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# CONFORMAL RICCI SOLITON IN SASAKIAN MANIFOLDS ADMITTING GENERAL CONNECTION

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Abstract. The object of the present paper is to study the Conformal Ricci soliton in Sasakian manifold admitting general connection, which is induced with quarter symmetric metric connection, generalized Tanaka Webster connection, Schouten-Van Kampen connection and Zamkovoy connection. Furthermore, we study $C^G$-semi symmetric and $C^G$-semi symmetric Sasakian manifolds admitting Conformal Ricci Soliton.

**Key Words:** Conformal Ricci soliton, quarter symmetric metric connection, Schouten-Van Kampen connection, Tanaka Webster connection, Zamkovoy connection, general connection.
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## 1. Introduction

In this paper, the symboles $\nabla^G$, $\nabla$, $\nabla^q$, $\nabla^z$, $\nabla^s$, and $\nabla^T$ are denoted for general connection, Levi-Civita connection, quarter-symmetric metric connection, Zamkovoy connection, Schouten-Van Kampen connection and generalized Tanaka-Webster connection respectively. In the context of Sasakian geometry the general connection is introduced by Biswas and Baishya ([5], [4]) and the general connection $\nabla^G$ is defined as

$$\nabla^G_X V = \nabla_X Y + k_1 \left[ (\nabla_X \eta)(Y)\,\xi - \eta(Y)\,\nabla_X \xi \right] + k_2 \eta(X)\,\phi Y, \tag{1.1}$$

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for all $U,V \in \chi(M)$ and the pair $(k_1,k_2)$ being real constants. The beauty of such connection $\nabla^G$ lies in the fact that it has the flavour of

$(i)$ quarter symmetric metric connection ([14], [3]) for $(k_1,k_2) \equiv (0,-1)$;

$(ii)$ Zamkovoy connection [10] for $(k_1,k_2) \equiv (1,1)$;

$(iii)$ Schouten-Van Kampen connection [8] for $(k_1,k_2) \equiv (1,0)$ and

$(iv)$ generalized Tanaka Webster connection [9] for $(k_1,k_2) \equiv (1,-1)$.

The torsion tensor $T$ of the connection $\nabla^G$ satisfies

$$\begin{aligned} & T^G(U,V) \\ = \; & \nabla^G_U V - \nabla^G_V U - [U,V] \\ = \; & 2k_1 g(U,\phi V)\xi + k_1\eta(V)\phi U - k_1\eta(U)\phi V \\ & + k_2\eta(U)\phi V - k_2\eta(V)\phi U. \end{aligned} \tag{1.2}$$

In 1982, R. S. Hamilton[6] introduced the idea of Ricci flow to investigate a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([15], [16]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.3}$$

In 2004, A. E. Fischer[2] introduced a new idea of conformal Ricci flow, it is a modified version of the Hamilton's Ricci flow equation. In the classical theory of Hamilton's Ricci flow equation, the unit volume constraint plays a crucial role. However, the primary distinction between the classical and conformal Ricci flow equations is the scalar curvature constraint. This new Ricci flow equation is defined as the conformal Ricci flow. For an $n$-dimensional, $n \geq 3$, closed connected oriented, smooth manifold $(M,g)$, the conformal Ricci flow equation is given by

$$\begin{aligned} \frac{\partial g}{\partial t} + 2\left(Ric + \frac{g}{n}\right) &= -pg, \\ r(g) &= -1, \end{aligned} \tag{1.4}$$

where $p$ is a scalar non-dynamical field (time dependent scalar field) and $r(g)$ is the scalar curvature of the manifold. and n is the dimension of manifold. In 2015, N. Basu and A. Bhattacharyya[1], have introduced the notion of Conformal Ricci soliton equation and it is given by

$$(\pounds_\xi g)(U,V) + 2S(U,V) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(U,V), \tag{1.5}$$

where $\pounds$ is the Lie derivative along the vector field $\xi$, $S$ is the Ricci tensor, $\lambda$ is a real-valued smooth function on $M$.

In an $n$-dimensional Riemannian manifold $(M^n, g)$ $(n > 3)$, the conformal curvature tensor $C$[13], projective curvature tensor $P$[12] are defined respectively by

$$\begin{aligned} & C(U,Y)W \\ = \; & R(U,Y)W - \frac{1}{n-2}\left[S(Y,W)U - S(U,W)Y\right] \\ & -\frac{1}{n-2}\left[g(Y,W)QU - g(U,W)QY\right] \\ + \; & \frac{r}{(n-1)(n-2)}\left[g(Y,W)U - g(U,W)Y\right], \end{aligned} \tag{1.6}$$

and

$$P(U,Y)W = R(U,Y)W - \frac{1}{n-1}\left[S(Y,W)U - S(U,W)Y\right]. \tag{1.7}$$

This paper is structered as follows: After introduction, a short description of Sasakian manifold and general connection are given in section 2. In section 3, we have highlighted Conformal Ricci Soliton in Sasakian manifold admitting general connection. Section 4 deals with $C^G$-semi symmetric Sasakian manifolds admitting Conformal Ricci Soliton. Finally in section 5, we discussed $P^G$-semi symmetric Sasakian manifolds admitting general connection.

## 2. Preliminaries

Let us consider $M$ be an $n$-dimensional almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$. Then

$$\phi^2 Y = -Y + \eta(Y)\xi, \eta(\xi) = 1, \eta(\phi U) = 0, \; \phi\xi = 0, \tag{2.1}$$

$$g(U,Y) = g(\phi U, \phi Y) + \eta(U)\eta(Y), \tag{2.2}$$

$$g(U,\phi Y) = -g(\phi U, Y), \eta(Y) = g(Y,\xi), \text{for all } U, Y \in \chi(M), \tag{2.3}$$

where $\chi(M)$ is set of all vector fields of the manifold $M$. An almost contact metric manifold $M$ is said to be (a) a contact metric manifold if

$$g(U,\phi Y)=d\eta(U,Y),\text{for all } U,Y\in\chi(M);\tag{2.4}$$

(b) a $K$-contact manifold if the vector field $\xi$ is Killing equivalently

$$\nabla_Y\xi=-\phi Y,\tag{2.5}$$

where $\nabla$ is Riemannian connection and (c) a Sasakian manifold if

$$(\nabla_U\phi)Y=g(U,Y)\xi-\eta(Y)U,\text{for all } U,\ Y\in\chi(M).\tag{2.6}$$

Further, for Sasakian manifold with structure ($\phi$, $\xi$, $\eta$, $g$), the following relations holds([7],[11]):

$$R(U,Y)\xi=\eta(Y)U-\eta(U)Y,\ \text{for all } U,Y\in\chi(M),\tag{2.7}$$

$$(\nabla_U\eta)Y = g(U,\phi Y),\tag{2.8}$$

$$R(\xi,U)Y = g(U,Y)\xi-\eta(Y)U,\tag{2.9}$$

$$S(U,\xi) = (n-1)\eta(U),\tag{2.10}$$

$$R(U,\xi)Y = \eta(Y)U-g(U,Y)\xi,\tag{2.11}$$

$$Q\xi=(n-1)\xi,\tag{2.12}$$

where $S$ and $Q$ are Ricci tensor and Ricci operator.where $Q$, $S$ and $r$ are the Ricci operator, the Ricci curvature tensor and the scalar curvature of $M^n$. The Ricci operator $Q$ and the $(0,2)$-tensor $S^2$ are defined as

$$S(U,Y)=g(QU,Y)\text{ and } S^2(U,Y)=S(QU,Y)=g(Q^2U,Y).\tag{2.13}$$

For an $n$-dimensional Sasakian manifold admitting general connection and if $R^G$, $S^G$, $r^G$, $Q^G$ are Riemannian curvature tensor, Ricci tensor, scalar curvature and Ricci operator in general connection, then following results ([5], [4]) hold.

$$\begin{aligned}
&R^G(X,Y)Z\\
={}& R(X,Y)Z+\left(k_1^2-2k_1\right)[g(Z,\phi X)\phi Y+g(Y,\phi Z)\phi X]\\
&-2k_2 g(Y,\phi X)\phi Z\\
&+(k_1-k_1k_2+k_2)[g(X,Z)\eta(Y)\xi-\eta(X)g(Y,Z)\xi]\\
&+(k_1-k_1k_2+k_2)[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X],
\end{aligned}\tag{2.14}$$

$$S^G(Y,Z) = S(Y,Z) - \overline{A}g(Y,Z) + \overline{B}\eta(Y)\eta(Z), \tag{2.15}$$
$$S^G(Y,\xi) = -(n-1)\overline{C}\eta(Y), \tag{2.16}$$
$$S^G(\xi,Z) = -(n-1)\overline{C}\eta(Z), \tag{2.17}$$
$$Q^G Y = QY - \overline{A}Y + \overline{B}\eta(Y)\xi, \tag{2.18}$$
$$Q^G\xi = -(n-1)\overline{C}\xi, \tag{2.19}$$
$$r^G = r - \overline{A}n + \overline{B}, \tag{2.20}$$

$$R^G(X,Y)\xi = \overline{C}[\eta(X)Y - \eta(Y)X], \tag{2.21}$$
$$R^G(\xi,Y)Z = \overline{C}[\eta(Z)Y - g(Y,Z)\xi], \tag{2.22}$$
$$R^G(X,\xi)Z = \overline{C}[g(X,Z)\xi - \eta(Z)X], \tag{2.23}$$

where

$$\overline{A} = \left(k_1^2 - k_1 - k_2 - k_1k_2\right), \tag{2.24}$$
$$\overline{B} = \left[k_1^2 + (n-2)k_1k_2 - n(k_1+k_2)\right], \tag{2.25}$$
$$\overline{C} = (k_1 - k_1k_2 + k_2 - 1). \tag{2.26}$$

Therefore for quarter-symmetric metric connection

$$\overline{A} = 1; \overline{B} = n; \overline{C} = -2, \tag{2.27}$$

for generalized Tanaka Webster connection

$$\overline{A} = 2; \overline{B} = 3 - n; \overline{C} = 0, \tag{2.28}$$

for Zamkovoy connection

$$\overline{A} = -2; \overline{B} = -1 - n; \overline{C} = 0, \tag{2.29}$$

and for Schouten-Van Kampen connection

$$\overline{A} = 0; \overline{B} = 1 - n; \overline{C} = 0. \tag{2.30}$$

## 3. Conformal Ricci soliton in Sasakian manifold admitting general connection

Conformal Ricci soliton equation is given by (1.5) but in general connection it becomes

$$\left(\pounds_\xi^G g\right)(X,Y) + 2S^G(X,Y) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y), \tag{3.1}$$

where $\pounds^G$ is the Lie derivative admitting general connection along the vector field $\xi$.

Now, we express the Lie derivative along $\xi$ on $M$ with respect to general connection as follows:

$$\begin{aligned} &\left(\pounds_{\xi}^{G} g\right)(X,Y) \\ &= \pounds_{\xi}^{G} g(X,Y) - g\left(\pounds_{\xi}^{G} X, Y\right) - g\left(X, \pounds_{\xi}^{G} Y\right) \\ &= \pounds_{\xi}^{G} g(X,Y) - g([\xi, X], Y) - g(X, [\xi, Y]). \end{aligned} \tag{3.2}$$

By the help of (1.1), (2.1), (2.2), (2.5) and (3.2) we obtain

$$\left(\pounds_{\xi}^{G} g\right)(X,Y) = 2g(X,Y) - 2\eta(X)\eta(Y). \tag{3.3}$$

Using (3.3) in (3.1), we get

$$2g(X,Y) - 2\eta(X)\eta(Y) + 2S^{G}(X,Y) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y). \tag{3.4}$$

Simlifying above equation

$$S^{G}(X,Y) = \left[\lambda - \frac{1}{2}\left(p + \frac{2}{n}\right) - 1\right] g(X,Y) + \eta(X)\eta(Y). \tag{3.5}$$

Setting $X = Y = \xi$ in (3.5) and using (2.15), we have

$$-2(n-1)\overline{C} = \left[2\lambda - \left(p + \frac{2}{n}\right)\right]. \tag{3.6}$$

**Theorem 3.1.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to general connection $\nabla^G$, then $M$ is -Einstein manifold with general connection.*

**Theorem 3.2.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to general connection $\nabla^G$, then $\lambda$ and $p$ are related by 3.6.*

**Theorem 3.3.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to quarter-symmetric metric connection $\nabla^q$, then $\lambda$ and $p$ are related by*

$$4(n-1) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right].$$

**Theorem 3.4.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to Zamkovoy connection $\nabla^z$, then $\lambda$ and $p$ are related by*

$$0 = \left[2\lambda - \left(p + \frac{2}{n}\right)\right].$$

**Theorem 3.5.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to generalized Tanaka Webster connection $\nabla^T$, then $\lambda$ and $p$ are related by*

$$0= \left[2\lambda - \left(p + \frac{2}{n}\right)\right].$$

**Theorem 3.6.** *Let $(M^n, g)$ be a Sasakian manifold admitting Conformal Ricci soliton with respect to Schouten-Van Kampen connection $\nabla^s$, then $\lambda$ and $p$ are related by*

$$0= \left[2\lambda - \left(p + \frac{2}{n}\right)\right].$$

## 4. $C^G$-semi symmetric Sasakian manifolds admitting Conformal Ricci soliton

In this section, we assume $C^G$-semi-symmetric Sasakian manifold,

$$\text{i.e. } R^G(V, X) \circ C^G(Y, Z) U = 0. \tag{4.1}$$

The conformal curvature tensor[13] in general connection is given by

$$\begin{aligned} C^G(X, Y) Z &= R^G(X, Y) Z - \frac{1}{n-2}\left[S^G(Y, Z) X - S^G(X, Z) Y\right] \\ &\quad - \frac{1}{n-2}\left[g(Y, Z) Q^G X - g(X, Z) Q^G Y\right] \\ &\quad + \frac{r}{(n-1)(n-2)}\left[g(Y, Z) X - g(X, Z) Y\right], \end{aligned} \tag{4.2}$$

for all $X$, $Y$ & $Z \in \chi(M)$, the set of all vector field of the manifold $M$.

The equation (4.1) can also be written as

$$\begin{aligned} 0 &= R^G(V, X) C^G(Y, Z) U - C^G(R^G(V, X) Y, Z) U \\ &\quad - C^G(Y, R^G(V, X) Z) U - C^G(Y, Z) R^G(V, X) U. \end{aligned} \tag{4.3}$$

Setting in $V = \xi$ in (4.3), we get

$$\begin{aligned} 0 &= R^G(\xi, X) C^G(Y, Z) U - C^G\left(R^G(\xi, X) Y, Z\right) U \\ &\quad - C^G\left(Y, R^G(\xi, X) Z\right) U - C^G(Y, Z) R^G(\xi, X) U. \end{aligned} \tag{4.4}$$

Using (2.21), (2.22) and (2.23) we obtain the following

$$\begin{aligned} & R^G(\xi, X) C^G(Y, Z) U \\ &= \overline{C}\left[C^G(Y, Z, U, \xi) X - g\left(X, C^G(Y, Z) U\right) \xi\right], \end{aligned} \tag{4.5}$$

$$\begin{aligned} & C^G\left(R^G(\xi, X) Y, Z\right) U \\ = & C^G\left(\overline{C}\left[\eta(Y) X-g(X, Y) \xi\right], Z\right) U \\ = & \overline{C}\left[\eta(Y) C^G(X, Z) U-g(X, Y) C^G(\xi, Z) U\right], \end{aligned} \tag{4.6}$$

$$\begin{aligned} & C^G\left(Y, R^G(\xi, X) Z\right) U \\ = & C^G\left(Y, \overline{C}\left[\eta(Z) X-g(X, Z) \xi\right]\right) U \\ = & \overline{C}\left[\eta(Z) C^G(Y, X) U-g(X, Z) C^G(Y, \xi) U\right], \end{aligned} \tag{4.7}$$

$$\begin{aligned} & C^G(Y, Z) R^G(\xi, X) U \\ = & C^G(Y, Z) R^G(\xi, X) U \\ = & C^G(Y, Z) \overline{C}\left[\eta(U) X-g(X, U) \xi\right] \\ = & \overline{C}\left[\eta(U) C^G(Y, Z) X-g(X, U) C^G(Y, Z) \xi\right]. \end{aligned} \tag{4.8}$$

By the help of (4.5), (4.6), (4.7), (4.8) in (4.4), we have

$$\begin{aligned} 0 = & \left[\eta\left(C^G(Y, Z) U\right) X-g\left(X, C^G(Y, Z) U\right) \xi\right] \\ & -\left[\eta(Y) C^G(X, Z) U-g(X, Y) C^G(\xi, Z) U\right] \\ & -\left[\eta(Z) C^G(Y, X) U-g(X, Z) C^G(Y, \xi) U\right] \\ & -\left[\eta(U) C^G(Y, Z) X-g(X, U) C^G(Y, Z) \xi\right], \end{aligned} \tag{4.9}$$

$$\begin{aligned} & \text{i.e. } C^G(Y, Z, U, \xi) X-g\left(X, C^G(Y, Z) U\right) \xi \\ = & \eta(Y) C^G(X, Z) U-g(X, Y) C^G(\xi, Z) U \\ & +\eta(Z) C^G(Y, X) U-g(X, Z) C^G(Y, \xi) U \\ & +\eta(U) C^G(Y, Z) X-g(X, U) C^G(Y, Z) \xi. \end{aligned} \tag{4.10}$$

Taking covariant derivative with $\xi$ in (4.10) and then contracting over $X$ and $Y$ we get

$$\begin{aligned} & g(e_i, e_i) C^G(\xi, Z, U, \xi)+g(e_i, Z) C^G(e_i, \xi, U, \xi) \\ & +g(e_i, U) C^G(e_i, Z, \xi, \xi) \\ = & C^G(e_i, Z, U, e_i)+\eta(Z) C^G(e_i, e_i, U, \xi) \\ & +\eta(U) C^G(e_i, Z, e_i, \xi). \end{aligned} \tag{4.11}$$

By the help of (2.9), (2.15), (2.16) and (4.2) we obtain the followings

$$\begin{aligned} & C^{G}\left(\xi, Y, Z, \xi\right) \\ = & \overline{C}\left[\eta\left(Z\right)\eta\left(Y\right) - g\left(Y, Z\right)\right] \\ & -\frac{1}{n-2}\left[S^{G}\left(Y, Z\right) + \overline{C}\left(n-1\right)\eta\left(Z\right)\eta\left(Y\right)\right] \\ & -\frac{1}{n-2}\left[-\overline{C}\left(n-1\right)g\left(Y, Z\right) + \overline{C}\left(n-1\right)\eta\left(Z\right)\eta\left(Y\right)\right] \\ & +\frac{r}{\left(n-1\right)\left(n-2\right)}\left[g\left(Y, Z\right) - \eta\left(Z\right)\eta\left(Y\right)\right], \end{aligned} \tag{4.12}$$

$$\begin{aligned} & C^{G}\left(e_{i}, Z, U, e_{i}\right) \\ = & S^{G}\left(Z, U\right) - \frac{1}{n-2}\left[nS^{G}\left(Z, U\right) - S^{G}\left(Z, U\right)\right] \\ & -\frac{1}{n-2}\left[g\left(Z, U\right)r^{G} - S^{G}\left(Z, U\right)\right] \\ + & \frac{r}{\left(n-1\right)\left(n-2\right)}\left[ng\left(Z, U\right) - g\left(Z, U\right)\right], \end{aligned} \tag{4.13}$$

$$\begin{aligned} & g\left(e_{i}, Z\right)C^{G}\left(e_{i}, \xi, U, \xi\right) \\ = & \overline{C}\left[g\left(U, Z\right) - \eta\left(Z\right)\eta\left(U\right)\right] \\ & -\frac{1}{n-2}\left[-\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) - S^{G}\left(Z, U\right)\right] \\ & -\frac{1}{n-2}\left[-\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) + \left(n-1\right)\overline{C}g\left(Z, U\right)\right] \\ + & \frac{r}{\left(n-1\right)\left(n-2\right)}\left[\eta\left(Z\right)\eta\left(U\right) - g\left(Z, U\right)\right], \end{aligned} \tag{4.14}$$

$$\begin{aligned} & C^{G}\left(e_{i}, Z, e_{i}, \xi\right) \\ = & \left(n-1\right)\overline{C}\eta\left(Z\right) - \frac{1}{n-2}\left[-\left(n-1\right)\overline{C}\eta\left(Z\right) - r^{G}\eta\left(Z\right)\right] \\ & -\frac{\left(n-1\right)\left(n-1\right)\overline{C}}{n-2}\eta\left(Z\right) \\ + & \frac{r\left(1-n\right)}{\left(n-1\right)\left(n-2\right)}\eta\left(Z\right), \end{aligned} \tag{4.15}$$

$$g\left(e_{i}, U\right)C^{G}\left(e_{i}, Z, \xi, \xi\right) = 0, \tag{4.16}$$

and

$$\eta(Z)\, C^G(e_i, e_i, U, \xi) = 0. \tag{4.17}$$

Using (4.12), (4.13), (4.14), (4.15), (4.16) and (4.17) in (4.11), we obtain

$$\begin{aligned}
& n\overline{C}\left[\eta(Z)\eta(Y) - g(Y,Z)\right] \\
& -\frac{n}{n-2}\left[S^G(Y,Z) + \overline{C}(n-1)\eta(Z)\eta(Y)\right] \\
& -\frac{n\overline{C}(n-1)}{n-2}\left[-g(Y,Z) + \eta(Z)\eta(Y)\right] \\
& +\frac{nr}{(n-1)(n-2)}\left[g(Y,Z) - \eta(Z)\eta(Y)\right] \\
& +\overline{C}\left[g(U,Z) - \eta(Z)\eta(U)\right] \\
& -\frac{1}{n-2}\left[-(n-1)\overline{C}\eta(U)\eta(Z) - S^G(Z,U)\right] \\
& -\frac{(n-1)\overline{C}}{n-2}\left[-\eta(U)\eta(Z) + g(Z,U)\right] \\
& +\frac{r}{(n-1)(n-2)}\left[\eta(Z)\eta(U) - g(Z,U)\right] \\
= \; & S^G(Z,U) - \frac{n-1}{n-2}S^G(Z,U) \\
& -\frac{1}{n-2}\left[g(Z,U)r^G - S^G(Z,U)\right] \\
& +\frac{r}{(n-1)(n-2)}\left[ng(Z,U) - g(Z,U)\right] \\
& +(n-1)\overline{C}\eta(Z)\eta(U) \\
& -\frac{1}{n-2}\left[-(n-1)\overline{C}\eta(Z)\eta(U) - r^G\eta(Z)\eta(U)\right] \\
& -\frac{(n-1)(n-1)\overline{C}}{n-2}\eta(Z)\eta(U) \\
& +\frac{r(1-n)}{(n-1)(n-2)}\eta(Z)\eta(U).
\end{aligned} \tag{4.18}$$

By the help of (2.9), (2.15), (2.16) and simplifying (4.18), we obtain

$$S^G(Z,U) = -\left[\left(\overline{A}+1\right) + \frac{r}{(n-1)}\right] g(Z,U) + \left[\left(\overline{B}+n\right) - \frac{r}{(n-1)}\right]\eta(Z)\eta(U). \tag{4.19}$$

Using (3.5) in (4.19), we get

$$\begin{aligned}
&\left[\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)-1\right]g\left(Z,U\right)+\eta\left(Z\right)\eta\left(U\right)\\
=\;& -\left[\left(\overline{A}+1\right)+\frac{r}{(n-1)}\right]g\left(Z,U\right)\\
&+\left[\left(\overline{B}+n\right)-\frac{r}{(n-1)}\right]\eta\left(Z\right)\eta\left(U\right). \qquad (4.20)
\end{aligned}$$

Putting $Z=U=\xi$ in (4.20), we get

$$\lambda-\frac{p}{2}=\overline{B}-\overline{A}+n+\frac{1}{n}-\frac{2r}{(n-1)}-1. \qquad (4.21)$$

Thus we can state

**Theorem 4.1.** *If $(g,\xi,\lambda,p)$ is an Conformal Ricci soliton admitting $R^G\cdot C^G=0$ on an Sasakian manifold $M$ with respect to the general connection $\nabla^G$, then $\lambda$ and $p$ are related by (4.21).*

**Theorem 4.2.** *If $(g,\xi,\lambda,p)$ is an Conformal Ricci soliton admitting $R^G\cdot C^G=0$ on an Sasakian manifold $M$ with respect to the quarter symmetric metric connection $\nabla^q$, then $\lambda$ and $p$ are related by*

$$\lambda-\frac{p}{2}=2n+\frac{1}{n}-\frac{2r}{(n-1)}-2.$$

**Theorem 4.3.** *If $(g,\xi,\lambda,p)$ is an Conformal Ricci soliton admitting $R^G\cdot C^G=0$ on an Sasakian manifold $M$ with respect to the Shouten-Van Kampen connection $\nabla^s$, then $\lambda$ and $p$ are related by*

$$\lambda-\frac{p}{2}=\frac{1}{n}-\frac{2r}{(n-1)}.$$

**Theorem 4.4.** *If $(g,\xi,\lambda,p)$ is an Conformal Ricci soliton admitting $R^G\cdot C^G=0$ on an Sasakian manifold $M$ with respect to generalized Tanaka Webster connection $\nabla^T$, then $\lambda$ and $p$ are related by*

$$\lambda-\frac{p}{2}=\frac{1}{n}-\frac{2r}{(n-1)}.$$

**Theorem 4.5.** *If $(g,\xi,\lambda,p)$ is an Conformal Ricci soliton admitting $R^G\cdot C^G=0$ on an Sasakian manifold $M$ with respect to Zamkovoy connection $\nabla^z$, then $\lambda$ and $p$ are related by*

$$\lambda-\frac{p}{2}=\frac{1}{n}-\frac{2r}{(n-1)}.$$

## 5. $P^G$-semi symmetric Sasakian manifolds admitting Conformal Ricci soliton

The projective curvature tensor $P^G$ [12] in general connection is given by

$$P^G(X,Y,Z,U) = R^G(X,Y,Z,U) - \frac{1}{n-1}\left[S^G(Y,Z)\,g(X,U) - S^G(X,Z)\,g(Y,U)\right]. \tag{5.1}$$

In this section, we assume $P^G$-semi-symmetric Sasakian manifold in general connection admitting a Conformal Ricci soliton $(g, V, \lambda, p)$. Then, we have

$$R^G(V,X) \circ P^G(Y,Z)\,U = 0, \tag{5.2}$$

for all $U, V \in \chi(M)$, set of all vector fields of the manifold $M$. The above equation can also be written as

$$\begin{aligned} 0 &= R^G(V,X)P^G(Y,Z)U - P^G(R^G(V,X)Y,Z)U \\ &\quad - P^G(Y,R^G(V,X)Z)U - P^G(Y,Z)R^G(V,X)U. \end{aligned} \tag{5.3}$$

Putting in $V = \xi$ in (5.3), we get

$$\begin{aligned} 0 &= R^G(\xi,X)\,P^G(Y,Z)\,U - P^G\left(R^G(\xi,X)\,Y,Z\right)U \\ &\quad - P^G\left(Y,R^G(\xi,X)\,Z\right)U - P^G(Y,Z)\,R^G(\xi,X)U. \end{aligned} \tag{5.4}$$

By the help of (2.21), (2.22) and (2.23) we obtain the followings

$$\begin{aligned} & R^G(\xi,X)\,P^G(Y,Z)\,U \\ &= \overline{C}\left[P^G(Y,Z,U,\xi)X - g\left(X,P^G(Y,Z)U\right)\xi\right], \end{aligned} \tag{5.5}$$

$$\begin{aligned} & P^G\left(R^G(\xi,X)\,Y,Z\right)U \\ &= P^G\left(\overline{C}\left[\eta(Y)\,X - g(X,Y)\,\xi\right],Z\right)U \\ &= \overline{C}\left[\eta(Y)\,P^G(X,Z)\,U - g(X,Y)\,P^G(\xi,Z)\,U\right], \end{aligned} \tag{5.6}$$

$$\begin{aligned} & P^G\left(Y,R^G(\xi,X)\,Z\right)U \\ &= P^G\left(Y,\overline{C}\left[\eta(Z)\,X - g(X,Z)\,\xi\right]\right)U \\ &= \overline{C}\left[\eta(Z)\,P^G(Y,X)\,U - g(X,Z)\,P^G(Y,\xi)\,U\right], \end{aligned} \tag{5.7}$$

$$\begin{aligned}
& P^G(Y,Z)\,R^G(\xi,X)U \\
= \;& P^G(Y,Z)\,R^G(\xi,X)U \\
= \;& P^G(Y,Z)\,\overline{C}\,[\eta(U)\,X - g(X,U)\,\xi] \\
= \;& \overline{C}\left[\eta(U)\,P^G(Y,Z)\,X - g(X,U)\,P^G(Y,Z)\,\xi\right]. \qquad (5.8)
\end{aligned}$$

Using (5.5), (5.6), (5.7), (5.8) in (5.4), we get

$$\begin{aligned}
0 \;=\; & \left[\eta\left(P^G(Y,Z)U\right)X - g\left(X, P^G(Y,Z)U\right)\xi\right] \\
& - \left[\eta(Y)\,P^G(X,Z)\,U - g(X,Y)\,P^G(\xi,Z)\,U\right] \\
& - \left[\eta(Z)\,P^G(Y,X)\,U - g(X,Z)\,P^G(Y,\xi)\,U\right] \\
& - \left[\eta(U)\,P^G(Y,Z)\,X - g(X,U)\,P^G(Y,Z)\,\xi\right], \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
& \text{i.e. } P^G(Y,Z,U,\xi)\,X - g\left(X, P^G(Y,Z)U\right)\xi \\
= \;& \eta(Y)\,P^G(X,Z)\,U - g(X,Y)\,P^G(\xi,Z)\,U \\
& + \eta(Z)\,P^G(Y,X)\,U - g(X,Z)\,P^G(Y,\xi)\,U \\
& + \eta(U)\,P^G(Y,Z)\,X - g(X,U)\,P^G(Y,Z)\,\xi. \qquad (5.10)
\end{aligned}$$

Taking covariant derivative with $\xi$ in (5.10) and then contracting over $X$ and $Y$ we get

$$\begin{aligned}
& g(e_i,e_i)\,P^G(\xi,Z,U,\xi) + g(e_i,Z)\,P^G(e_i,\xi,U,\xi) \\
& + g(e_i,U)\,P^G(e_i,Z,\xi,\xi) \\
= \;& P^G(e_i,Z,U,e_i) + \eta(Z)\,P^G(e_i,e_i,U,\xi) \\
& + \eta(U)\,P^G(e_i,Z,e_i,\xi). \qquad (5.11)
\end{aligned}$$

Using (2.9), (2.15), (2.16) and (5.1) we obtain the followings

$$\begin{aligned}
& g(e_i,e_i)\,P^G(\xi,Z,U,\xi) \\
= \;& ng(Z,U) - n\eta(Z)\,\eta(U) \\
& + n(k_1 - k_1k_2 + k_2)\,[\eta(Z)\,\eta(U) - g(Z,U)] \\
& - \frac{n}{n-1}\left[S^G(Z,U) + (n-1)\,\overline{C}\eta(Z)\,\eta(U)\right], \qquad (5.12)
\end{aligned}$$

$$\begin{aligned}
& g(e_i,Z)\,P^G(e_i,\xi,U,\xi) \\
= \;& \eta(U)\,\eta(Z) - g(Z,U) + (k_1 - k_1k_2 + k_2)\,[g(Z,U) - \eta(U)\,\eta(Z)] \\
& + \frac{1}{n-1}\left[(n-1)\,\overline{C}\eta(Z)\,\eta(U) + S^G(Z,U)\right], \qquad (5.13)
\end{aligned}$$

$$P^{G}\left(e_{i},Z,U,e_{i}\right)=S^{G}\left(Z,U\right)-\frac{1}{n-1}\left[nS^{G}\left(Z,U\right)-S^{G}\left(Z,U\right)\right], \tag{5.14}$$

$$\begin{aligned} & P^{G}\left(e_{i},Z,e_{i},\xi\right) \\ = \; & \eta\left(Z\right)-n\eta\left(Z\right)+\left(k_{1}-k_{1}k_{2}+k_{2}\right)\left[n\eta\left(Z\right)-\eta\left(Z\right)\right] \\ & -\frac{1}{n-1}\left[-\left(n-1\right)\overline{C}\eta\left(Z\right)-r^{G}\eta\left(Z\right)\right], \end{aligned} \tag{5.15}$$

$$g\left(e_{i},U\right)P^{G}\left(e_{i},Z,\xi,\xi\right)=0, \tag{5.16}$$

$$\eta\left(Z\right)P^{G}\left(e_{i},e_{i},U,\xi\right)=0. \tag{5.17}$$

By the help of (5.12), (5.13), (5.14), (5.15), (5.16) and (5.17) in (5.11), we have

$$\begin{aligned} & ng\left(Z,U\right)-n\eta\left(Z\right)\eta\left(U\right) \\ & +n\left(k_{1}-k_{1}k_{2}+k_{2}\right)\left[\eta\left(Z\right)\eta\left(U\right)-g\left(Z,U\right)\right] \\ & -\frac{n}{n-1}\left[S^{G}\left(Z,U\right)+\left(n-1\right)\overline{C}\eta\left(Z\right)\eta\left(U\right)\right]+\eta\left(U\right)\eta\left(Z\right) \\ & -g\left(Z,U\right)+\left(k_{1}-k_{1}k_{2}+k_{2}\right)\left[g\left(Z,U\right)-\eta\left(U\right)\eta\left(Z\right)\right] \\ & +\frac{1}{n-1}\left[\left(n-1\right)\overline{C}\eta\left(Z\right)\eta\left(U\right)+S^{G}\left(Z,U\right)\right] \\ = \; & S^{G}\left(Z,U\right)-\frac{1}{n-1}\left[nS^{G}\left(Z,U\right)-S^{G}\left(Z,U\right)\right]+\eta\left(Z\right)\eta\left(U\right) \\ & +\left(k_{1}-k_{1}k_{2}+k_{2}\right)\left[n\eta\left(Z\right)\eta\left(U\right)-\eta\left(Z\right)\eta\left(U\right)\right]-n\eta\left(Z\right)\eta\left(U\right) \\ & -\frac{1}{n-1}\left[-\left(n-1\right)\overline{C}\eta\left(Z\right)\eta\left(U\right)-r^{G}\eta\left(Z\right)\eta\left(U\right)\right]. \end{aligned} \tag{5.18}$$

By the help of (2.9), (2.15), (2.16) and simplifying (5.18), we get

$$\begin{aligned} & S^{G}\left(Z,U\right) \\ = \; & \left[-\left(n-1\right)\left(\overline{B}+n\right)+r\right]\eta\left(Z\right)\eta\left(U\right)-\left(n-1\right)\overline{C}g\left(Z,U\right). \end{aligned} \tag{5.19}$$

Using (3.5) in (5.19), we obtain

$$\begin{aligned} & \left[-\left(n-1\right)\left(\overline{B}+n\right)+r\right]\eta\left(Z\right)\eta\left(U\right)-\left(n-1\right)\overline{C}g\left(Z,U\right) \\ = \; & \left[\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right)-1\right]g\left(X,Y\right)+\eta\left(X\right)\eta\left(Y\right). \end{aligned} \tag{5.20}$$

Setting $Z=U=\xi$ in (5.20), we have

$$\left[-\left(n-1\right)\left(\overline{B}+n\right)+r\right]-\left(n-1\right)\overline{C}=\lambda-\frac{1}{2}\left(p+\frac{2}{n}\right).$$

Thus we can state

**Theorem 5.1.** *If* $(g, \xi, \lambda, p)$ *is an Conformal Ricci soliton admitting* $R^G \cdot P^G = 0$ *on an Sasakian manifold* $M$ *with respect to the general connection* $\nabla^G$*, then* $\lambda$ *and* $p$ *are related by (5.21).*

**Theorem 5.2.** *If* $(g, \xi, \lambda, p)$ *is an Conformal Ricci soliton admitting* $R^G \cdot P^G = 0$ *on an Sasakian manifold* $M$ *with respect to the quarter symmetric metric connection* $\nabla^q$*, then* $\lambda$ *and* $p$ *are related by*

$$-2(n-1)^2 + r = \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right).$$

**Theorem 5.3.** *If* $(g, \xi, \lambda, p)$ *is an Conformal Ricci soliton admitting* $R^G \cdot P^G = 0$ *on an Sasakian manifold* $M$ *with respect to the Shouten-Van Kampen connection* $\nabla^s$*, then* $\lambda$ *and* $p$ *are related by*

$$[-(n-1) + r] = \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right).$$

**Theorem 5.4.** *If* $(g, \xi, \lambda, p)$ *is an Conformal Ricci soliton admitting* $R^G \cdot P^G = 0$ *on an Sasakian manifold* $M$ *with respect to generalized Tanaka Webster connection* $\nabla^T$*, then* $\lambda$ *and* $p$ *are related by*

$$[-3(n-1) + r] = \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right).$$

**Theorem 5.5.** *If* $(g, \xi, \lambda, p)$ *is an Conformal Ricci soliton admitting* $R^G \cdot P^G = 0$ *on an Sasakian manifold* $M$ *with respect to Zamkovoy connection* $\nabla^z$*, then* $\lambda$ *and* $p$ *are related by*

$$(n-1) + r = \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right).$$

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