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Developing fixed point literature on the Branciari-Bakhtin-metric space

Basel Hardan[a*], Alaa A. Abdullah[b], Kirtiwant P. Ghadle[c], Ahmed A. Hamoud[d] and Homan Emadifar[e]

[a,b]Department of Mathematics, Abyan University, Abyan, Yemen.
[a,b,c]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar, Maharashtra, India
[d]Department of Mathematics, Taiz University, Taiz P.O. Box 6803, Yemen
[e]Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran

E-mail: bassil2003@gmail.com
E-mail: maths.aab@bamu.ac.in
E-mail: ghadle.maths@bamu.ac.in
E-mail: drahmedselwi985@gmail.com
E-mail: homan_emadi@yahoo.com

Abstract. In the generalized Branciari space, this paper develops and reproves some conclusions from the literature.

Keywords: Complete metric spaces, Banach theory, generalized space.

1. Introduction

Banach contraction, widely used in several fields of science and engineering, originated from creating a contractive mapping or a metric space. Reducing or altering the metric conditions establishes the development of type metric space. Go to [1, 15] to learn more. It should be noted that certain topological advantages are lost when certain metric requirements are abused or weakened, which makes the proof of certain fixed-point theorems difficult. Due to these

*Corresponding Author
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challenges, the authors have had to invent new techniques for developing fixed-point theorems to tackle more specialized applications. Branciari [6] established the notion of rectangular metric space and obtained the analog of the Banach contraction principle in this space. A fixed point theory in the Branciari-metric spaces has been the subject of numerous papers, (see [2, 8, 13].

On the other hand, since Bakhtin [3] established a generalization of metric space and demonstrated the symmetric Banach contraction principle, numerous studies of fixed point theorem or the changeful notion for sole-evaluative and multi-evaluative mappings in Bakhtin-metric space have been conducted (see [5, 9, 10].

This study re-shows a contraction type in the Branciari-Bakhtin-metric space, as the parallel of the Banach fixed point theorem literature.

2. Preliminaries

This section presents the foundation for our primary findings.

Definition 2.1. [3] *Suppose $\mathcal{E}$ is a non-empty set and $\lambda > 1$, is a given real number. A function $\sigma : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ is a Bakhtin on $\mathcal{E}$ if for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{E}$, satisfied :*

- **(i):** $\sigma(\mathfrak{u}_1, \mathfrak{u}_2) = 0$ *if and only if* $\mathfrak{u}_1 = \mathfrak{u}_2$;
- **(ii):** $\sigma(\mathfrak{u}_1, \mathfrak{u}_2) = \sigma(\mathfrak{u}_2, \mathfrak{u}_1)$;
- **(iii):** $\sigma(\mathfrak{u}_1, \mathfrak{u}_2) \leq \sigma(\mathfrak{u}_1, \mathfrak{u}_3) + \sigma(\mathfrak{u}_3, \mathfrak{u}_2)$.

The pair $(\mathcal{E}, \sigma)$ is called a Bakhtin metric space.

Definition 2.2. [7] *Suppose $\mathcal{E}$ is a non-empty set and $\lambda > 1$, is a given real number. A function $\sigma : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ is Branciari on $\mathcal{E}$ if for all $\mathfrak{u}_1, \mathfrak{u}_2 \in mathcalE$ and all distinct points $\mathfrak{v}_1, \mathfrak{v}_2 \in mathcalE$ each distinct from $\mathfrak{u}_1, \mathfrak{u}_2$, the following terms are satisfied:*

- **(i):** $\sigma(\mathfrak{u}_1, \mathfrak{u}_2) = 0$ *if and only if* $\mathfrak{u}_1 = \mathfrak{u}_2$;
- **(ii):** $\sigma(\mathfrak{u}_1, \mathfrak{u}_2) = \sigma(\mathfrak{u}_2, \mathfrak{u}_1)$;
- **(iii):** $\sigma(\mathfrak{u}_1, \mathfrak{v}_1) \leq \sigma(\mathfrak{u}_1, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_1)$.

The pair $(\mathcal{E}, \sigma)$ is called a Branciari metric space.

Remark 2.3. [12] *In Definition 2.2 if the third term becomes to*

$$(iii)\sigma(\mathfrak{u}_1, \mathfrak{v}_1) \leq \lambda(\sigma(\mathfrak{u}_1, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_1)).$$

Then $(\mathcal{E}, \sigma)$ is called a Branciari-Bakhtin metric space.

Lemma 2.4. [14] *Suppose, $(\mathcal{E}, \sigma)$ be a Branciari-Bakhtin metric space with $\lambda > 1$ and suppose $\mathfrak{u}_i$ be a Cauchy sequence in $\mathcal{E}$ such that $\mathfrak{u}_i = \mathfrak{u}_j$ when it was $i \neq j$. Then $\mathfrak{u}_i$ be able to converge at most one point.*

3. **Main Results**

The parallel of the Banach [4] contraction principle in the Branciari-Bakhtin metric spaces is the next theorem

Theorem 3.1. *Suppose $(\mathcal{E}, \sigma)$ be a complete Branciari-Bakhtin metric space with $\lambda > 1$ and suppose ζ is a self-mapping on $\mathcal{E}$ for all $\mathfrak{u}_1, \mathfrak{v}_1) + \sigma(\mathfrak{u}_2, \mathfrak{v}_2) \in \mathcal{E}$ satisfies*

$$\sigma(\zeta\mathfrak{u}_1, \zeta\mathfrak{u}_2) \leq \lambda^{(-1)}\Big[\sigma(\mathfrak{u}_1, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_2) + \sigma(\mathfrak{u}_2, \mathfrak{v}_1)\Big]. \tag{3.1}$$

Then ζ has a fixed point.

Proof. Assume that $\mathfrak{u}_0 \in \mathcal{E}$, consider the iteration $\zeta\mathfrak{u}_i = \mathfrak{u}_{i+1}$ for all $i \leq 1$. We will prove that $\{\mathfrak{u}_i\}$ is a Cauchy sequence, such that $\mathfrak{u}_i \neq \mathfrak{u}_{i+1}$. Using (3.1), to get

$$\sigma(\mathfrak{u}_i, \mathfrak{u}_{i+1}) = \sigma(\zeta\mathfrak{u}_{i-1}, \zeta\mathfrak{u}_i) \leq \lambda^{(-1)}\Big[\sigma(\mathfrak{u}_{i-1}, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_i) + \sigma(\mathfrak{u}_i, \mathfrak{v}_1)\Big]$$
$$\sigma_i \leq \lambda^{(-1)}\sigma_{(i-1)}.$$

Using this procedure i times, we get

$$\sigma_i \leq \lambda^{(-1)}\sigma_0 \tag{3.2}$$

Allowing the assumption that $\mathfrak{u}_0$ is not a cyclic point of ζ. Actually, if $\mathfrak{u}_0 = \mathfrak{u}_i$ for all $i \geq 2$, we get

$$\begin{aligned} \sigma(\mathfrak{u}_0, \zeta\mathfrak{u}_0) &= \sigma(\mathfrak{u}_i, \zeta\mathfrak{u}_i) \\ \sigma(\mathfrak{u}_0, \zeta\mathfrak{u}_1) &= \sigma(\mathfrak{u}_i, \zeta\mathfrak{u}_{i+1}) \\ \sigma_0 &= \sigma_i \\ \sigma_0 &= \lambda^{(-1)}\sigma_0. \end{aligned}$$

Thus, $\sigma_0 = 0$, hence $\mathfrak{u}_0$ is a fixed point of ζ. Letting $\mathfrak{u}_i \neq \mathfrak{u}_j$ for all $i \neq j \in \mathcal{N}$ and $(\mathfrak{u}_i, \mathfrak{u}_{i+2}) = \delta$, getting

$$\sigma(\mathfrak{u}_i, \mathfrak{u}_{i+2}) = \sigma(\zeta\mathfrak{u}_{i-1}, \zeta\mathfrak{u}_{i+1}) \leq \lambda^{-1}\Big[\sigma(\mathfrak{u}_{i-1}, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \mathfrak{u}_{i+1}) + \sigma(\mathfrak{u}_{i+1}, \mathfrak{v}_1)\Big]$$
$$\delta_i \leq \lambda^{-1}\delta a_{i-1}$$
$$\delta_i \leq \lambda^{-1}\delta_{i_0}.$$

Thus, $\mathfrak{u}_i$ ia a Cauchy sequence. Since $(\mathcal{E}, \sigma)$ is a complete Branciari-Bakhtin metric space then there exists $\mathfrak{u} \in \mathcal{E}$ satisfies $\mathfrak{u}_i \to \mathfrak{u}$ as $i \to \infty$.

To proving that $\mathfrak{u}$ is a fixed point of ζ, take

$$\sigma(\mathfrak{u}, \zeta\mathfrak{u}) \leq \lambda\Big[\sigma(\mathfrak{u}, \mathfrak{u}_i) + \sigma(\mathfrak{u}_i, \mathfrak{u}_{i+1}) + \sigma(\mathfrak{u}_{i+1}, \zeta\mathfrak{u}_{i+1})\Big].$$

Thus $\mathfrak{u} = \zeta\mathfrak{u}$. Therefore, ζ has a fixed point. □

Let $\zeta : \mathcal{E} \to \mathcal{E}$ such that $F(\zeta) = F(\zeta^i), \forall i \in \mathcal{N}$. Then ζ has ρ property (see [11]) where, $F(\zeta) = Y \in \mathcal{E} : \zeta Y = Y$. So, we get the following result.

Corollary 3.2. *Suppose $(\mathcal{E}, \sigma)$ be a complete Branciari-Bakhtin metric space with $\lambda > 1$ and let $\zeta : \mathcal{E} \to \mathcal{E}$ satisfies for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{v}_1, \mathfrak{v}_2 \in \mathcal{E}$*

$$\sigma(\zeta^i \mathfrak{u}_1, \zeta^i \mathfrak{u}_2) \leq \lambda^{-1}[\sigma(\mathfrak{u}_1, \mathfrak{u}_i) + \sigma(\mathfrak{u}_i, \mathfrak{v}_2) + \sigma(\mathfrak{v}_2, \zeta \mathfrak{u}_2) + \sigma(\mathfrak{u}_2, \zeta \mathfrak{v}_1)]. \tag{3.3}$$

Then $\zeta^i \mathfrak{u} = \mathfrak{u}, \forall I \in \mathcal{N}$, where $\mathfrak{u}$ is a fixed point of ζ.

Example 3.3. *Suppose that $\mathcal{E} = \mathfrak{u}_1 \cup \mathfrak{u}_2$ such that $\mathfrak{u}_1 = \{0.5, 0.3, 0.25, 0.2\}, \mathfrak{u}_2 = [1, 2]$. Consider $\sigma : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ and $\sigma(\chi_1, \chi_2) = 0$, where $\chi_1 = \zeta_2$ and $\sigma(\chi_1, \chi_2) = \sigma(\chi_2, \chi_1)$ where $\chi_1 = \chi_2$ such that $\chi_1, \chi_2 \in \mathcal{E}$, as*

$$\begin{cases} \sigma(0.3, 0.5) = \sigma(0.25, 0.2) = 0.03, \\ \sigma(0.2, 0.5) = \sigma(0.3, 0.25) = 0.02, \\ \sigma(0.25, 0.5) = \sigma(0.25, 0.2) = 0.6, \\ \quad \sigma(\chi_1, \chi_2) = |\chi_1 - \chi_2|^2 \quad else \end{cases}$$

Therefore $(\mathcal{E}, \sigma)$ is a complete Branciari-Bakhtin metric space with $\lambda = 4$. Define $\zeta : \mathcal{E} \to \mathcal{E}$ as

$$\zeta \chi = \begin{cases} 0.25, & \chi \in \mathfrak{u}_1, \\ 0.2, & \chi \in \mathfrak{u}_2 \end{cases}$$

Hence, the Condition of Theorem 3.1 is satisfied and ζ has a unique fixed point of 0.25.

4. **Conclusion**

This paper has generalized an important theorem from the literature and re-proved it in the context of Branciari-Bakhtin metric spaces.

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