Journal of Hyperstructures 13 *(1)* (*2024*), *94-108.*
ISSN: 2322-1666 print/2251-8436 online
https://doi.org/10.22098/jhs.2024.14426.1005

CONVERGENCE OF PANIGRAHY ITERATION PROCESS FOR SUZUKI GENERALIZED NONEXPANSIVE MAPPING IN UNIFORMLY CONVEX BANACH SPACE

OMPRAKASH SAHU AND AMITABH BANERJEE

Abstract. In this paper, we establish strong and weak convergence theorems for Suzuki's generalized nonexpansive mapping in uniformly convex Banach spaces using the iterative scheme introduced by Panigrahy et al [9]. Next, we see an example of Suzuki's generalized nonexpansive mapping, which is not a nonexpansive mapping. Using this example and some numerical tests, we infer empirically that the Panigrahy iteration process converges faster than the Krasnoselskij, Thakur, and M-iteration processes.

Key Words: Fixed Point, Uniformly convex Banach space, Suzuki generalized nonexpansive mapping, Panigrahy iteration, Convergence Theorem.
2010 Mathematics Subject Classification: Primary: 47H09, 47H10, 47H20.

1. Introduction

Let X be a Banach space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a mapping. We denote by $F(T)$ the fixed point of T, i.e.

$$F(T) = \{x \in C : Tx = x\}.$$

Research Paper
Received: 11 January 2024, Accepted: 14 May 2024. Communicated by Hoger Ghahramani;
∗Address correspondence to O.P. Sahu; E-mail: om2261995@yahoo.com.

A mapping $T : C \to C$ is called a nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||$, $\forall x, y \in C$. The Concept of nonexpansive mapping can be defined accordingly in the general settings of a metric space. Suzuki [15] introduced the concept of generalized nonexpansive single-valued mapping which is called Suzuki generalized nonexpansive mappings. Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ is said to be Suzuki generalized nonexpansive if for all $x, y \in C$, we have

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||.$$

There exist some iteration processes that are often used to approximate fixed points of nonexpansive mapping see Picard iteration, Mann iteration [16] and Ishikawa iteration [14]. In 2014, Gursoy et al. [4] introduced new iteration process called the Picard-S iteration process and proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. In 2016, Thakur et al. [3] introduced a new iteration process and proved strong and weak convergence results for Suzuki generalized mapping in Banach space. In 2018, Ullah et al.[10] introduced a new iteration process called the M-iteration process. They proved that the M-iteration process can be used to approximate the fixed point of Suzuki generalized nonexpansive mapping and obtain weak convergence and strong convergence results on Banach space. In 2020, Hassan et al.[13] introduced a new iteration process named S^*-iteration process and established strong and weak convergence theorems for Suzuki generalized nonexpansive mapping in Banach space.
In 2022, Panigrahy et al. [9] introduced the following iteration process:

$$\begin{cases} x_0 \in C \\ u_n = T((1 - \gamma_n)x_n + \gamma_n T x_n) \\ v_n = T((1 - \beta_n)u_n + \beta_n T u_n) \\ w_n = T((1 - \alpha_n)v_n + \alpha_n T v_n) \\ x_{n+1} = T w_n \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). They proved convergence and stability results for contractive-like mapping in Uniformly convex Banach space.
Motivated by all these facts, we study some fixed points results for Suzuki generalized nonexpansive mappings in uniformly convex Banach

space and establish strong and weak convergence theorems for approximating a fixed point of Suzuki generalized nonexpansive mappings using the iterative scheme introduced by Panigrahy et al. [9].

2. Preliminaries

In this section, we shall discuss some definitions and lemmas to be used in the main results,

Definition 2.1. [8] A Banach space X is said to be uniformly convex if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that $x, y \in X$

$$||(x+y)/2|| \leq 1-\delta \textit{ whenever } \quad ||x-y|| \geq \epsilon \textit{ and } ||x|| = ||y|| = 1.$$

Definition 2.2. [17]A Banach space X is said to satisfy Opial's property if for each sequence $\{x_n\}$ in X converging weakly to $x \in X$, we have

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||,$$

for all $y \in X$ s.t. $x \neq y$.

Definition 2.3. [5]A mapping $T : X \to X$ is said to satisfy condition I, if $\exists$ a non decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(c) > 0$ for all $c > 0$ s.t. $||x - Tx|| \geq f(d(x_n, F(T)))$, for all $x \in X$, where $d(x, F(T)) = \inf\{||x-p|| : p \in F(T)\}$.

Definition 2.4. [15] Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ is said to be Suzuki generalized nonexpansive if for all $x, y \in C$, we have

$$\frac{1}{2}||x - Tx|| \leq ||x-y|| \Rightarrow ||Tx - Ty|| \leq ||x-y||$$

Definition 2.5. [7] Let X be a Banach space and C be any nonempty subset of X. A mapping $T : C \to C$ is said to be quasi-nonexpansive if for each $x \in C$ and $y \in F(T)$

$$||Tx - y|| \leq ||x - y||.$$

where $F(T)$ is the set of fixed points of T.

Lemma 2.6. [6]*Let X be a uniformly convex Banach space and $0 < u \leq t_n \leq v < 1$ $\forall n \in N$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X s.t. $\limsup_{n\to\infty} ||x_n|| \leq a$, $\limsup_{n\to\infty} ||y_n|| \leq a$ and $\limsup_{n\to\infty} ||t_n x_n + (1-t_n)y_n|| = a$ hold for some $a \geq 0$. Then $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

Lemma 2.7. [8] *Let X be a uniformly convex Banach space and C be any nonempty closed convex subset of X. Let T be a nonexpansive mapping on X. Then, $I - T$ is demiclosed at zero.*

Proposition 2.8. [15] *Let C be a nonempty closed subset of a Banach space X with the Opial property and $T : C \to C$ a Suzuki generalized nonexpansive mapping. If $\{x_n\}$ converges weakly to a point z and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$, then $T(z) = z$.*

Proposition 2.9. [15]*Let C be a nonempty subset of a Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping and $F(T) \neq \phi$ then T is quasi- nonexpansive.*

3. Main Results

We first state and prove the following lemmas which will be needed in the proof of our main theorems:

Lemma 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \phi$. Suppose that $\{x_n\}$ is defined by (1.1), then $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. By Proposition 2.9, T is quasi nonexpansive mapping. So we have

$$\begin{aligned} ||u_n - p|| &= ||T((1-\gamma_n)x_n + \gamma_n Tx_n) - p|| \\ &\leq ||(1-\gamma_n)x_n + \gamma_n Tx_n - p|| \\ &= (1-\gamma_n)||x_n - p|| + \gamma_n||Tx_n - p|| \\ &\leq (1-\gamma_n)||x_n - p|| + \gamma_n||x_n - p|| \\ &= ||x_n - p|| \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} ||v_n - p|| &= ||T((1-\beta_n)u_n + \beta_n Tu_n) - p|| \\ &\leq ||(1-\beta_n)u_n + \beta_n Tu_n - p|| \\ &= (1-\beta_n)||u_n - p|| + \beta_n||Tu_n - p|| \\ &\leq (1-\beta_n)||u_n - p|| + \beta_n||u_n - p|| \\ &= ||u_n - p|| \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
||w_n - p|| &= ||T((1-\alpha_n)v_n + \alpha_n T v_n) - p|| \\
&\le ||(1-\alpha_n)v_n + \alpha_n T v_n - p|| \\
&= (1-\alpha_n)||v_n - p|| + \alpha_n ||T v_n - p|| \\
&\le (1-\alpha_n)||v_n - p|| + \alpha_n ||v_n - p|| \\
&= ||v_n - p|| \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
||x_{n+1} - p|| &= ||T w_n - p|| \\
&\le ||w_n - p|| \qquad (3.4)
\end{aligned}
$$

From equation (3.1), (3.2), (3.3) and (3.4), we get

$$
||x_{n+1} - p|| \le ||x_n - p|| \qquad (3.5)
$$

This implies that $\{||x_n - p||\}$ is bounded and non-increasing. Hence $\lim_{n\to\infty} ||x_n - p||$ exists $\forall\ p \in F(T)$. □

Lemma 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \phi$. Suppose that $\{x_n\}$ is defined by (1.1), then $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$.*

Proof. Since $F(T) \neq \phi$. Suppose $p \in F(T)$, then by Lemma 3.1 $\lim_{n\to\infty} ||x_n - p||$ exists $\forall\ p \in F(T)$ and $\{x_n\}$ is bounded. Put

$$
\lim_{n\to\infty} ||x_n - p|| = k \qquad (3.6)
$$

Case I: If $k = 0$, then we are done.
Case II: If $k > 0$, From equation (3.1) in Lemma 3.1, we have

$$
||u_n - p|| \le ||x_n - p||
$$

It follows from

$$
\begin{aligned}
&\limsup_{n\to\infty} ||u_n - p|| \le \limsup_{n\to\infty} ||x_n - p|| = k \\
&\Rightarrow \limsup_{n\to\infty} ||u_n - p|| \le k \qquad (3.7)
\end{aligned}
$$

By using Proposition 2.9, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} ||Tx_n - p|| \le \limsup_{n\to\infty} ||x_n - p|| = k \\
&\Rightarrow \limsup_{n\to\infty} ||Tx_n - p|| \le k \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
||x_{n+1} - p|| &= ||T w_n - p|| \\
&\le ||w_n - p|| \qquad (3.9)
\end{aligned}
$$

From equation (3.3), (3.4) in Lemma 3.1, we have

$$||x_{n+1} - p|| \leq ||u_n - p|| \tag{3.10}$$

So we can get from equation (3.6) and (3.10), we have

$$k \leq \liminf_{n\to\infty} ||u_n - p|| \tag{3.11}$$

From equation (3.7) and (3.11), we have

$$\begin{aligned} k &= \lim_{n\to\infty} ||u_n - p|| \\ &= \lim_{n\to\infty} ||T((1-\gamma_n)x_n + \gamma_n T x_n) - p|| \\ &= \lim_{n\to\infty} ||(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)|| \end{aligned}$$

Using Lemma 2.6, we have

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0. \tag{3.12}$$

□

Theorem 3.3. *Let X be a uniformly convex Banach space which satifies the Opial's condition and C a nonempty closed convex subset of X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \phi$ and $\{x_n\}$ be a sequence defined by iteration* (1.1)*. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. From Lemma 3.1, we proved that $\lim_{n\to\infty} ||x_n - p||$ exists and that $\{x_n\}$ is bounded. Since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We have to show that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let u and v be weak limits of the sequence $\{x_{n_k}\}$ and $\{x_{n_m}\}$ of $\{x_n\}$ respectively. By Lemma 3.2, $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $I - T$ is demiclosed with respect to zero by Proposition 2.8, we therefore have that $Tu = u$. Similarly we can show that $Tv = v$. Next we establish uniqueness. From Lemma 3.1, we have that $\lim_{n\to\infty} ||x_n - v||$ exists. Now Suppose that $u \neq v$, then by Opial condition

$$\begin{aligned}\lim_{n\to\infty}||x_n-u|| &= \lim_{n\to\infty}||x_{n_k}-u||\\ &< \lim_{k\to\infty}||x_{n_k}-v||\\ &= \lim_{n\to\infty}||x_n-v||\\ &= \lim_{m\to\infty}||x_{n_m}-v||\\ &< \lim_{m\to\infty}||x_{n_m}-u||\\ &= \lim_{m\to\infty}||x_n-u||\end{aligned}$$

Which is a contradiction, so $u=v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$. □

Theorem 3.4. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T: C\to C$ be a Suzuki generalized nonexpansive mapping with $F(T)\neq\phi$. Suppose that $\{x_n\}$ is defined by* (1.1). *Then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$, where $d(x,F(T))=\inf\{||x-p||: p\in F(T)\}$.*

Proof. Let $\{x_n\}$ be converged to a fixed point, say p of T. Then $\lim_{n\to\infty} d(x_n,p)=0$ and since $0\leq d(x_n,F(T))\leq d(x_n,p)$. It follows that $\lim_{n\to\infty} d(x_n,F(T))=0$. Therefore $\liminf_{n\to\infty} d(x_n,F(T))=0$.
Conversely: suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$. From Lemma 3.1, we have that $\lim_{n\to\infty}||x_n-p||$ exists and that $\lim_{n\to\infty} d(x_n,F(T))$ exists for all $p\in F(T)$. Our assumption, $\liminf_{n\to\infty} d(x_n,F(T))=0$.so for any give $\epsilon>0$, there exists $n\in N$ such that for all $n\geq n_0$, we have $d(x_n;F(T))\leq\epsilon$. We now show that $\{x_n\}$ is a Cauchy sequence in C.Since, $\lim_{n\to\infty} d(x_n,F(T))=0$, for any give $\epsilon>0$, there exist $n_0\in N$ such that for $n,m\geq n_0$, we have

$$\begin{aligned}d(x_m,F(T)) &\leq \frac{\epsilon}{2}\\ d(x_n,F(T)) &\leq \frac{\epsilon}{2}\end{aligned}$$

Therefore we have

$$\begin{aligned} ||x_m - x_n|| &= ||x_m - p + p - x_n|| \\ &\leq ||x_m - p|| + ||x_n - p|| \\ &= d(x_m, F(T)) + d(x_n, F(T)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence,$\{x_n\}$ is a Cauchy sequence in C. Since C is closed, then there exists a point $x_1 \in C$ such that $\lim_{n\to\infty} x_n = x_1$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, it follows that $\lim_{n\to\infty} d(x_1, F(T)) = 0$. Since, $F(T)$ is closed, $x_1 \in F(T)$. □

Theorem 3.5. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \phi$ and $\{x_n\}$ is defined by* (1.1)*. Let T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By using Lemma 3.2, we have

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{3.13}$$

From condition I and equation (3.13), we have

$$\begin{aligned} \lim_{n\to\infty} f(d(x_n, F(T))) &\leq \lim_{n\to\infty} ||x_n - Tx_n|| \\ &\Rightarrow \lim_{n\to\infty} f(d(x_n, F(T))) = 0. \end{aligned}$$

Since $f : [0, \infty) \to [0, \infty)$ is a non decreasing function satisfying $f(0) = 0$ and $f(c) > 0$ for all $c \in (0, \infty)$, therefore, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

By Theorem 3.4, the sequence $\{x_n\}$ strongly converges to a fixed point of $F(T)$. □

4. Numerical Examples

In this section, we see an example of Suzuki's generalized nonexpansive mapping, which is not a nonexpansive mapping. Using this example, we compare our iterative process with three other iterative processes in the literature.

Example 4.1. Define a mapping $T : [0,1] \to [0,1]$ as

$$Tx = \begin{cases} 1 - x \ if x \in [0, \frac{1}{7}) \\ \frac{x+6}{7} \ if x \in [\frac{1}{7}, 1] \end{cases}$$

Then T is a Suzuki generalized nonexpansive mapping but not non-expansive mapping.

Proof. Consider the following cases:
Cases I: Let $x \in [0, \frac{1}{7})$, then $\frac{1}{2}||x - Tx|| = \frac{1-2x}{2} \in (\frac{5}{14}, \frac{1}{2}]$. For $\frac{1}{2}||x - Tx|| \leq ||x - y||$ we must have $\frac{1-2x}{2} \leq y - x$, $\frac{1}{2} \leq y$. Hence $y \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned} ||Tx - Ty|| &= |\frac{y+6}{7} - (1 - x)| \\ &= |\frac{y + 7x - 1}{7}| \\ &\leq \frac{1}{7} \end{aligned}$$

and

$$||x - y|| = |x - y| > |\frac{1}{7} - \frac{1}{2}| = \frac{5}{14}.$$

Hence $\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||$.
Case II: Let $x \in [\frac{1}{7}, 1]$, then

$$\begin{aligned} \frac{1}{2}||x - Tx|| &= \frac{1}{2}|\frac{x+6}{7} - x| \\ &= \frac{1}{2}|\frac{6 - 6x}{7}| \\ &= \frac{3 - 3x}{7} \end{aligned}$$

$\frac{3-3x}{7} \in [0, \frac{18}{49}]$. for $\frac{1}{2}||x - Tx|| \leq ||x - y||$ we must have $\frac{3-3x}{7} \leq |y - x|$, which goes two possibilities:
(a) Let $x < y$, $\frac{3-3x}{7} \leq y - x \Rightarrow y \geq \frac{3-3x}{7} + x$, $y \geq \frac{3+4x}{7} \in [\frac{25}{49}, 1] \subset [\frac{1}{7}, 1]$
So

$$\begin{aligned} ||Tx - Ty|| &= ||\frac{x+6}{7} - \frac{y+6}{7}|| \\ &= \frac{1}{7}||x - y|| \\ &\leq ||x - y|| \end{aligned}$$

Hence $\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||$.
(b) Let $x > y$, $\frac{3-3x}{7} \leq x - y \Rightarrow y \leq x - \frac{3-3x}{7}$. $y \leq \frac{10x-3}{7} \Rightarrow y \in [\frac{-11}{49}, 1]$. Since $y \in [0,1]$ so $y \leq \frac{10x-3}{7} \Rightarrow x \geq \frac{7y+3}{10} \Rightarrow x \in [\frac{3}{10}, 1]$. So the case is $x \in [\frac{3}{10}, 1]$ and $y \in [0,1]$. Now $x \in [\frac{3}{10}, 1]$ and $y \in [0,1]$.
Now $x \in [\frac{3}{10}, 1]$ and $y \in [\frac{1}{7}, 1]$ is already in **Case I**. So we consider $x \in [\frac{3}{10}, 1]$ and $y \in [0, \frac{1}{7})$. To start with suppose $x \in [\frac{3}{10}, \frac{1}{2}]$ and $y \in [0, \frac{1}{7}]$ we have

$$\begin{aligned} ||Tx - Ty|| &= |\frac{x+6}{7} - (1-y)| \\ &= |\frac{x+7y-1}{7}| \\ &\leq \frac{3}{70} \end{aligned}$$

and

$$\begin{aligned} ||x - y|| &= |x - y| \\ &> |\frac{3}{10} - \frac{1}{7}| \\ &= \frac{11}{70} \end{aligned}$$

Thus we have that $\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||$.
Also for $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{7})$, we therefore have that

$$\begin{aligned} ||Tx - Ty|| &= |\frac{x+6}{7} - (1-y)| \\ &= |\frac{x+7y-1}{7}| \\ &\leq \frac{1}{14} \end{aligned}$$

and

$$\begin{aligned} ||x - y|| &= |x - y| \\ &> |\frac{1}{2} - \frac{1}{7}| \\ &= \frac{5}{14} \end{aligned}$$

Thus we have that $\frac{1}{2}||x-Tx|| \leq ||x-y|| \Rightarrow ||Tx-Ty|| \leq ||x-y||$. Hence T is a Suzuki generalized nonexpansive mapping. However to show that

T is not nonexpansive, we take $x = \frac{3}{22}$ and $y = \frac{1}{7}$, we then have that

$$\begin{aligned} ||Tx - Ty|| &= |1 - \frac{3}{22} - \frac{43}{49}| \\ &= |\frac{931 - 946}{1078}| \\ &= \frac{15}{1078} \\ &> |\frac{3}{22} - \frac{1}{7}| \\ &= \frac{1}{154} \\ &= ||x - y|| \end{aligned}$$

Hence T is not nonexpansive mapping. □

In what follows, we numerically compare the Panigrahy iteration process with some existing iteration processes.

Case I: Taking $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{n+1}$, $\gamma_n = \frac{1}{n^3}$ and $x_0 = 0.7$.

Case II: Taking $\alpha_n = \frac{1}{\sqrt{n^3+5}}$, $\beta_n = \frac{1}{\sqrt{n^3+7}}$, $\gamma_n = \frac{5}{n^3+150}$ and $x_0 = 0.5$.

Table 1. Comparison of the rate of convergence with different iteration processes.

Iteration	Panigarhy iteration	M-iteration	Thakur	Krasnoeslskij
0	0.70000000	0.70000000	0.70000000	0.70000000
1	0.99999417	0.99650146	0.99518950	0.82857143
2	1.00000000	0.99994900	0.99991118	0.87755102
3	1.00000000	0.99999918	0.99999828	0.90379009
4	1.00000000	0.99999999	0.99999997	0.92028322
5	1.00000000	1.00000000	1.00000000	0.93167133
6	1.00000000	1.00000000	1.00000000	0.94003810
7	1.00000000	1.00000000	1.00000000	0.94646259
8	1.00000000	1.00000000	1.00000000	0.95156139
9	1.00000000	1.00000000	1.00000000	0.95571327
10	1.00000000	1.00000000	1.00000000	0.95916419

Case III: Taking $\alpha_n = \frac{1}{\sqrt{5^n+1}}$, $\beta_n = \frac{1}{n+1}$, $\gamma_n = \frac{5}{189}$ and $x_0 = 0.4$.

Case IV: Taking $\alpha_n = \frac{2}{3}$, $\beta_n = \frac{2}{3}$, $\gamma_n = \frac{3}{4}$ and $x_0 = 0.8$.

Case V: Taking $\alpha_n = \frac{5}{389}$, $\beta_n = \frac{1}{\sqrt{n^{20}+5}}$, $\gamma_n = \frac{1}{\sqrt{n^2+5}}$ and $x_0 = 0.6$.

TABLE 2. Comparison of the rate of convergence with different iteration processes.

Iteration	Panigarhy iteration	M-iteration	Thakur	Krasnoeslskij
0	0.50000000	0.50000000	0.50000000	0.50000000
1	0.99990833	0.99336660	0.99105835	0.67496355
2	0.99999998	0.99989681	0.99982872	0.75223403
3	1.00000000	0.99999821	0.99999660	0.78977624
4	1.00000000	0.99999997	0.99999993	0.81146879
5	1.00000000	1.00000000	1.00000000	0.82564188
6	1.00000000	1.00000000	1.00000000	0.83569497
7	1.00000000	1.00000000	1.00000000	0.84324440
8	1.00000000	1.00000000	1.00000000	0.84915363
9	1.00000000	1.00000000	1.00000000	0.85392607
10	1.00000000	1.00000000	1.00000000	0.85787558

TABLE 3. Comparison of the rate of convergence with different iteration processes.

Iteration	Panigarhy iteration	M-iteration	Thakur	Krasnoeslskij
0	0.40000000	0.40000000	0.40000000	0.40000000
1	0.99990928	0.99203992	0.98989751	0.60995626
2	0.99999998	0.99986486	0.99980538	0.67552244
3	1.00000000	0.99999745	0.99999610	0.70029966
4	1.00000000	0.99999995	0.99999992	0.71056689
5	1.00000000	1.00000000	1.00000000	0.71500407
6	1.00000000	1.00000000	1.00000000	0.71695826
7	1.00000000	1.00000000	1.00000000	0.71782624
8	1.00000000	1.00000000	1.00000000	0.71821322
9	1.00000000	1.00000000	1.00000000	0.71838604
10	1.00000000	1.00000000	1.00000000	0.71846329

Note 4.1. Table 1, Case I; Table 2, Case II; Table 3, Case III; Table 4, Case IV; Table 5, Case V.

The comparison shows that the Panigrahy iteration (1.1) converges to $x^* = 1$ faster than M- iteration[10], Thakur[3] and Krasnoeslskij [11].

Table 4. Comparison of the rate of convergence with different iteration processes.

Iteration	Panigarhy iteration	M-iteration	Thakur	Krasnoeslskij
0	0.80000000	0.80000000	0.80000000	0.80000000
1	0.99999454	0.99825073	0.99747328	0.91428571
2	1.00000000	0.99998470	0.99996808	0.96326531
3	1.00000000	0.99999987	0.99999960	0.98425656
4	1.00000000	1.00000000	0.99999999	0.99325281
5	1.00000000	1.00000000	1.00000000	0.99710835
6	1.00000000	1.00000000	1.00000000	0.99876072
7	1.00000000	1.00000000	1.00000000	0.99946888
8	1.00000000	1.00000000	1.00000000	0.99977238
9	1.00000000	1.00000000	1.00000000	0.99990245
10	1.00000000	1.00000000	1.00000000	0.99995819

Table 5. Comparison of the rate of convergence with different iteration processes.

Iteration	Panigarhy iteration	M-iteration	Thakur	Krasnoeslskij
0	0.60000000	0.60000000	0.60000000	0.60000000
1	0.99993037	0.99192667	0.99187345	0.60440690
2	0.99999998	0.99983705	0.99983415	0.60876526
3	1.00000000	0.99999671	0.99999662	0.61307559
4	1.00000000	0.99999993	0.99999993	0.61733844
5	1.00000000	1.00000000	1.00000000	0.62155432
6	1.00000000	1.00000000	1.00000000	0.62572375
7	1.00000000	1.00000000	1.00000000	0.62984725
8	1.00000000	1.00000000	1.00000000	0.63392532
9	1.00000000	1.00000000	1.00000000	0.63795846
10	1.00000000	1.00000000	1.00000000	0.64194717

5. Conclusion

The main contribution of our work:

(1) We have established strong and weak convergence theorems for the Panigrahy iteration process in the class of Suzuki's generalized nonexpansive mapping in uniformly convex Banach spaces.

(2) We provide an example of Suzuki's generalized nonexpansive mapping which is not nonexpansive mapping. Using this example and some numerical tests, we infer empirically that the Panigrahy iteration process converges faster than the Krasnoselskij, Thakur, and M-iteration processes.

Acknowledgments

The authors thank all the professors, friends, and family members who encouraged us to work.

References

[1] A. A. Mebawondu, and O.T. Mewomo, *Fixed point results for a new three steps iteration process*, Annals of the University of Craiova, Mathematics and Computer Science Series, **46**, no.2, (2019), 298-319.

[2] A. Aghanians, K. Fallahi, and K. Nourouzi, *Fixed points for G-contractions on uniform spaces endowed with a graph*, Fixed Point Theory Appl., (2012), **2012, 182**.

[3] B. S. Thakur, D. Thakur, and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzukis generalized nonexpansive mappings*, App. Math. Comp., **275** (2016), 147155.

[4] F. Gursoy, and V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, (2014) arXiv:1403.2546v2.

[5] H. F. Senter, and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc., **44**, no.2, (1974), 375380.

[6] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc., **43**, no.1, (1991),153159.

[7] J. B. Diaz, and F.T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Am. Math. Soc., **73**, (1967), 516519.

[8] K. Goebel, and W. A. Kirk, *Topic in metric fixed point theory*, Cambridge University Press, (1990).

[9] K. Panigrahy, and D. Mishra, *A note on a faster-fixed point iterative method*, The Journal of Analysis, (2022), DOI: https://doi.org/10.1007/s41478-022-00485-z.

[10] K. Ullah, and M. Arshad, *Numerical reckoning fixed point for Suzuki generalized nonexpansive mappings via new iteration process*, Filomat, **32**, no.1, (2018), 187-196.

[11] M.A.Krasnoselskij, *Two remarks on the method of successive approximations (Russian)* , Uspehi Mat. Nauk., **10**, no.1, (1955), 123-127.

[12] O. Sahu, and A. Banerjee, *Convergence Results for Mean Nonexpansive Mappings in Uniformly Convex Banach Space*, Creat. Math. Inform., **32**, no.2, (2023), 219-227.

[13] S. Hassan, M. De la Sen, P.Agarwal, Q. Ali, and A. Hussain, *A New Faster Iterative Scheme for Numerical Fixed Points Estimation of Suzukis Generalized Nonexpansive Mappings*, Math. Probl. Eng., (2020), Article ID 3863819, 9 pp.
[14] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**, (1974), 147150.
[15] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340**, (2008), 1088-1095.
[16] W.R.Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**,(1953), 506510.
[17] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc., **73**, no.4, (1967), 591598.

Omprakash Sahu
Department of Mathematics, Babu Pandhri Rao Kridatt Govt.College Silouti, Dhamtari, Raipur(C.G.), India
Email: om2261995@yahoo.com

Amitabh Banerjee
Principal, Govt. J.Y. Chhattisgarh College, Raipur, India
Email: amitabh_61@yahoo.com