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Research Paper

THE DISTANCE SEIDEL SPECTRUM OF SOME GRAPH OPERATIONS

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ABSTRACT. The distance matrix, distance eigenvalue, and distance energy of a connected graph have been studied in detail in literature where as the study on distance seidel matrix associated with a connected graph is in progress. The eigenvalues $\partial_1^S \geq \partial_2^S \geqslant ...\partial_n^S$ of the distance seidel matrix $D^S(G)$ of a graph G forms the distance seidel spectrum of G. We describe here the distance seidel spectrum of some types of subdivision related graphs of a regular graph in terms of its adjacency spectrum. We also derive analytic expressions for the distance seidel energy of $\bar{S}(C_p)$, the partial complement of the subdivision graph of a cycle C_p and the distance seidel energy of $\overline{S(C_p)}$, the complement of the even cycle C_{2p}.

Key Words: Distance Seidel Matrix; Distance Seidel Spectrum; Distance Seidel Energy
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1. Introduction

Let G be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and edge set $E(G)$. In [11] the authors introduce distance seidel matrix defined as $D^S(G) = J-I-2D$ where D is the distance matrix of G. The eigenvalues of $D^S(G)$ be $\partial_1^S \geq \partial_2^S \geqslant ...\partial_p^S$, called the distance seidel eigenvalues (D^S eigenvalues) and the set of all D^S- eigenvalues of G is said to be the distance seidel spectrum

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$Spec_{D^S}(G)$ of G . The distance seidel energy $\mathcal{E}_{D^S}(G)$ of a graph G is defined as

$$\mathcal{E}_{D^S}(G) = \sum_{i=1}^{p} \mid \partial_i^S \mid .$$

In [11] the authors obtain the distance Seidel spectra of different graph operations such as join, cartesian product, lexicographic product, and unary operations like the double graph and extended double cover graph and some other related results on bounds on eigenvalues. In this paper we obtain some results on distance seidel spectrum of some subdivision related graphs.

The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. We denote the set of such new vertices by $I(G)$.

All graphs considered in this paper are simple and we follow [3] for spectral graph theoretic terminology.

The discussions in the subsequent sections are based upon the following definitions and lemmas:

Definition 1.1. [4] The subdivision-vertex join of two vertex disjoint graphs G_1 and G_2, denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$.

Definition 1.2. [4] The subdivision-edge join of two vertex disjoint graphs G_1 and G_2, denoted by $G_1 \underline{\vee} G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$.

Definition 1.3. [5] The Indu - Bala product of G_1 and G_2 is denoted by $G_1 \blacktriangledown G_2$ and is obtained from two disjoint copies of the join $G_1 \vee G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2.

Definition 1.4. [21] Let G be a (p, q) graph. Corresponding to every edge e of G introduce a vertex and make it adjacent with all the vertices not incident with e in G. Delete the edges of G only. The resulting graph is called the partial complement of the subdivision graph(PCSD) of G denoted by $S(G)$.

Lemma 1.5. [3] *Let G be a $r-$ regular (p, q) graph with an adjacency matrix A and an incidence matrix R. Let $L(G)$ be its line graph. Then $RR^T = A + rI$, $R^T R = A(L(G)) + 2I$. Also if J denote an all one matrix of appropriate order then $JR = 2J = R^T J$ and $JR^T = rJ = RJ$*

Lemma 1.6. [3] *Let G be an $r-$ regular (p, q) graph with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then*
$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2+r-2 & .. & \lambda_p+r-2 & -2 \\ 1 & 1 & .. & 1 & q-p \end{pmatrix}$. *Also Z is an eigenvector belonging to the eigenvalue -2 if and only if $RZ = 0$.*

2. The distance seidel spectrum of $G_1\dot{\vee}G_2$ and $G_1\underline{\vee}G_2$

The distance spectrum of the graphs $G_1\dot{\vee}G_2$ and $G_1\underline{\vee}G_2$ were obtained in [23]. However, the distance seidel spectrum of these two graphs have not yet been studied. In this section we obtain the distance seidel spectrum of the graphs $G_1\dot{\vee}G_2$ and $G_1\underline{\vee}G_2$ when G_1 and G_2 are regular graphs.

Theorem 2.1. *Let G_i be an r_i regular graph on p_i vertices and q_i edges with an adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, ..., \lambda_{ip_i}\}, i = 1, 2$. Then the distance seidel spectrum of $G_1\dot{\vee}G_2$ consists of the following numbers:*

$(4\lambda_{1j} + 4r_1 + 3), j = 2, 3, ..., p_1; -1$ *of multiplicity* $q_1 - 1$;
$(2\lambda_{2j} + 3); j = 2, 3, ..., p_2$
together with the three eigenvalues of

$$\begin{bmatrix} 3-3p_1 & -5q_1+4r_1 & -p_2 \\ -5p_1+8 & 8r_1-7q_1-1 & -3p_2 \\ -p_1 & -3q_1 & 2r_2-3p_2+3 \end{bmatrix}$$

Proof. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1 and B be the adjacency matrix of $L(G_1)$. Then by a proper ordering of vertices of $G_1\dot{\vee}G_2$, its distance seidel matrix D^S can be written as

$$D^S = \begin{bmatrix} 3(I-J) & 4R-5J & -J \\ 4R^T-5J & 7(I-J)+4B & -3J \\ -J & -3J & 3(I-J)+2A_2 \end{bmatrix}$$

where J and I denote the all one matrix and identity matrix respectively of appropriate orders. Thus D^S is a square matrix of order $p_1 + q_1 + p_2$.

Let $\lambda \neq r_1$ be an eigenvalue of A_1 with an eigenvector X. Then by the theorem of Perron - Frobenius, X is orthogonal to the all one matrix J and $A_1X = \lambda X$. Now by Lemma 1.5, we have the following

$$\begin{aligned}
RR^T &= A_1 + r_1 I \\
RR^T X &= (A_1 + r_1 I)\, X \\
&= (\lambda + r_1) X \\
B &= R^T R - 2I \\
BR^T X &= \left(R^T R - 2I\right) R^T X \\
&= R^T (A_1 + r_1 I)\, X - 2R^T X \\
&= (\lambda + r_1 - 2)\, R^T X
\end{aligned}$$

Now $\varphi = \begin{bmatrix} X \\ R^T X \\ 0 \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue $4\lambda + 4r_1 + 3$. This is because

$$\begin{aligned}
D^S \varphi &= \begin{bmatrix} 3(I-J) & 4R - 5J & -J \\ 4R^T - 5J & 7(I-J) + 4B & -3J \\ -J & -3J & 3(I-J) + 2A_2 \end{bmatrix} \begin{bmatrix} X \\ R^T X \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3X + 4(\lambda + r_1)X \\ 4R^T X + 7R^T X + 4(\lambda + r_1 - 2)R^T X \\ 0 \end{bmatrix} \\
&= (3 + 4\lambda + 4r_1)\varphi
\end{aligned}$$

Now let Y be an eigenvector of $A(L(G_1))$ corresponding to the eigenvalue $\lambda + r_1 - 2$, different from the regularity of $L(G_1)$. Then Y is orthogonal to J . Now using Lemma 1.5, we can show that $\phi = \begin{bmatrix} RY \\ -Y \\ 0 \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue -1. This is because

$$\begin{aligned} D^S\phi &= \begin{bmatrix} 3(I-J) & 4R-5J & -J \\ 4R^T-5J & 7(I-J)+4B & -3J \\ -J & -3J & 3(I-J)+2A_2 \end{bmatrix} \begin{bmatrix} RY \\ -Y \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3RY-4RY \\ 4(\lambda+r_1)Y-7Y-4(\lambda+r_1-2)Y \\ 0 \end{bmatrix} \\ &= -1\phi \end{aligned}$$

Now -2 is an eigenvalue of $A(L(G_1))$ with multiplicity q_1-p_1 times. Let Z be an eigenvector of $A(L(G_1))$with eigenvalue -2. Then by Lemma 1.6, $RZ=0$. Now $\chi = \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue -1. This is because

$$\begin{aligned} D^S\chi &= \begin{bmatrix} 3(I-J) & 4R-5J & -J \\ 4R^T-5J & 7(I-J)+4B & -3J \\ -J & -3J & 3(I-J)+2A_2 \end{bmatrix} \begin{bmatrix} 0 \\ Z \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \\ &= -1\chi \end{aligned}$$

Now let $\mu \neq r_2$ be an eigenvalue of G_2 with an eigenvector W. Then $\eta = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue $(2\mu+3)$. This is because

$$D^S\eta = \begin{bmatrix} 3(I-J) & 4R-5J & -J \\ 4R^T-5J & 7(I-J)+4B & -3J \\ -J & -3J & 3(I-J)+2A_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ (2\mu+3)\,W \end{bmatrix} = (2\mu+3))\,\eta$$

Thus we get a totality of $p_1-1+p_1-1+q_1-p_1+p_2-1 = p_1+q_1+p_2-3$ eigenvalues and there remains three. By the very form of the eigenvectors constructed so far, they all are orthogonal to $\begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ J \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ J \end{bmatrix}$. Also since D^S is symmetric, $R^{p_1+p_2+q_1}$ has an orthogonal basis consisting of eigenvectors of D^S and hence the remaining three eigenvectors are spanned by these three vectors and is of the form $\rho = \begin{bmatrix} \alpha J \\ \beta J \\ \gamma J \end{bmatrix}$ for some $(\alpha,\beta,\gamma) \neq (0,0,0)$. Thus if σ is an eigenvalue of D^S with an eigenvector ρ then from $D^S\rho = \sigma\rho$ we can see that the remaining three are the eigenvalues of the matrix $\begin{bmatrix} 3(1-p_1) & -5q_1+4r_1 & -p_2 \\ -5p_1+8 & -7q_1+8r_1-1 & -3p_2 \\ -p_1 & -3q_1 & 2r_2-3p_2+3 \end{bmatrix}$ and this completes the proof.

□

Theorem 2.2. *Let G_i be an r_i regular graph on p_i vertices and q_i edges with an adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, ..., \lambda_{ip_i}\}$, $i = 1,2$. Then the distance seidel spectrum of $G_1 \underline{\vee} G_2$ consists of the following numbers:*

$\left(2\lambda_{1j}+5\pm 2\sqrt{(\lambda_{1j}+1)^2+4(\lambda_{1j}+r_1)}\right), j=2,3,...,p_1;$
3 *of multiplicity* q_1-p_1;
$(2\lambda_{2j}+3)\,; j=2,3,...,p_2$
together with the three eigenvalues of

$$\begin{bmatrix} -7(p_1-1)+4r_1 & -5q_1+4r_1 & -3p_2 \\ -5p_1+8 & 3(-q_1+1) & -p_2 \\ -3p_1 & -q_1 & -3(p_2-1)+2r_2 \end{bmatrix}$$

Proof. The proof is on similar lines as that of Theorem 2.1 □

3. The distance seidel spectrum of Indu - Bala product of graphs

The distance spectrum of $G_1 \blacktriangledown G_2$ is obtained in [5]. This section focuses on the determination of the distance seidel spectrum of $G_1 \blacktriangledown G_2$ when G_1 and G_2 are two regular graphs. Using this we obtain a pair of distance seidel equienergetic graphs of diameter 3 on p vertices for every $p = 18+2k, k \geq 1$.

Theorem 3.1. *Let G_i be an r_i regular graph on p_i vertices with an adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, ..., \lambda_{ip_i}\}, i = 1,2$. Then the distance seidel spectrum of $G_1 \blacktriangledown G_2$ consists of the following numbers: $(2\lambda_{1j}+3), j = 2,3,....,p_1$, each with multiplicity 2; -1 of multiplicity p_2-1; $(4\lambda_{2j}+7), j = 2,3,....,p_2$ together with the four eigenvalues of the matrix*

$$\begin{bmatrix} 3-3p_1+2r_1 & -p_2 & -3p_2 & -5p_1 \\ -p_1 & 3-3p_2+2r_2 & 4-5p_2+2r_2 & -3p_1 \\ -3p_1 & 4-5p_2+2r_2 & 3-3p_2+2r_2 & -p_1 \\ -5p_1 & -3p_2 & -p_2 & 3-3p_1+2r_1 \end{bmatrix}$$

Proof. Given that G_1 and G_2 are regular graphs with regularity r_1 and r_2 respectively. Let R be the incidence matrix of G_1. Then by a proper ordering of vertices of $G_1 \blacktriangledown G_2$, its distance seidel matrix D^S can be written as

$$\begin{bmatrix} 3(I-J)+2A_{G_1} & -J & -3J & -5J \\ -J & 3(I-J)+2A_{G_2} & 4I-5J+2A_{G_2} & -3J \\ -3J & 4I-5J+2A_{G_2} & 3(I-J)+2A_{G_2} & -J \\ -5J & -3J & -J & 3(I-J)+2A_{G_1} \end{bmatrix}$$

where J and I denote the all one matrix and identity matrix respectively of appropriate orders.

Let λ be an eigenvalue of the adjacency matrix of G_1 with an eigenvector X. Then by the theorem of Perron - Frobenius, X is orthogonal to the all one matrix J and $A_1X = \lambda X$. Now $\varphi = \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue $(2\lambda_i + 3)$. This is because

$$D^S\varphi = \begin{bmatrix} 3(I-J)+2A_{G_1} & -J & -3J & -5J \\ -J & 3(I-J)+2A_{G_2} & 4I-5J+2A_{G_2} & -3J \\ -3J & 4I-5J+2A_{G_2} & 3(I-J)+2A_{G_2} & -J \\ -5J & -3J & -J & 3(I-J)+2A_{G_1} \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3X+2\lambda X \\ 0 \\ 0 \\ 0 \end{bmatrix} = (2\lambda+3)\,\varphi$$

In a similar way the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ X^T \end{bmatrix}$ is an eigenvector of D^S corresponding to the eigenvalue $(2\lambda + 3)$ Now let Y be an eigenvector of $A(G_2)$ corresponding to the eigenvalue μ. Then Y is orthogonal to J . Now using Lemma 1.5, we can show that $\phi = \begin{bmatrix} 0 \\ Y^T \\ Y^T \\ 0 \end{bmatrix}$ is an eigenvector of D^S with an eigenvalue $7 + 4\mu_i$. This is because

$$D^S\phi = \begin{bmatrix} 3(I-J)+2A_{G_1} & -J & -3J & -5J \\ -J & 3(I-J)+2A_{G_2} & 4I-5J+2A_{G_2} & -3J \\ -3J & 4I-5J+2A_{G_2} & 3(I-J)+2A_{G_2} & -J \\ -5J & -3J & -J & 3(I-J)+2A_{G_1} \end{bmatrix} \begin{bmatrix} 0 \\ Y^T \\ Y^T \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ (7+4\mu)Y^T \\ (7+4\mu)Y^T \\ 0 \end{bmatrix} = (7+4\mu)\phi$$

In a similar way the vector $\begin{bmatrix} 0 \\ Y^T \\ -Y^T \\ 0 \end{bmatrix}$ is an eigenvector of D^S corresponding to the eigenvalue -1

Thus we get a totality of $p_1-1+p_1-1+p_2-1+p_2-1=2p_1+2p_2-4$ eigenvalues and there remains four. By the very form of the eigenvectors constructed so far, they all are orthogonal to $\begin{bmatrix} J \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ J \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ J \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ J \end{bmatrix}$. This means that these four vectors span the space spanned by the remaining four eigenvectors of D^S and is of the form $\rho = \begin{bmatrix} \alpha J \\ \beta J \\ \gamma J \\ \delta J \end{bmatrix}$ for some $(\alpha, \beta, \gamma, \delta) \neq (0,0,0,0)$. Thus if σ is an eigenvalue of D^S with an eigenvector ρ then from $D^S\rho = \sigma\rho$ we can see that the remaining four are the eigenvalues of the matrix

$$\begin{bmatrix} 3-3p_1+2r_1 & -p_2 & -3p_2 & -5p_1 \\ -p_1 & 3-3p_2+2r_2 & 4-5p_2+2r_2 & -3p_1 \\ -3p_1 & 4-5p_2+2r_2 & 3-3p_2+2r_2 & -p_1 \\ -5p_1 & -3p_2 & -p_2 & 3-3p_1+2r_1 \end{bmatrix}$$

and this completes the proof. □

Corollary 3.2. *The D^S- spectrum of $\overline{K_p} \blacktriangledown \overline{K_{p+1}}$ is*

$$\begin{pmatrix} -12p-1 & -4p+3 & 4p+3 & 1 & 7 & -1 & 3 \\ 1 & 1 & 1 & 1 & p & p & 2(p-1) \end{pmatrix}$$

Proof. The corollary follows from Theorem 3.1 by setting $p_1 = p; p_2 = p+1$ and $r_1 = r_2 = 0$. Clearly $\overline{K_p} \blacktriangledown \overline{K_{p+1}}$ is distance seidel integral. □

The distance seidel spectrum of $\overline{K_p} \blacktriangledown \overline{K_{p+1}}$ is integral while its distance spectrum has been proved to be also integral in [5]. This gives rise to an infinite family of regular graphs of diameter 3 all of which having integral distance spectrum and integral distance seidel spectrum.

Corollary 3.3. *There exists a pair of graphs of diameter 3 on p vertices for every $p = 18+2k, k \geq 1, \mathcal{E}_{D^S}(F_1^k) = \mathcal{E}_{D^S}(F_2^k)$.*

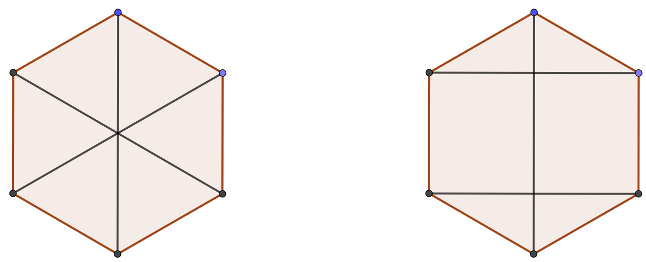

FIGURE 1. The graphs H_1 and H_2

Proof. Let H_1 and H_2 be the graphs of $Fig.1$. H_1 and H_2 are both 3 - regular graphs on 6 vertices with adjacency spectra

$$\begin{pmatrix} 3 & 0 & 3 \\ 1 & 4 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

respectively.

Let G_1 and G_2 denote respectively their line graphs. Then G_1 and G_2 both have 9 vertices and are $4-$ regular. The adjacency spectra of G_1 and G_2 are

$$\begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & 4 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 & 2 & 1 & -1 & -2 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}$$

respectively. Thus G_1 and G_2 are not cospectral.
Let $F_1^k = \overline{K_k} \blacktriangledown G_1$ and $F_2^k = \overline{K_k} \blacktriangledown G_2$. Then each of the graphs F_1^k and F_2^k has $p = 18 + 2k$ vertices. Further by Theorem 3.1, we have $\mathcal{E}_{D^S}(F_1^k) = 2(68 + 3k)$ and $\mathcal{E}_{D^S}(F_1^k) = \mathcal{E}_{D^S}(F_2^k)$. $\square$

In the following sections we obtain the three kind of spectra of the subdivision graph of K_p, partial complement of the subdivision graph of a regular graph G and complement of the subdivision graph of a regular graph. We also derive the distance seidel energy of $\bar{S}(C_p)$, partial complement of the subdivision of a cycle C_p and the distance seidel energy of $\overline{S(C_p)}$, complement of the subdivision of a cycle C_p.

4. Distance seidel spectrum of the subdivision graph of K_p

In [22] the Distance spectrum, Distance laplacian spectrum and Distance signless laplacian spectrum of the subdivision graph of K_p are obtained. In this section we obtain the Distance seidel spectrum of the subdivision graph of K_p .

Theorem 4.1. *The Distance seidel spectrum of the subdivision graph of K_p consists of the numbers:*
(i) $(4\lambda_i + 4r + 3)$ *of multiplicity* $p - 1$
(ii) -1 *of multiplicity* $\dfrac{p^2 - p - 2}{2}$ *and*
(iii) $\dfrac{(8r - 3p - 7q + 2) \pm \sqrt{(8r - 3p - 7q + 2)^2 + 32r - 76q + 16pr + 16pq - 12p + 12}}{2}$

Proof. Let K denote the subdivision graph of K_p. Then the distance seidel matrix D^S of K has the form

$$D^S = \begin{bmatrix} 3(I - J) & 4R - 5J \\ 4R^T - 5J & 7(I - J) + 4B \\ & \end{bmatrix}$$

where B is the adjacency matrix of the line graph of K_p.

Let X_i, $i = 1, 2, ...q - p$ be $q - p$ linearly independent eigenvectors corresponding to the eigenvalue -2 of $L(G)$. Since $L(G)$ is regular $J_{1\times q}$ is orthogonal to X_i and $R.\ X_i = 0$. Now we claim that $\zeta_i = \begin{bmatrix} 0_{p\times 1} \\ X_i \end{bmatrix}$, $i = 1, 2, ...q - p$ are eigenvectors of D^S. This is because

$$D^S.\ \zeta_i = \begin{bmatrix} 3(I - J) & 4R - 5J \\ 4R^T - 5J & 7(I - J) + 4B \end{bmatrix} \begin{bmatrix} 0 \\ X_i \end{bmatrix}$$

$$= (-1)\ .\ \zeta_i$$

Let $\lambda_i \neq r$ be an eigenvalue of A with an eigenvector X_i. Now we look into the condition under which $\Gamma_i = \begin{bmatrix} tX_i \\ R^T X_i \end{bmatrix}$ is an eigenvector of D^S. If μ is an eigenvalue of D^S with Γ_i as an eigenvector, then from the equation $D^S.\ \Gamma_i = \mu.\ \Gamma_i$, we get

$$3t + 4(\lambda_i + r) = \mu t....(1)$$

$$4t + 4(\lambda_i + r) - 1 = \mu....(2)$$

Eliminiating μ from (1) and (2), we get

$$t^2 + (\lambda_i + r - 1)t - (\lambda_i + r) = 0$$

So that t has two values

$$\begin{aligned} t_1 &= 1 \\ t_2 &= -(\lambda_i + r) \end{aligned}$$

Thus we get $4t_1+4(\lambda_i+r)-1$ and $4t_2+4(\lambda_i+r)-1$ for $i = 2, 3, ..., p$ eigenvalues and there remains two. From the hitherto constructed form of the eigenvectors it is proved that they all are orthogonal to, $\begin{bmatrix} J \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ J \end{bmatrix}$. Therefore these two vectors span the remaining two eigenvectors of D^S. Thus the remaining eigenvectors of D^S are then of the form $\rho = \begin{bmatrix} \alpha J \\ \beta J \end{bmatrix}$ for some $(\alpha, \beta) \neq (0, 0)$. Therefore if σ is an eigenvalue of D^S with an eigenvector $\begin{bmatrix} \alpha J \\ \beta J \end{bmatrix}$, from $D^S \begin{bmatrix} \alpha J \\ \beta J \end{bmatrix} = \sigma \begin{bmatrix} \alpha J \\ \beta J \end{bmatrix}$, we get the system of equations:

$$3(1 - p)\alpha + (4r - 5q)\beta = 0$$

$$(8 - 5p)\alpha + (8r - 7q - 1)\beta = 0$$

Then ρ is a non- trivial solution to the above system of equations, which in turn are the eigenvalues of the matrix $\begin{bmatrix} 3(1-p) & 4r - 5q \\ 8 - 5p & 8r - 7q - 1 \end{bmatrix}$, which completes the proof. □

5. The Distance seidel spectrum of the partial complement of the subdivision graph ($PCSD$) of a regular graph G

In [21] the adjacency spectrum of the partial complement of the subdivision graph of a regular graph G in terms of that of G is derived. In [22] the Distance spectrum, Distance laplacian spectrum and Distance signless laplacian spectrum of the partial complement of the subdivision graph of a regular graph G are obtained. In this section we obtain the Distance seidel spectrum of the $PCSD$ of a regular graph G of order $p \geqslant 5$.

Theorem 5.1. *Let G a connected $r-$ regular (p,q) graph with adjacency spectrum $\{r, \lambda_2, ...\lambda_p\}$. Then $spec_{D^S}(\bar{S}(G))$ is*

$$\begin{pmatrix} \frac{(-3p-3q+6)\pm\sqrt{9p^2+9q^2-14pq+8(2pr+4q+16r)}}{2} & 3\pm 4\sqrt{\lambda_i+r} & 3 \\ 1 & 1 & q-p \end{pmatrix},$$

$i = 2, 3, ..., p$

Corollary 5.2. *The distance seidel spectrum of the partial complement of the subdivision graph of a cycle*

$$spec_{D^S}(\bar{S}(C_p)) = \begin{pmatrix} 11-2p & -5-4p & 3\pm 4\sqrt{\lambda_i+r} \\ 1 & 1 & 1 \end{pmatrix}, j = 1, 2, ..., p-1$$

Corollary 5.3. *For $p \geq 6$*

$$\mathcal{E}_{D^S}\left(\overline{S}(C_p)\right) = \begin{cases} 8\sqrt{3}\ \cot\frac{\pi}{2p} + 4(2p-5); p\equiv 0(\text{mod}3) for\ p \leq 21 \\ 16\cos(\frac{p-9}{3}\times\frac{\pi}{2p})cosec\frac{\pi}{2p} + 8(p-5); p\equiv 0(mod\ 3)\ for\ p > 21 \\ 16\cos\left(\frac{p-1}{3}\times\frac{\pi}{2p}\right)\csc\frac{\pi}{2p} + 8\left(p-3\right); p\equiv 1(mod\ 3) for\ p \leq 13 \\ 16\cos(\frac{p-7}{3}\times\frac{\pi}{2p})cosec\frac{\pi}{2p} + 4(2p-9); p\equiv 1(mod\ 3)\ for\ p > 13 \\ 16\cos\left(\frac{p-5}{3}\times\frac{\pi}{2p}\right)\csc\frac{\pi}{2p} + 8\left(p-4\right); p\equiv 2(mod\ 3) \end{cases}$$

Proof. The distance energy of $\bar{S}(C_p)$ is the sum of the absolute values of all of its eigenvalues which in turn is twice of the sum of its positive eigenvalues. First we find the negative eigenvalues other than $11-2p$ and $-5-4p$

We consider the following cases

Case 1: $p \equiv 0(mod\ 3), p \leq 21$.

The numbers $3 -\ 8\cos\frac{\pi j}{p}$ are negative only for $\frac{\pi}{p}j \leq\ \frac{\pi}{3}$ and consequently for $j = 1, 2......\frac{p}{3}$

Thus the negative values from $3 -\ 8\cos\frac{\pi j}{p}$ are

$$3 -\ 8\cos\frac{\pi}{p}, 3 -\ 8\cos\frac{2\pi}{p}, ...3 -\ 8\cos(\frac{p}{3}\times\frac{\pi}{p}).$$

Let $C = 8\cos\frac{\pi}{p} +\ 8\cos\frac{2\pi}{p} +\ ......+8\cos(\frac{p}{3}\times\frac{\pi}{p})$ and

$S = 8\sin\frac{\pi}{p} + 8\sin\frac{2\pi}{p} + \ldots\ldots + 8\sin(\frac{p}{3}\times\frac{\pi}{p})$

$C + i\,S = 8\,e^{i\frac{\pi}{p}} + 8\,e^{i\frac{2\pi}{p}} + \ldots. + 8\,e^{i(\frac{p}{3}\times\frac{\pi}{p})}$

$= 8\{\alpha + \alpha + \ldots. + \alpha^{\frac{p}{3}}\} = 8\alpha\frac{1-\alpha^{\frac{p}{3}}}{1-\alpha}$ where $\alpha = e^{i\frac{\pi}{p}}$

Equating real parts, we get $C = 2\sqrt{3}\cot\frac{\pi}{2p} - 2$

The negative contribution from $3 - 8\cos\frac{\pi j}{p}$ is $\frac{3p}{3} + 2 - 2\sqrt{3}\cot\frac{\pi}{2p}$.The numbers $3 + 8\cos\frac{\pi j}{p}$ are negative only for $\frac{\pi}{p}j \geq \frac{2\pi}{3}$ Therefore the negative contribution from $3 + 8\cos\frac{\pi j}{p}$ is $\frac{3p}{3} + 2 - 2\sqrt{3}\cot\frac{\pi}{2p}$

Therefore the total negative contributions from these eigenvalues $= 2p + 4 - 4\sqrt{3}\cot\frac{\pi}{2p}$

Distance seidel energy $= 2\times$ sum of positive eigenvalues $= 2\times\{2p - 11 + 5 + 4p - 2p - 4 + 4\sqrt{3}\cot\frac{\pi}{2p}\}$

$\mathcal{E}_{D^S}(\bar{S}(C_p))$=$8\sqrt{3}\cot\frac{\pi}{2p} + 4(2p-5)$

Case 2: $p \equiv 0(mod\ 3), p > 21$.

The numbers $3 - 8\cos\frac{\pi j}{p}$ are negative only for $j \leq \frac{p+3}{3}$

Thus the negative values from $3 - 8\cos\frac{\pi j}{p}$ are

$3 - 8\cos\frac{\pi}{p}$, $3 - 8\cos\frac{2\pi}{p}$, ...$3 - 8\cos(\frac{p+3}{3}\times\frac{\pi}{p})$.

Let $C = 8\cos\frac{\pi}{p} + 8\cos\frac{2\pi}{p} + \ldots\ldots + 8\cos(\frac{p+3}{3}\times\frac{\pi}{p})$ and

$S = 8\sin\frac{\pi}{p} + 8\sin\frac{2\pi}{p} + \ldots\ldots + 8\sin(\frac{p+3}{3}\times\frac{\pi}{p})$

$C + i\,S = 8\,e^{i\frac{\pi}{p}} + 8\,e^{i\frac{2\pi}{p}} + \ldots. + 8\,e^{i(\frac{p+3}{3}\times\frac{\pi}{p})}$

$= 8\alpha\{1 + \alpha + \ldots. + \alpha^{\frac{p+3}{3}}\} = 8\alpha\,\frac{1-\alpha^{\frac{p+3}{3}}}{1-\alpha}$ where $\alpha = e^{i\frac{\pi}{p}}$

Equating real parts, we get $C = 4\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p} - 4$

The negative contribution from $3 - 8\cos\frac{\pi j}{p}$ is $3(\frac{p+3}{3}) + 4 - 4\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p}$.The numbers $3 + 8\cos\frac{\pi j}{p}$ are negative only for $j \geq \frac{2p-3}{3}$

Therefore the negative contribution from $3 + 8\cos\frac{\pi j}{p}$ is $3(\frac{p+3}{3}) + 4 - 4\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p}$.

Therefore the total negative contributions from these eigenvalues $= 2p + 14 - 8\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p}$

Distance seidel energy $= 2\times$ sum of positive eigenvalues $= 2 \times \{2p - 11 + 5 + 4p - 2p - 14 + 8\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p}\}$

$\mathcal{E}_{D^S}(\bar{S}(C_p)) = 16\cos(\frac{p-9}{3} \times \frac{\pi}{2p}) \csc \frac{\pi}{2p} + 8(p-5)$

The other two cases $p \equiv 1(mod\ 3)$ and $p \equiv 2(mod\ 3)$ can be similarly proved.

□

6. Distance seidel spectrum of the complement of the subdivision graph of a regular graph

In [22] the Distance spectrum, Distance laplacian spectrum and Distance signless laplacian spectrum of the complement of the subdivision graph of a regular graph are obtained. In this section, the Distance seidel spectrum of the complement of the subdivision graph of a regular graph is obtained.

Theorem 6.1. *Let G be an $r-$ regular (p,q) graph with adjacency spectrum $\{r, \lambda_2, ...\lambda_p\}$. Then Distance seidel spectrum of the complement of the subdivision graph of G is*

$$\begin{pmatrix} \frac{-(p+q-2) \pm \sqrt{(p+q)^2 + 8(pr + 2q + 4r)}}{2} & (1 \pm 2\sqrt{\lambda_i + r}\ ) & 1 \\ 1 & 1 & q-p \end{pmatrix}$$

for $i = 2, 3, ..., p$

Corollary 6.2. *The distance seidel spectrum of the complement of the subdivision graph of a cycle*

$$spec_{D^S}\left(\overline{S\left(C_p\right)}\right) = \begin{pmatrix} 5 & -3-2p & 1 \pm 2\sqrt{\lambda_i + r} \\ 1 & 1 & 1 \end{pmatrix}, j = 1, 2, ..., p-1$$

Corollary 6.3. *For* $p \geq 6$

$$\mathcal{E}_{D^S}\left(\overline{S\left(C_p\right)}\right) = \begin{cases} 4\sqrt{3}\ \cot\dfrac{\pi}{2p} + \dfrac{8}{3}(p+\dfrac{3}{4})\ ; p \equiv 0(mod\ 3) for\ p \leq 9 \\ 8\cos(\dfrac{p-9}{3} \times \dfrac{\pi}{2p}) cosec\dfrac{\pi}{2p} + \dfrac{2}{3}(4p-9); p \equiv 0(mod\ 3)\ for\ p > 9 \\ 8\cos(\dfrac{p-1}{3} \times \dfrac{\pi}{2p}) cosec\dfrac{\pi}{2p} + \dfrac{2}{3}(4p-1); p \equiv 1(mod\ 3) for\ p \leq 7 \\ 8\cos(\dfrac{p-7}{3} \times \dfrac{\pi}{2p}) cosec\dfrac{\pi}{2p} + \dfrac{2}{3}(4p-7); p \equiv 1(mod\ 3)\ for\ p > 7 \\ 8\cos(\dfrac{p-5}{3} \times \dfrac{\pi}{2p}) cosec\dfrac{\pi}{2p} + \dfrac{2}{3}(4p-5); p \equiv 2(mod\ 3) for\ p \leq 14 \\ 8\cos(\dfrac{p-11}{3} \times \dfrac{\pi}{2p}) cosec\dfrac{\pi}{2p} + \dfrac{2}{3}(4p-11); p \equiv 2(mod\ 3) for\ p > 14 \end{cases}$$

7. Conclusion

In conclusion, our exploration into the derivation of the distance Seidel spectrum and energy for a specific class of graphs has paved the way for promising avenues of research. The findings presented herein open the door to a plethora of opportunities for extending these analyses to diverse graph operations and classes, thereby contributing to the broader field of spectral graph theory.

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