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On projectively flat Finsler space with $n-$power (α,β)- metric

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Abstract. In this paper we have taken the $n-$power (α,β)-metric and obtained the condition for projectively flatness and further find the the some special cases..

Keywords: (α,β)- metric, Projectively flat Finsler space, Randers metric, Kropina metric.

1. **Introduction**

An $n-$ dimensional Finsler space $F^n = (M^n, \mathcal{L})$ is known as a locally Minkowskian space [3] if the manifold M^n is covered by coordinate neighbourhood system (x^i) in each of which the metric $\mathcal{L}$ is the function of y^i only. Further the Finsler space F^n is known as projectively flat if F^n is projective to a locally Minskowski space. Matsumoto [6] introduced a condition for a Finsler space with Randers metric and Kropina metric to be projectively flat. The projective flatness property for the Finsler space with various important (α,β)-metric had been studied by various authors [1], [5], [7],[8], [9], [10], [11],

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[12] [13] and obtained fruitful and beneficial results in the field of Finsler spaces. Initially the concept and importance of (α,β)-metric has been introduced and explained by Matsumoto [6] in detail and the metric $\mathcal{L}=\mathcal{L}(\alpha,\beta)$ is an n- dimensional manifold M^n, which is positively homogeneous function of degree one in α and β, where α is a regular Riemannian metric $\alpha=\sqrt{\alpha_{ij}(x)y^iy^j}$, i.e $det(\alpha_{ij})\neq 0$ and β is $1-$ form, $\beta=b_i(x)y^i$. It is generalization of Randers metric $\mathcal{L}=\alpha+\beta$. We know that there are many types of important (α,β)-metrics namely Kropina metric, Matsumoto metric, generalized Kropina metric, and Z. shen's square metric, infinite series metric and many more metrices [2], [3], [4] [12], [13] , [14] discussed and obtained various fruitful results in field of Finsler geometry. Matsumoto [5] used the following notation, which we have applied in this research and took γ^i_{jk} to repersent the Christoffel symboles in the Riemannian space (M^n,α)-metric

$$\begin{aligned}
&r_{ij}=\frac{1}{2}\Big\{b_{i;j}+b_{j;i}\Big\},\quad r^i_j=a^{ih}r_{hj},\quad r_j=b_ir^i_j,\\
&\mathcal{S}_{ij}=\frac{1}{2}\Big\{b_{i;j}-b_{j;i}\Big\},\quad \mathcal{S}^i_j=a^{ih}\mathcal{S}_{hj},\quad \mathcal{S}_j=b_i\mathcal{S}^i_j,\\
&b^i=a^{ih}b_h,\quad b^2=b^ib_i,
\end{aligned}$$

where $b_{i;j}$ is the covariant derivative of the vector field b_i related to the Riemannian connection γ^i_{jk}, i.e.,

$$b_{i;j}=\frac{\partial b_i}{\partial x^j}-b_k\gamma^i_{jk}.$$

It has been shown by Matsumoto [5] that a Finsler space $F^n=(M^n,\mathcal{L})$ with an (α,β)-metric is projectively flat if and only if for every point of the manifold M^n there is a local co-ordinate neighbourhood that includes the point such that christoffel symbols γ^i_{jk} in the Riemannian space (M^n,α) satisfies:

$$\frac{1}{2}\Big(\gamma^i_{00}-\frac{\gamma_{000}y^i}{\alpha^2}\Big)+\Big(\frac{\alpha\mathcal{L}_\beta}{\mathcal{L}_\alpha}\Big)\mathcal{S}^i_0+\Big(\frac{\mathcal{L}_{\alpha\alpha}}{\mathcal{L}_\alpha}\Big)\Big(C+\frac{\alpha r_{00}}{2\beta}\Big)\Big(\frac{\alpha^2b^i}{\beta}-y^i\Big)=0, \quad (1.1)$$

where $'0'$ stands contraction by y^i and C is given by

$$C+\Big(\frac{\alpha^2\mathcal{L}_\beta}{\beta\mathcal{L}_\alpha}\Big)\mathcal{S}_0+\Big(\frac{\alpha\mathcal{L}_{\alpha\alpha}}{\beta^2\mathcal{L}_\alpha}\Big)\Big(\alpha^2b^2-\beta^2\Big)\Big(C+\frac{\alpha r_{00}}{2\beta}\Big)=0. \quad (1.2)$$

Since $\alpha^2\mathcal{L}_{\alpha\alpha}=\beta^2\mathcal{L}_{\beta\beta}$, due to homogeneity of $\mathcal{L}$ equation (1.2) may be rewritten as

$$\{1+(\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2b^2-\beta^2)\}(C+\frac{\alpha r_{00}}{2\beta})=(\frac{\alpha}{2\beta})\{r_{00}-(\frac{2\alpha\mathcal{L}_\beta}{\mathcal{L}_\beta})\mathcal{S}_0\}. \quad (1.3)$$

The term $(C + \frac{\alpha r_{00}}{2\beta})$ in (1.3) can be eliminated if $\{1 + (\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2 b^2 - \beta^2)\} \neq 0$, it is expreesed as :

$$\Big\{1 + \frac{\mathcal{L}_{\beta\beta}(\alpha^2 b^2 - \beta^2)}{\alpha\mathcal{L}_\alpha}\Big\}\Big\{\frac{1}{2}\Big(\gamma^i_{00} - \frac{\gamma_{000}y^i}{\alpha^2}\Big) + (\frac{\alpha\mathcal{L}_\beta}{\mathcal{L}_\alpha})\mathcal{S}^i_0\Big\} + \Big(\frac{\mathcal{L}_{\alpha\alpha}}{\mathcal{L}_\alpha}\Big)\Big(\frac{\alpha}{2\beta}\Big)\Big\{r_{00} - \Big(\frac{2\alpha\mathcal{L}_\beta}{\mathcal{L}_\alpha}\Big)\mathcal{S}_0\Big\}\Big(\frac{\alpha^2 b^i}{\beta} - y^i\Big) = 0. \tag{1.4}$$

Thus we have [6] :

Theorem 1.1. *Let*

$$\Big\{1 + (\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2 b^2 - \beta^2)\Big\} \neq 0.$$

Then a Finsler space F^n equipped with (α,β)-metric is projectively flat if and only if (1.4) *is satisfied.*

In this research paper, we have considered a generalized form of an (α,β)-metric which is known as $n-$power (α,β)-metric [15] on an $n-$ dimensional manifold M^n, defind as

$$\mathcal{L} = \alpha\Big(1 + \frac{\beta}{\alpha}\Big)^n. \tag{1.5}$$

Further we shall discuss and find out the projectively flatness condition of (1.5) and also try to obtain the special conditions on some particular cases by taking $n = 0, 1, 2, 3$ and 4.

2. Projectively Flat Finsler Space with $n-$ Power (α,β)-Metric

In this section, we have taken $n-$power (α,β)-metric as defined in equation (1.5).

It has been obtained [1] if α^2 contains β as a factor, then the dimension is equal to 2 and $b^2 = 0$.

Here we have assumed that the dimension is more than two, and $b^2 \neq 0$, i.e $\alpha^2 \not\equiv 0(mod\beta)$. Taking the partial derivative of (1.5) with respect to α, β, $\alpha\alpha$ and $\beta\beta$, we have

$$\begin{cases} \mathcal{L}_\alpha = \frac{(\alpha+\beta)^{n-1}(\alpha-(n-1)\beta)}{\alpha^n}, \\ \mathcal{L}_\beta = \frac{n(\alpha+\beta)^{n-1}}{\alpha^{n-1}}, \\ \mathcal{L}_{\alpha\alpha} = \frac{(n^2-n)\beta^2(\alpha+\beta)^{n-2}}{\alpha^{n+1}}, \\ \mathcal{L}_{\beta\beta} = \frac{n(n-1)(\alpha+\beta)^{n-2}}{\alpha^{n-1}}. \end{cases} \tag{2.1}$$

By virtue of theorem (1.1), $\{1 + (\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2 b^2 - \beta^2)\} = 0$ then we have $\{\alpha^2(1 + (n^2 - n)b^2) + (2 - n)\alpha\beta + (1 - n^2)\beta^2\} = 0$, which is contradiction. Hence theorem (1.1) can be applied.

Putting the values of $\mathcal{L}_\alpha$, $\mathcal{L}_\beta$, $\mathcal{L}_{\alpha\alpha}$ and $\mathcal{L}_{\beta\beta}$, in equation (1.4), we obtain

$$(\alpha^2(1+(n^2-n)b^2)+(2-n)\alpha\beta+(1-n^2)\beta^2)\{(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)(\alpha-(n-1)\beta)$$
$$+2n\alpha^4\mathcal{S}^i_0\}+(n^2-n)\alpha^2\{(\alpha-(n-1)\beta)r_{00}-2n\alpha^2\mathcal{S}_0\}(\alpha^2b^i-\beta y^i)=0. \tag{2.2}$$

The above equation can be rewritten as a polynomial of degree 6 in 'α', which is given as

$$A_6\alpha^6+A_4\alpha^4+A_2\alpha^2+A_0+\alpha(A_5\alpha^4+A_3\alpha^2+A_1)=0, \tag{2.3}$$

where

$A_0=-(n-1)(n^2-1)\beta^3y^i\gamma_{000}$,

$A_1=(3n-3)\beta^2\gamma_{000}y^i$,

$A_2=(2n-3)\beta y^i\gamma_{000}+n(n-1)^2b^2\beta y^i\gamma_{000}+(n-1)^2(n+1)\beta^3\gamma^i_{00}+(n^2-n)(n-1)\beta^2y^ir_{00}$,

$A_3=-y^i\gamma_{000}-2nb^2y^i\gamma_{000}+(3-3n)\beta^2\gamma^i_{00}-(n^2-1)\beta y^ir_{00}$,

$A_4=n(n^2-1)b^2\beta\gamma^i_{00}+(3-2n)\beta\gamma^i_{00}-2n(n^2-1)\beta^2\mathcal{S}^i_0-(n^2-n)(n-1)b^i\beta r_{00}+2n(n^2-n)\beta y^i$,

$A_5=\gamma^i_{00}+(n^2-n)\gamma^i_{00}b^2+(n^2-n)b^ir_{00}+2n(2-n)\beta\mathcal{S}^i_0$,

$A_6=2n\{(1+(n^2-n)b^2)\mathcal{S}^i_0-(n^2-n)b^i\mathcal{S}_0\}$.

Since $A_6\alpha^6+A_4\alpha^4+A_2\alpha^2+A_0$ and $A_5\alpha^4+A_3\alpha^2+A_1$ are rational and α is irrational in y^i, therefore we have

$$A_6\alpha^6+A_4\alpha^4+A_2\alpha^2+A_0=0. \tag{2.4}$$

$$A_5\alpha^4+A_3\alpha^2+A_1=0. \tag{2.5}$$

Since the term which does not contains β is $A_6\alpha^6$, therefore there exists a homogeneous polynomial V_6 of degree 6 in y^i, such that

$2n\{(1+(n^2-n)b^2)\mathcal{S}^i_0-(n^2-n)b^i\mathcal{S}_0\}\alpha^6=\beta V_6$.

Since $\alpha^2\not\equiv 0(mod\beta)$, then we must have $u^i=u^i(x)$ satisfying

$$2n\{(1+(n^2-n)b^2)\mathcal{S}^i_0-(n^2-n)b^i\mathcal{S}_0\}=u^i\beta. \tag{2.6}$$

Contracting the above equation by b_i, we have
$2n\{(1+(n^2-n)b^2)\mathcal{S}_0-(n^2-n)b^i\mathcal{S}_0\}=u^i\beta b_i$, i.e.

$$2n\mathcal{S}_0=u^i\beta b_i. \tag{2.7}$$

Again contracting this by b_j, we have $2n\mathcal{S}_j=u^ib_ib_j$, further contracting this equation by b^j, we obtain
$u^ib_ib^2=0$, i.e $u^ib_i=0$.
Putting this value in equation (2.7), we obtain
$\mathcal{S}_0=0$.
Therefore from (2.6), we get

$$2n(1+(n^2-n)b^2)\mathcal{S}_{ij}=u^ib_j, \tag{2.8}$$

which implies $u_ib_j+u_jb_i=0$.
Contracting this equation by b^j, we have $u_ib^2=0$ by virtue of $b^iu_j=0$. Therefore we get, $u_i=0$. Hence from (2.8), we have $\mathcal{S}_{ij}=0$.
Conversely, from (2.5) we have $1-$form $v_0=v_i(x)y^i$, such that

$$\gamma_{000}=v_0\alpha^2. \tag{2.9}$$

Putting $\mathcal{S}_0=0$, $\mathcal{S}^i_0$ and $\gamma_{000}=v_0\alpha^2$ into (2.2), we have

$$\{\alpha^2(1+(n^2-n)b^2)-(2-n)\alpha\beta-(n^2-1)\beta^2\}(\gamma^i_{00}-v_0y^i)+(n^2-n)r_{00}(\alpha^2b^i-\beta y^i)=0. \tag{2.10}$$

Since $(\alpha-(n-1)\beta)\neq 0$, the equation (2.10) may be expressed as follows

$$P\alpha+Q=0,$$

where

$$P=(2-n)\beta(\gamma^i_{00}-v_0y^i),$$
$$Q=\{\alpha^2(1+(n^2-n)b^2)-(n^2-1)\beta^2\}(\gamma^i_{00}-v_0y^i)+(n^2-n)r_{00}(\alpha^2b^i-\beta y^i).$$

Since P and Q are rational and α is irrational in y^i we have $P=0$ and $Q=0$.
Initially, $P=0$ implies that

$$\gamma^i_{00}-v_0y^i=0. \tag{2.11}$$

i.e.

$$2\gamma^i_{jk}=v_j\delta^i_k+v_k\delta^i_j, \tag{2.12}$$

which implies that the associated Riemannian space (M^n,α) is projectively flat.
Next, from $Q=0$ and from $\gamma^i_{00}-v_0y^i=0$, we have

$$(n^2-n)r_{00}(\alpha^2b^i-\beta y^i)=0. \tag{2.13}$$

Contracting the equation (2.13) by b_i, we have $(n^2-n)r_{00}(\alpha^2b^2-\beta^2)=0$, from which we obtain $r_{00}=0$ i.e. $r_{ij}=0$.

From $\mathcal{S}_{ij}=0$ and $r_{ij}=0$, we have $b_{i;j}=0$.
On the other hand if $b_{i;j}=0$, then

$$2r_{ij}=b_{j;i}, \tag{2.14}$$

$$2\mathcal{S}_{ij}=-b_{j;i}. \tag{2.15}$$

By adding (2.14) and (2.15), we have $2r_{ij}+2\mathcal{S}_{ij}=0$ i.e. $2\mathcal{S}_{ij}=0$ and $2r_{ij}=0$, then we have $r_{00}=\mathcal{S}^i_0=\mathcal{S}_0$. So (2.2) is a result of (2.11). Hence we have:

Theorem 2.1. *A Finsler space F^n equipped with $n-$power $(\alpha,\beta)-$metric and the associated Riemannian space (M^n,α) is projectively flat if and only if the covariant derivative of b_i with respect to $'j'$ is zero.*

Some special cases:

Case(a): Put $n=0$ in equation (1.5), we have

$$\mathcal{L}=\alpha \tag{2.16}$$

Differentiating equation (2.16) partially with respect to α, β, $\alpha\alpha$ and $\beta\beta$, we have

$$\begin{cases}\mathcal{L}_\alpha=1,\\ \mathcal{L}_\beta=0,\\ \mathcal{L}_{\alpha\alpha}=0,\\ \mathcal{L}_{\beta\beta}=0.\end{cases} \tag{2.17}$$

Since $1+(\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2b^2-\beta^2)\neq 0$, then putting the these values of $\mathcal{L}_\alpha$, $\mathcal{L}_\beta$, $\mathcal{L}_{\alpha\alpha}$ and $\mathcal{L}_{\beta\beta}$ in the equation (1.4) we obtain
$(1+0)\{\frac{(\gamma^i_{00}-\frac{\gamma_{000}y^i}{\alpha^2}}{2}\}=0.$

This implies that

$$\alpha^2\gamma^i_{00}=\gamma_{000}y^i. \tag{2.18}$$

Hence:

Theorem 2.2. *If we take $n=0$, then the $n-$power $(\alpha,\beta)-$metric is neither projectively flat nor the associated Riemannian space (M^n,α).*

Case(b): Put $n=1$ in equation (1.5), we obtain

$$\mathcal{L}=\alpha+\beta. \tag{2.19}$$

If we put $n=1$ in equation (1.5), then equation (2.19) is known as a Randers change of $(\alpha,\beta)-$metric. It has been studied by Matsumoto [5].
Case(c): Put $n=2$ in equation (1.5), we obtain

$$\mathcal{L}=\frac{(\alpha+\beta)^2}{\alpha}. \tag{2.20}$$

Differentiating equation (2.20) partially with respect to α, β, $\alpha\alpha$ and $\beta\beta$, we have

$$\begin{cases} \mathcal{L}_\alpha = \frac{(\alpha^2-\beta^2)}{\alpha^2}, \\ \mathcal{L}_\beta = \frac{2\beta^2}{\alpha}, \\ \mathcal{L}_{\alpha\alpha} = \frac{2(\alpha+\beta)}{\alpha^3}, \\ \mathcal{L}_{\beta\beta} = \frac{2}{\alpha}, \end{cases} \tag{2.21}$$

Since $1 + (\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha})(\alpha^2 b^2 - \beta^2) \neq 0$, then putting the these values of $\mathcal{L}_\alpha$, $\mathcal{L}_\beta$, $\mathcal{L}_{\alpha\alpha}$ and $\mathcal{L}_{\beta\beta}$ in the equation (1.4), we obtain

$$(\alpha^2(1+2b^2) - 3\beta^2)\{(\alpha^2\gamma^i_{00} - \gamma_{000}y^i)(\alpha-\beta) + 4\alpha^4\mathcal{S}^i_0\} + 2\alpha^2(\alpha^2 b^i - \beta y^i)\{(\alpha-\beta)r_{00} - 4\alpha^2\mathcal{S}_0\} = 0 \tag{2.22}$$

The above equation can be rewritten as a polynomial of degree 6 in 'α', which is given as

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 + \alpha(A_5\alpha^4 + A_3\alpha^2 + A_1) = 0, \tag{2.23}$$

where

$$A_0 = -3\beta^3 y^i\gamma_{000},$$

$$A_1 = 3\beta^2\gamma_{000}y^i,$$

$$A_2 = \beta y^i\gamma_{000} + 2b^2\beta y^i\gamma_{000} + 3\beta^3\gamma^i_{00} + 2\beta^2 y^i r_{00},$$

$$A_3 = -y^i\gamma^i_{00} - 2b^2 y^i\gamma_{000} - 3\beta^2\gamma^i_{00} - 2\beta y^i r_{00},$$

$$A_4 = -2b^2\beta\gamma^i_{00} - \beta\gamma^i_{00} - 12\beta^2\mathcal{S}^i_0 - 2b^i\beta r_{00} + 8\beta y^i\mathcal{S}_0,$$

$$A_5 = \gamma^i_{00} + 2\gamma^i_{00}b^2\beta + 2b^i r_{00},$$

$$A_6 = 4\{(1+2b^2)\mathcal{S}^i_0 - 2b^i\mathcal{S}_0\}.$$

Since $A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0$ and $A_5\alpha^4 + A_3\alpha^2 + A_1$ are rational and α is irrational in y^i, therefore we have

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0, \tag{2.24}$$

$$A_5\alpha^4 + A_3\alpha^2 + A_1 = 0. \tag{2.25}$$

Since the term which does not contains β is $A_6\alpha^6$, therefore there exists a homogeneous polynomial V_6 of degree 6 in y^i, such that

$4\{(1+2b^2)\mathcal{S}^i_0 - 2b^i\mathcal{S}_0\}\alpha^6 = \beta V_6$.
Since $\alpha^2 \not\equiv 0(mod\beta)$, then we must have $u^i = u^i(x)$ satisfying

$$4\{(1+2b^2)\mathcal{S}^i_0 - 2b^i\mathcal{S}_0\} = u^i\beta. \tag{2.26}$$

Contracting the above equation by b_i, we have
$4\mathcal{S}_0 = u^i\beta b_i$,
i.e.

$$4\mathcal{S}_j = u^i b_i b_j. \tag{2.27}$$

Further contracting this equation by b^j, we obtain $u^i b_i b^2 = 0$ i.e. $u^i b_i = 0$.

Putting these value in equation (2.27), we obtain

$\mathcal{S}_0 = 0$.

Therefore from (2.26), we get

$$4(1+2b^2)\mathcal{S}_{ij} = u_i b_j, \tag{2.28}$$

which implies $u_i b_j + u_j b_i = 0$. Contracting this equation by b^j, we have $u_i b^2 = 0$ by virtue of $b^i u_j = 0$.
Therefore we get $u_i = 0$, hence from (2.28), we have $\mathcal{S}_{ij} = 0$.
Conversely, from (2.25), we have $1-$ form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{2.29}$$

Putting $\mathcal{S}_0 = 0$, $\mathcal{S}^i_0$ and $\gamma_{000} = v_0\alpha^2$ into (2.22), we have

$$\{\alpha^2(1+2b^2) - 3\beta^2\}(\gamma^i_{00} - v_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i) = 0. \tag{2.30}$$

Since $(\alpha-\beta) \neq 0$, the equation (2.30) may be expressed as:

$P\alpha + Q = 0$,

where
$P = 0$,
$Q = \{\alpha^2(1+2b^2) - 3\beta^2\}(\gamma^i_{00} - v_0 y^i) + 2r_{00}(\alpha^2 b^i - \beta y^i)\}$.

Here P and Q are rational and α is irrational in y^i, we have $P = Q = 0$.
Since rational part of this equation has already vanished so it is not showing that the associated Riemannian space $(M^n,\alpha)$is projectively flat and $b_{i;j} \neq 0$.
Hence, we have

Theorem 2.3. *A Finsler space F^n equipped with a square $(\alpha,\beta)-$metric is neither the associated Riemannian space (M^n,α) nor projectively flat.*

Case(d): Put $n = 3$ in equation (1.5), we obtain

$$\mathcal{L} = \frac{(\alpha+\beta)^3}{\alpha^2}. \tag{2.31}$$

If we put $n = 3$ in equation (1.5), then equation (2.31) is known as cubic $(\alpha,\beta)-$metric. It has been studied by Brijesh Tripathi, Sadika Khan and V. K. Chaubey [14].

Case(e): If we put $n = 4$ in equation (1.5), we obtain

$$\mathcal{L} = \frac{(\alpha+\beta)^4}{\alpha^3}. \tag{2.32}$$

Taking the partial derivative of (2.32) with respect to α, β, $\alpha\alpha$ and $\beta\beta$, we have

$$\begin{cases} \mathcal{L}_\alpha = \frac{(\alpha+\beta)^3(\alpha-3\beta)}{\alpha^4}, \\ \mathcal{L}_\beta = \frac{4(\alpha+\beta)^3}{\alpha^3}, \\ \mathcal{L}_{\alpha\alpha} = \frac{12\beta^2(\alpha+\beta)^3}{\alpha^5}, \\ \mathcal{L}_{\beta\beta} = \frac{12(\alpha+\beta)^2}{\alpha^3}. \end{cases} \tag{2.33}$$

If $\{1+\frac{\mathcal{L}_{\beta\beta}}{\alpha\mathcal{L}_\alpha}(\alpha^2b^2-\beta^2)\} \neq 0$, then we have $\{\alpha^2(1+12b^2)-2\alpha\beta-15\beta^2\} \neq 0$. Putting the values of $\mathcal{L}_\alpha$, $\mathcal{L}_\beta$, $\mathcal{L}_{\alpha\alpha}$ and $\mathcal{L}_{\beta\beta}$ in equation (1.4), we obtain

$$(\alpha^2(1+12b^2)-2\alpha\beta-15\beta^2)\{(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)(\alpha-3\beta)+8\alpha^4\mathcal{S}^i_0)+12\alpha^2(\alpha^2b^i - \beta y^i)\}\{(\alpha-3\beta)r_{00}-8\alpha^2\mathcal{S}_0\}=0. \tag{2.34}$$

The above equation can be rewritten as a polynomial of degree 6 in 'α', which is given as

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 + \alpha(A_5\alpha^4 + A_3\alpha^2 + A_1) = 0, \tag{2.35}$$

where

$$A_0 = -45\beta^3y^i\gamma_{000},$$

$$A_1 = 9\beta^2\gamma_{000}y^i,$$

$$A_2 = 5\beta y^i\gamma_{000} + 36b^2\beta y^i\gamma_{000} + 45\beta^3\gamma^i_{00} + 36\beta^2y^ir_{00},$$

$$A_3 = -y^i\gamma_{000} - 8b^2y^i\gamma_{000} - 9\beta^2\gamma^i_{00} - 15\beta y^ir_{00},$$

$$A_4 = 60b^2\beta\gamma^i_{00} + 5\beta\gamma^i_{00} - 120\beta^2\mathcal{S}^i_0 - 36b^i\beta r_{00} + 96\beta y^i,$$

$$A_5 = \gamma^i_{00} + 12\gamma^i_{00}b^2 + 12b^ir_{00} - 16\beta\mathcal{S}^i_0,$$

$$A_6 = 8\{(1+12b^2)\mathcal{S}^i_0 - 12b^i\mathcal{S}_0\}.$$

Since $A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0$ and $A_5\alpha^4 + A_3\alpha^2 + A_1$ are rational and α is irrational in y^i, therefore we have

$$A_6\alpha^6 + A_4\alpha^4 + A_2\alpha^2 + A_0 = 0. \tag{2.36}$$

$$A_5\alpha^4 + A_3\alpha^2 + A_1 = 0. \tag{2.37}$$

Since the term which does not contains β is $A_6\alpha^6$, therefore there exists a homogeneous polynomial V_6 of degree 6 in y^i, such that

$$8\{(1+12b^2)\mathcal{S}^i_0 - 12b^i\mathcal{S}_0\}\alpha^6 = \beta V_6\ .$$

Since $\alpha^2 \not\equiv 0 (mod \beta)$, then we must have $u^i = u^i(x)$ satisfying

$$8\{(1+12b^2)\mathcal{S}^i_0 - 12b^i\mathcal{S}_0\} = u^i\beta. \tag{2.38}$$

Contracting the above equation by b_i, we have

$$8\mathcal{S}_0 = u^i\beta b_i. \tag{2.39}$$

Again contracting this by b_j, we have

$$8\mathcal{S}_j = u^i b_i b_j.$$

Further contracting this equation by b^j, we obtain
$u^i b_i b^2 = 0$, i.e $u^i b_i = 0$.
Putting this value in equation (2.39), we obtain
$\mathcal{S}_0 = 0$.
Therefore from (2.38), we get

$$8(1+12b^2)\mathcal{S}_{ij} = u_i b_j, \tag{2.40}$$

which implies $u_i b_j + u_j b_i = 0$. Contracting this equation by b^j, we have $u_i b^2 = 0$ by virtue of $b^i u_j = 0$.
Therefore we get $u_i = 0$, hence from (2.40), we have
$\mathcal{S}_{ij} = 0$.
Conversely, from (2.37), we have $1-$ form $v_0 = v_i(x)y^i$, such that

$$\gamma_{000} = v_0\alpha^2. \tag{2.41}$$

Putting $\mathcal{S}_0 = 0$, $\mathcal{S}^i_0$ and $\gamma_{000} = v_0\alpha^2$ into (2.34), we have

$$\{\alpha^2(1+12b^2) - 2\alpha\beta - 15\beta^2\}(\gamma^i_{00} - v_0 y^i) + 12r_{00}(\alpha^2 b^i - \beta y^i) = 0. \tag{2.42}$$

Since $(\alpha - 3\beta) \neq 0$, the equation (2.42) may be expressed as:

$$P\alpha + Q = 0,$$

where

$P = -2\beta(\gamma^i_{00} - v_0 y^i)$,

$Q = \{\alpha^2(1 + 12b^2) - 15\beta^2\}(\gamma^i_{00} - v_0 y^i) + 12r_{00}(\alpha^2 b^i - \beta y^i)$.

Since P and Q are rational and α is irrational in y^i, we have $P = Q = 0$. Initially, $P = 0$ implies that

$$\gamma^i_{00} - v_0 y^i = 0, \tag{2.43}$$

that is

$$2\gamma^i_{jk} = v_j \delta^i_k + v_k \delta^i_j, \tag{2.44}$$

which implies that the associated Riemannian space (M^n, α) is projectively flat.

Next , from $Q = 0$ and from $\gamma^i_{00} - v_0 y^i = 0$, we have

$$12r_{00}(\alpha^2 b^i - \beta y^i) = 0. \tag{2.45}$$

Contracting the equation (2.45) by b_i, we have $12r_{00}(\alpha^2 b^2 - \beta^2) = 0$, we obtain $r_{00} = 0$, i.e. $r_{ij} = 0$.
From $\mathcal{S}_{ij} = 0$ and $r_{ij} = 0$, we have $b_{i;j} = 0$.
On the other hand, if $b_{i;j} = 0$, then we have $r_{00} = \mathcal{S}^i_0 = \mathcal{S}_0$. So (2.34) is a result of (2.43). Thus we have:

Theorem 2.4. *A Finsler space F^n equipped with quartic $(\alpha, \beta)-$metric and the associated Riemannian space (M^n, α) is projectively flat if and only if the covariant derivative of b_i with respect to $'j'$ is zero.*

Conclusion

A Finsler space F^n equipped with n-power $(\alpha, \beta)-$metric and associated Riemannian space (M^n, α) is projectively flat if and only if covariant derivative of b_i with respect to $'j'$ is zero. If we take $n = 0, 2$, the condition of projectively flatness is not satisfied but for $n = 1, 3, 4$ the condition of projectively flatness are satisfied.

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