h_4, 0.2, 0.7)\}$.
So, $\mathcal{L}_1(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.2, 0.7)\}$.
Now let $(\mathcal{F}, A)\,\widetilde{\triangle}\,(\mathcal{L}_1, B \cap C) = (\mathcal{L}_2, A \cap B \cap C)$, *where* $A \cap B \cap C = \{e_2\}$ *and by Definitions 2.25, 2.17, 2.24,*
we have $\mathcal{L}_2(e_2) = [\mathcal{F}(e_2) \cap \mathcal{L}_1^c(e_2)] \cup [\mathcal{L}_1(e_2) \cap \mathcal{F}^c(e_2)]$. *Now*
$\mathcal{F}(e_2) \cap \mathcal{L}_1^c(e_2) = \{(h_1, 0.3, 0.2), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$
and $\mathcal{L}_1(e_2) \cap \mathcal{F}^c(e_2) = \{(h_1, 0.1, 0.7), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.2, 0.7)\}$.
Therefore

$\mathcal{L}_2(e_2) = \{(h_1, 0.3, 0.2), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$. *Now we consider the right hand side of the inequality and let* $(F, A) \widetilde{\triangle} (G, B) = (\mathcal{M}_1, A \cap B)$, *where* $A \cap B = \{e_2\}$ *and by Definitions 2.25, 2.17, 2.24, we have* $\mathcal{M}_1(e_2) = [\mathcal{F}(e_2) \cap \mathcal{G}^c(e_2)] \cup [\mathcal{G}(e_2) \cap \mathcal{F}^c(e_2)]$. *Now* $\mathcal{F}(e_2) \cap \mathcal{G}^c(e_2) = \{(h_1, 0.1, 0.2), (h_2, 0.1, 0.3), (h_3, 0, 0.9), (h_4, 0.2, 0.7)\}$ *and* $\mathcal{G}(e_2) \cap \mathcal{F}^c(e_2) = \{(h_1, 0.1, 0.7), (h_2, 0.2, 0.2), (h_3, 0.5, 0.4), (h_4, 0.3, 0.3)\}$. *Therefore* $\mathcal{M}_1(e_2) = \{(h_1, 0.1, 0.2), (h_2, 0.2, 0.2), (h_3, 0.5, 0.4), (h_4, 0.3, 0.3)\}$.
Now let $(\mathcal{M}_1, A \cap B) \widetilde{\triangle} (\mathcal{H}, C) = (\mathcal{M}_2, A \cap B \cap C)$, *where* $A \cap B \cap C = \{e_2\}$ *and by Definitions 2.25, 2.17, 2.24, we have* $\mathcal{M}_2(e_2) = [\mathcal{M}_1(e_2) \cap \mathcal{H}^c(e_2)] \cup [\mathcal{H}(e_2) \cap \mathcal{M}_1^c(e_2)]$, *where* $\mathcal{M}_1(e_2) \cap \mathcal{H}^c(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.1, 0.9)\}$ *and* $\mathcal{H}(e_2) \cap \mathcal{M}_1^c(e_2) = \{(h_1, 0.2, 0.7), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$. *Therefore* $\mathcal{M}_2(e_2) = \{(h_1, 0.2, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$. *This shows that* $\mathcal{L}_2(e_2) \neq \mathcal{M}_2(e_2)$. *Therefore in general,* $(\mathcal{F}, A) \widetilde{\triangle} ((\mathcal{G}, B) \widetilde{\triangle} (\mathcal{H}, C)) \neq ((\mathcal{F}, A) \widetilde{\triangle} (\mathcal{G}, B)) \widetilde{\triangle} (\mathcal{H}, C)$.
(5) Now we shall check that $(\mathcal{F}, A) \Cap ((\mathcal{G}, B) \widetilde{\triangle} (\mathcal{H}, C)) \neq ((\mathcal{F}, A) \Cap (\mathcal{G}, B)) \widetilde{\triangle} ((\mathcal{F}, A) \Cap (\mathcal{H}, C))$, *in general. Let* $(\mathcal{G}, B) \widetilde{\triangle} (\mathcal{H}, C) = (\mathcal{L}_1, B \cap C)$, *where* $B \cap C = \{e_2\}$ *and by previous part (4) of this example* $\mathcal{L}_1(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.2, 0.7)\}$. *Now let* $(\mathcal{F}, A) \Cap (\mathcal{L}_1, B \cap C) = (\mathcal{L}_2, A \cap B \cap C)$, *where* $A \cap B \cap C = \{e_2\}$ *and by Definition 2.15,* $\mathcal{L}_2(e_2) = \mathcal{F}(e_2) \cap \mathcal{L}_1(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.2, 0.7)\}$. *Now we consider the right hand side of the inequality and let* $(\mathcal{F}, A) \Cap (\mathcal{G}, B) = (\mathcal{M}_1, A \cap B)$ *and* $(\mathcal{F}, A) \Cap (\mathcal{H}, C) = (\mathcal{M}_2, A \cap C)$, *where* $A \cap B = \{e_2\}, A \cap C = \{e_2\}$ *and by Definition 2.15,* $\mathcal{M}_1(e_2) = \{(h_1, 0.1, 0.2), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$ *and* $\mathcal{M}_2(e_2) = \{(h_1, 0.3, 0.7), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$.
Now let $(\mathcal{M}_1, A \cap B) \widetilde{\triangle} (\mathcal{M}_2, A \cap C) = (\mathcal{M}_3, A \cap B \cap C)$, *where* $A \cap B \cap C = \{e_2\}$ *and by Definitions 2.25, 2.17, 2.24, we have* $\mathcal{M}_3(e_2) = [\mathcal{M}_1(e_2) \cap \mathcal{M}_2^c(e_2)] \cup [\mathcal{M}_2(e_2) \cap \mathcal{M}_1^c(e_2)]$, *where* $\mathcal{M}_1(e_2) \cap \mathcal{M}_2^c(e_2) = \{(h_1, 0.1, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$ *and* $\mathcal{M}_2(e_2) \cap \mathcal{M}_1^c(e_2) = \{(h_1, 0.2, 0.7), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$. *Therefore* $\mathcal{M}_3(e_2) = \{(h_1, 0.2, 0.3), (h_2, 0.1, 0.2), (h_3, 0.4, 0.5), (h_4, 0.3, 0.3)\}$. *This shows that* $\mathcal{L}_2(e_2) \neq \mathcal{M}_3(e_2)$. *Therefore in general*
$(\mathcal{F}, A) \Cap ((\mathcal{G}, B) \widetilde{\triangle} (\mathcal{H}, C)) \neq ((\mathcal{F}, A) \Cap (\mathcal{G}, B)) \widetilde{\triangle} ((\mathcal{F}, A) \Cap (\mathcal{H}, C))$.

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