


Developing fixed point literature on the Branciari-Bakhtin-metric space

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Abstract. In the generalized Branciari space, this paper develops and re-
proves some conclusions from the literature.

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1. Introduction

Banach contraction, widely used in several fields of science and engineering,
originated from creating a contractive mapping on a metric space. Reducing

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or altering the metric conditions establishes the development of type metric space. Go to [1, 15] to learn more. It should be noted that certain topological advantages are lost when certain metric requirements are abused or weakened, which makes the proof of certain fixed-point theorems difficult. Due to these challenges, the authors have had to invent new techniques for developing fixed-point theorems to tackle more specialized applications. Branciari [6] established the notion of rectangular metric space and obtained the analog of the Banach contraction principle in this space. A fixed point theory in the Branciari-metric spaces has been the subject of numerous papers, (see [2, 8, 13]).

On the other hand, since Bakhtin [3] established a generalization of metric space and demonstrated the symmetric Banach contraction principle, numerous studies of fixed point theorem or the changeful notion for sole-evaluative and multi-evaluative mappings in Bakhtin-metric space have been conducted (see [5, 9, 10]).

This study re-shows a contraction type in the Branciari-Bakhtin-metric space, as the parallel of the Banach fixed point theorem literature.

2. Preliminaries

This section presents the foundation for our primary findings.

Definition 2.1. [3] *Suppose \mathcal{E} is a non-empty set and $\lambda > 1$, is a given real number. A function $\sigma : \mathcal{E} \times \mathcal{E} \rightarrow [0, \infty)$ is a Bakhtin on \mathcal{E} if for all $u_1, u_2, u_3 \in \mathcal{E}$, satisfied :*

- (i): $\sigma(u_1, u_2) = 0$ if and only if $u_1 = u_2$;
- (ii): $\sigma(u_1, u_2) = \sigma(u_2, u_1)$;
- (iii): $\sigma(u_1, u_2) \leq \sigma(u_1, u_3) + \sigma(u_3, u_2)$.

The pair (\mathcal{E}, σ) is called a Bakhtin metric space.

Definition 2.2. [7] *Suppose \mathcal{E} is a non-empty set and $\lambda > 1$, is a given real number. A function $\sigma : \mathcal{E} \times \mathcal{E} \rightarrow [0, \infty)$ is Branciari on \mathcal{E} if for all $u_1, u_2 \in \mathcal{E}$ and all distinct points $v_1, v_2 \in \mathcal{E}$ each distinct from u_1, u_2 , the following terms are satisfied:*

- (i): $\sigma(u_1, u_2) = 0$ if and only if $u_1 = u_2$;
- (ii): $\sigma(u_1, u_2) = \sigma(u_2, u_1)$;
- (iii): $\sigma(u_1, v_1) \leq \sigma(u_1, v_2) + \sigma(v_2, u_2) + \sigma(v_2, u_1)$.

The pair (\mathcal{E}, σ) is called a Branciari metric space.

Remark 2.3. [12] *In Definition 2.2 if the third term becomes to*

$$(iii) \sigma(u_1, v_1) \leq \lambda(\sigma(u_1, v_2) + \sigma(v_2, u_2) + \sigma(v_2, u_1)).$$

Then (\mathcal{E}, σ) is called a Branciari-Bakhtin metric space.

Lemma 2.4. [14] *Suppose, (\mathcal{E}, σ) be a Branciari-Bakhtin metric space with $\lambda > 1$ and suppose u_i be a Cauchy sequence in \mathcal{E} such that $u_i = u_j$ when it was $i \neq j$. Then u_i be able to converge at most one point.*

3. Main Results

The parallel of the Banach [4] contraction principle in the Branciari-Bakhtin metric spaces is the next theorem

Theorem 3.1. *Suppose (\mathcal{E}, σ) be a complete Branciari-Bakhtin metric space with $\lambda > 1$ and suppose ζ is a self-mapping on \mathcal{E} for all $u_1, v_1) + \sigma(u_2, v_2) \in \mathcal{E}$ satisfies*

$$\sigma(\zeta u_1, \zeta u_2) \leq \lambda^{(-1)} \left[\sigma(u_1, v_2) + \sigma(v_2, u_2) + \sigma(u_2, v_1) \right]. \quad (3.1)$$

Then ζ has a fixed point.

Proof. Assume that $u_0 \in \mathcal{E}$, consider the iteration $\zeta u_i = u_{i+1}$ for all $i \leq 1$. We will prove that $\{u_i\}$ is a Cauchy sequence, such that $u_i \neq u_{i+1}$. Using (3.1), to get

$$\begin{aligned} \sigma(u_i, u_{i+1}) &= \sigma(\zeta u_{i-1}, \zeta u_i) \leq \lambda^{(-1)} \left[\sigma(u_{i-1}, v_2) + \sigma(v_2, u_i) + \sigma(u_i, v_1) \right] \\ \sigma_i &\leq \lambda^{(-1)} \sigma_{(i-1)}. \end{aligned}$$

Using this procedure i times, we get

$$\sigma_i \leq \lambda^{(-1)} \sigma_0 \quad (3.2)$$

Allowing the assumption that u_0 is not a cyclic point of ζ . Actually, if $u_0 = u_i$ for all $i \geq 2$, we get

$$\begin{aligned} \sigma(u_0, \zeta u_0) &= \sigma(u_i, \zeta u_i) \\ \sigma(u_0, \zeta u_1) &= \sigma(u_i, \zeta u_{i+1}) \\ \sigma_0 &= \sigma_i \\ \sigma_0 &= \lambda^{(-1)} \sigma_0. \end{aligned}$$

Thus, $\sigma_0 = 0$, hence u_0 is a fixed point of ζ . Letting $u_i \neq u_j$ for all $i \neq j \in \mathcal{N}$ and $(u_i, u_{i+2}) = \delta$, getting

$$\begin{aligned} \sigma(u_i, u_{i+2}) &= \sigma(\zeta u_{i-1}, \zeta u_{i+1}) \leq \lambda^{-1} \left[\sigma(u_{i-1}, v_2) + \sigma(v_2, u_{i+1}) + \sigma(u_{i+1}, v_1) \right] \\ \delta_i &\leq \lambda^{-1} \delta_{i-1} \\ \delta_i &\leq \lambda^{-1} \delta_{i_0}. \end{aligned}$$

Thus, u_i ia a Cauchy sequence. Since (\mathcal{E}, σ) is a complete Branciari-Bakhtin metric space then there exists $u \in \mathcal{E}$ satisfies $u_i \rightarrow u$ as $i \rightarrow \infty$.

To proving that \mathbf{u} is a fixed point of ζ , take

$$\sigma(\mathbf{u}, \zeta\mathbf{u}) \leq \lambda \left[\sigma(\mathbf{u}, \mathbf{u}_i) + \sigma(\mathbf{u}_i, \mathbf{u}_{i+1}) + \sigma(\mathbf{u}_{i+1}, \zeta\mathbf{u}_{i+1}) \right].$$

Thus $\mathbf{u} = \zeta\mathbf{u}$. Therefore, ζ has a fixed point. \square

Let $\zeta : \mathcal{E} \rightarrow \mathcal{E}$ such that $F(\zeta) = F(\zeta^i), \forall i \in \mathcal{N}$. Then ζ has ρ property (see [11]) where, $F(\zeta) = Y \in \mathcal{E} : \zeta Y = Y$. So, we get the following result.

Corollary 3.2. *Suppose (\mathcal{E}, σ) be a complete Branciari-Bakhtin metric space with $\lambda > 1$ and let $\zeta : \mathcal{E} \rightarrow \mathcal{E}$ satisfies for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{E}$*

$$\sigma(\zeta^i \mathbf{u}_1, \zeta^i \mathbf{u}_2) \leq \lambda^{-1} [\sigma(\mathbf{u}_1, \mathbf{u}_i) + \sigma(\mathbf{u}_i, \mathbf{v}_2) + \sigma(\mathbf{v}_2, \zeta \mathbf{u}_2) + \sigma(\mathbf{u}_2, \zeta \mathbf{v}_1)]. \quad (3.3)$$

Then $\zeta^i \mathbf{u} = \mathbf{u}, \forall i \in \mathcal{N}$, where \mathbf{u} is a fixed point of ζ .

Example 3.3. *Suppose that $\mathcal{E} = \mathbf{u}_1 \cup \mathbf{u}_2$ such that $\mathbf{u}_1 = \{0.5, 0.3, 0.25, 0.2\}, \mathbf{u}_2 = [1, 2]$. Consider $\sigma : \mathcal{E} \times \mathcal{E} \rightarrow [0, \infty)$ and $\sigma(\chi_1, \chi_2) = 0$, where $\chi_1 = \zeta_2$ and $\sigma(\chi_1, \chi_2) = \sigma(\chi_2, \chi_1)$ where $\chi_1 = \chi_2$ such that $\chi_1, \chi_2 \in \mathcal{E}$, as*

$$\left\{ \begin{array}{l} \sigma(0.3, 0.5) = \sigma(0.25, 0.2) = 0.03, \\ \sigma(0.2, 0.5) = \sigma(0.3, 0.25) = 0.02, \\ \sigma(0.25, 0.5) = \sigma(0.25, 0.2) = 0.6, \\ \sigma(\chi_1, \chi_2) = |\chi_1 - \chi_2|^2 \quad \text{else} \end{array} \right.$$

Therefore (\mathcal{E}, σ) is a complete Branciari-Bakhtin metric space with $\lambda = 4$. Define $\zeta : \mathcal{E} \rightarrow \mathcal{E}$ as

$$\zeta\chi = \begin{cases} 0.25, & \chi \in \mathbf{u}_1, \\ 0.2, & \chi \in \mathbf{u}_2 \end{cases}$$

Hence, the Condition of Theorem 3.1 is satisfied and ζ has a unique fixed point of 0.25.

4. Conclusion

This paper has generalized an important theorem from the literature and re-proved it in the context of Branciari-Bakhtin metric spaces.

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REFERENCES

1. T. Abeljawad, K. Abodayeh and N. Mlaiki, *On fixed point generalizations to partial b-metric spaces*, Journal of Computational Analysis and Applications, **19**(5)(2015), 883-891.
2. A. Azam, M. Arshad and I. Beg, *Banach contraction principle on cone rectangular metric spaces*, Appl. Anal. Discrete Math., **3**(2)(2009), 236-241.
3. I. Bakhtin, *The contraction mapping principle in quasi metric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., **30**(1989), 26-37.
4. S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fundam. Math., **3**(1922), 133-181.
5. M. Boriceanu, *Strictly fixed point theorems for multivalued operators in b-metric spaces*, Inter. J. Mod. Math., **4**(3)(2009), 285-301.
6. A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57**(2000), 31-37.
7. R. George, S. Radenovic, K. P. Reshma and S. Shukla, *Rectangular b-metric spaces and contraction principle*, J. Nonlinear Sci. Appl., **8**(6)(2015), 1005-1013.
8. R. George, R. Rajagopalan, *Common fixed point results for (η, α) -contractions in rectangular metric spaces*, Bull. Math. Anal. Appl., **5**(1)(2013), 44-52.
9. A. A. Hamoud, J. Patil, B. Hardan, A. Bachhav, H. Emadifar and H. Gunerhan, *Generalization Contractive Mappings on Rectangular b-Metric Space*, Advances in Mathematical Physics, 2022(2022), 10 pages.
10. B. Hardan, J. Patil, A. A. Hamoud, A. Bachhav, H. Emadifar, A. Ghanizadeh, S. A. Edalatpanah and H. Azizi, *On $(\eta, \gamma)_{f,g}$ -contractions in extended b-metric spaces*, Advances in Mathematical Physics, 2022(2022), 8 pages.
11. G. Jeong, B. E. Rhoades, *Maps for which $F(T) = F(T_n)$* , Fixed Point Theory, **6**(2007), 71-105.
12. Z. Kadelburg, S. Radenovic, *Pata-type common fixed point results in b-metric and b-rectangular metric spaces*, J. Nonlinear Sci. Appl., **8**(6)(2015), 944-954.
13. J. Patil, B. Hardan and A. Bachhav, *On generalized metric space type and related fixed point results*, Int. J of Adv. Innovative. Research. **4**(4)(2021), 273-278.
14. JR. Roshan, V. Parvaneh, Z. Kadelburg and N. Hussain, *New fixed point results in b-rectangular metric spaces*, Nonlinear Anal. Model. Control, **21**(5)(2016), 614-634.
15. S. Shukla, *Partial Rectangular Metric Spaces and Fixed Point Theorems*, The Scientific World Journal, 2014(2014), 7 pages.

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