Journal of Finsler Geometry and its Applications
Vol. 5, No. 1 (2024), pp 97-114.
https://doi.org/10.22098/jfga.2024.14548.1115

Characteristics of T–conformal mappings

Mehran Aminian[a*] and Mehran Namjoo[b]

[a,b]Department of Mathematics, Vali-e-Asr University of Rafsanjan,
Rafsanjan, Iran

E-mail: mehran.aminian@vru.ac.ir
E-mail: namjoo@vru.ac.ir

Abstract. In this paper, we introduce the notion of T-conformal transformations and T-conformal maps between Riemannian manifolds. Here, T stands for a smooth (1,1)-tensor field defined on the domain of these maps. We start by defining what it means for a map to be T-conformal and also dwell on some basic properties of such type maps. We next specialize our discussion to the situation when the map T satisfies the condition $\nabla T = 0$. Accordingly, we prove Liouville's theorem for T-conformal maps between space forms $R^n(c)$ as an application under the condition $\nabla T = 0$. The proof relies upon properties of T-conformal maps proved earlier. Broadly, the paper seeks to provide a general understanding of conformal mappings in the presence of a tensor field T and show how classical results such as Liouville's theorem apply.

Keywords: Conformal map, isometry, $(1, 1)$–tensor field.

1. Introduction

Conformal mappings have a significant position in the world of mathematics, physics as well as engineering. They are important in addressing complex problems that are characterized with geometries that pose challenges. In effect, such mappings are effectively put in use by many scientific disciplines since due to expression through functions of a complex variable, they are often encountered in natural phenomena. This is related to wide applicability supported by large studies and references [5, 10, 18, 21, 20, 14, 7, 15].

---

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 58E20, 53C43

Essentially, one can think of a function that is a conformal map as being a function across $\mathbb{C}$, the set of complex numbers, that respects angles between curves. This inherent characteristic makes conformal maps useful in various applications where the preservation of angles is important [11, 2, 4, 6, 16]. The property of length preservation is one of the most central properties in mathematics describing conformal maps as those angle-preserving transformations. However, it should be noted that the same maps do not preserve lengths.

In order to formalize more rigorously what conformal mappings are, let U, V be open subsets of $\mathbb{R}^n$. A function $f : U \to V$ is said to be conformal at a point $u_0 \in U$ if it does not change the angles between the directed curves going through u_0 and does not reverse their orientation.

The idea of conformality goes further, intrinsically, to the maps between Riemannian manifolds. In particular, for Riemannian geometry two metrics g and h on a smooth manifold M, will be conformally equivalent if they relate by means $g = uh$ for a positive function u on M, hence the named conformal factor [17].

Conformal map is a smooth map that under the context of Riemannian manifolds called conformal with respect to the metric if its pulled back metric should be conformally equivalent with the original one [3, 19]. An example of stereographic projection of a sphere to a plane demonstrating these concepts find play in the real world, augmented by a point at infinity, would be a classic example [12].

One of Liouville's theorems reveals one of the key differences between two dimensions and higher dimensions. Actually, in dimensions three and above, an arbitrary conformal map between open subsets of Euclidean spaces factorizes into the composition of three transformations: a homothety, an isometry, and a special conformal transformation [13, 8, 9].

Building on classical conformal maps, we would like to introduce the concept of T-conformal maps as a result of research [1]. Concretely, define T-conformality with respect to vector spaces endowed with scalar product where T is linear transformation. This innovative approach allows the generation of T-conformal mappings, thus enabling the study of intrinsic properties.

We will focus on a few aspects of the properties exuded by T-conformal transformations and particularly how they differ from the well-known conformal maps. This study includes the systematic investigation of its mathematical properties in depth as well as recognising more precise applications. For the reason that exploring these properties offers an enhancement understanding of T-conformality and its importance in a higher science discipline.

To make the idea even more concrete, we also extend our notion of T-conformality to maps between Riemannian manifolds. If fact, it turns out that this extension is in just the right progression that one should follow after the initial realization of the ideas applied initially on vector spaces. This is mainly due to the fact that it makes it possible to apply the T-conformal map in an even richer and more geometrical context.

A major contribution of our work is the derivation of a system of Partial Differential Equations (PDEs) satisfied by T-conformal maps on space forms $R^n(c)$, with some additional properties of maps like $\nabla T = 0$. This PDE is developed on the premise of foundational principles "Liouville's theorem for conformal maps" wherein a very common approach followed nowadays. By understanding and applying T-conformal maps, these PDEs are established as a crucial step toward their wider practical use.

While note that through our work we extend as well as remain motivated by existing literature [3, 19], the results that we put forth with respect to T-conformality would be new and original. The novelty as well as the innovative manner in which we approach the matter of conformal mapping has main implications in extension of existing theoretical work.

As we proceed further in the discussion of the concept, we are optimistic about digging up more and more results and developments in our application of the study of T-conformal maps.

The field of conformal mappings, which has applications in various scientific fields, has now become richer and satisfied with the notion of T-conformal maps. The possibilities for future researches and applications from this new concept not only broaden the scope of conformal mapping. The deep research of properties of T-conformal maps will show enormous prospects for numerous applications, which will generate new findings and thereby to innovations far not only in respect of mathematics.

2. Fundamentals of T–conformality

In this section, we focus on a critical concept named as T-conformality, which is relevant to understanding linear transformations that operate on vector spaces with an inner product. We start off by explaining W^T, a symbol that stands for a subspace perpendicular to another subspace W, as influenced by a transformation labeled T. There's an important finding we discuss, 2.2. This finding connects the dimensions of W and W^T, proves that $W^\perp$ equals the result of applying T to W^T, and also shows that if T is symmetric, then $(W^T)^T = W$. From here, we go on to establish a theorem that says if the outcome of applying T to W stays within W, then the space V splits as a direct sum of W and W^T.

Next up, we investigate how a linear transformation $T : V \to V$ on a space V, interacts with functions that map V to a different space $\overline{V}$. At the heart of our exploration are the notions of being T-conformal and becoming a T-linear isometry. We detail what it means for a linear map, say $A : V \to \overline{V}$, to align perfectly with T, either by being T-conformal or a T-linear isometry. What this basically boils down to is how well A keeps the inner product relationships the same, in sync with T. Several key points about these special types of transformations, A, come out in our analysis, particularly when it comes to whether T is symmetric or invertible. For instance, if A preserves the conformity with T, then it turns out that T has to be isomorphism. As well, we lay out under what conditions the one might find a T-linear isometry linking two spaces equipped with an inner product, V and $\overline{V}$. By formally connecting T and A in this manner, give us a deeper understanding of interplays between linear mappings on a vector space.

Definition 2.1. *Let g be a scalar product on a vector space V, $T : V \to V$ be a linear transformation, and W be a subspace of V. We put*

$$W^T = \{v \in V : g(Tv, w) = 0 \ \forall w \in W\}.$$

As we see, W^T is a subspace of V. Note that the nondegeneracy of g implies that $V^T = \ker T$.

Lemma 2.2. *If W be a subspace of a scalar product space (V, g), $\dim V = n$, $T : V \to V$ be a linear isomorphism, then*

(1) *$dim(W) + dim(W^T) = n$,*
(2) *$W^\perp = TW^T$,*
(3) *if T is symmetric relative to g, then $(W^T)^T = W$.*

Proof. (i) Let $e_1, ..., e_n$ be a basis for V, which for a k , $e_1, ..., e_k$ is a basis for W. Now $v \in W^T$ if and only if $g(Tv, e_i) = 0$ for $1 \le i \le k$, which in coordinate terms is $\sum_{A,B=1}^{n} g_{iB} T_{BA} v_A = 0$. This is k linear equations in n unknowns, and since the rows of the coefficient matrix are linearly independent, so the matrix has rank k. Hence by linear algebra the space of solutions has dimension $n-k$.
(ii) Let $v \in W^\perp$, then for some x, $Tx = v$ and $g(Tx, w) = 0$ for every $w \in W$. So $x \in W^T$ and $v = Tx \in TW^T$. Therefore $W^\perp \subset TW^T$. Also obviously $TW^T \subset W^\perp$. Hence $W^\perp = TW^T$.
(iii) Let $v \in W$ and $w \in W^T$, then $g(Tv, w) = g(v, Tw) = 0$. Therefore $W \subset (W^T)^T$ and by (i) these two subspaces have the same dimension, hence they are equal. □

Theorem 2.3. *Let subspace W of a scalar product space V be nondegenerate, $T : V \to V$ be a linear isomorphism and $TW \subset W$ then V is the direct sum of W and W^T and $TW = W$.*

Proof. By a standard vector space identity $\dim(W + W^T) + \dim(W \cap W^T) = \dim W + \dim W^T$. According to 2.2, the right-hand side is $n = \dim V$. Hence $W + W^T = V$ if and only if $W \cap W^T = 0$. Thus either of these two conditions is equivalent to $V = W \oplus W^T$. If $v \in W \cap W^T$, then for every $w \in W$, $g(Tv, w) = 0$. Hence by nondegeneracy of W, $Tv = 0$ and so $v = 0$. Now since

T is isomorphism, the two subspaces W and TW have the same dimension, hence they are equal. $\square$

Definition 2.4. *Let V and $\overline{V}$ have scalar products g and h respectively, and $T : V \to V$ be a linear transformation. We call a linear transformation $A : V \to \overline{V}$, a T-conformal if there is a number $\Lambda \neq 0$ such that*

$$h(ATv, Aw) = \Lambda\ g(v, w) \ \text{ for all } v, w \in V. \tag{2.1}$$

Remark 2.5. *Equation (2.1) is equivalent to $A^*AT = \Lambda I$ and so $\Lambda = \frac{1}{n}\sum_{i=1}^{n} h(ATe_i, Ae_i) = \frac{1}{n}\mathrm{tr}_g(A^*AT)$ where A^* is the adjoint operator and $\{e_i\}_{i=1}^{n}$ is the orthonormal basis for V.*

Lemma 2.6. *Let V and $\overline{V}$ have scalar products g and h respectively, and $T : V \to V$ be a linear transformation. If $A : V \to \overline{V}$, is a T-conformal transformation, then T is symmetric relative to g, T is isomorphism and A is one-to-one, and so*

$$h(Av, Aw) = \Lambda\ g(T^{-1}v, w) \ \text{ for all } v, w \in V, \tag{2.2}$$

and when h is an inner product, then ΛT is positive definite relative to g.

Proof. T is symmetric relative to g, because

$$g(w, Tv) = \frac{1}{\Lambda}h(ATw, ATv) = \frac{1}{\Lambda}h(ATv, ATw) = g(v, Tw).$$

By non-degeneracy of g, T and A are one-to-one and so T is an isomorphism. Also when h is an inner product, if $v \neq 0$, then

$$g(v, \Lambda Tv) = h(ATv, ATv) > 0.$$

Therefore ΛT is positive definite relative to g. $\square$

Remark 2.7. *Let $A : V \to V$ be a linear isomorphism and consider an orthonormal basis for positive definite scalar product space V. In this basis by equation (2.2), we have A is T-conformal if and only if $A^tA = \Lambda T^{-1}$.*

Example 2.8. *Let ΛT and so ΛT^{-1} be a positive definite symmetric real matrix and $\sqrt{\Lambda T^{-1}}$ be its unique positive definite symmetric real square root, and T and A be $n \times n$ real matrices. Then $A^tA = \Lambda T^{-1}$ if and only if $A = B\sqrt{\Lambda T^{-1}}$ for an unique orthogonal matrix $B \in O(n)$. In special case, consider*

$$\Lambda T = \Lambda \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

be a positive definite real matrix, then algebra shows that $A^tA = \Lambda T^{-1}$ if and only if

$$A = \begin{pmatrix} \sqrt{\alpha}\cos\theta & \sqrt{\beta}\cos\phi \\ \sqrt{\alpha}\sin\theta & \sqrt{\beta}\sin\phi \end{pmatrix},$$

where $\alpha = \frac{\Lambda c}{ac-b^2}$, $\beta = \frac{\Lambda a}{ac-b^2}$, *and* $\cos(\theta - \phi) = \frac{\Lambda}{|\Lambda|}\frac{-b}{\sqrt{ac}}$.

Definition 2.9. *Let V and $\overline{V}$ have scalar products g and h respectively, and $T : V \to V$ be a linear transformation. We call a linear isomorphism $A : V \to \overline{V}$, a T-linear isometry provided*

$$h(ATv, Aw) = g(v, w) \ \text{ for all} \, v, w \in V. \tag{2.3}$$

Lemma 2.10. *Let V and $\overline{V}$ have scalar products g and h respectively, and $T : V \to V$ be a linear transformation. If $A : V \to \overline{V}$, is a T-linear isometry, then T is symmetric relative to g, T is isomorphism, and so*

$$h(Av, Aw) = g(T^{-1}v, w) \ \text{ for all} \, v, w \in V, \tag{2.4}$$

and when g and h are inner products, then T is positive definite relative to g.

Proof. T is symmetric relative to g, because $g(w, Tv) = h(ATw, ATv) = h(ATv, ATw) = g(v, Tw)$. By nondegeneracy of g, T is one-to-one and so T is an isomorphism. Also when g and h are inner products, if $v \neq 0$, then $g(v, Tv) = h(ATv, ATv) > 0$, therefore T is positive definite relative to g. $\square$

Remark 2.11. *Let $A : V \to V$ be a linear isomorphism and consider an orthonormal basis for positive definite scalar product space V. In this basis by equation* (2.4)*, we have A is a T-linear isometry if and only if $A^tA = T^{-1}$.*

Lemma 2.12. *Consider scalar product spaces (V, g), $(\overline{V}, h)$ and $(\overline{\overline{V}}, l)$. If $A_1 : V \to \overline{V}$ is a T-linear isometry and $A_2 : \overline{V} \to \overline{\overline{V}}$ is a linear isometry then A_2A_1 is a T-linear isometry of V into $\overline{\overline{V}}$.*

Proposition 2.13. *Consider inner product spaces (V, g) and $(\overline{V}, h)$ which have the same dimension and suppose $T : V \to V$ be a linear isomorphism, positive definite and symmetric relative to g. Then there exists a T-linear isometry from V to $\overline{V}$.*

Proof. By Lemma 2.10, we can choose an orthonormal basis for V which diagonalize T. Let $\{e_i\}_{i=1}^n$ be the basis that makes diagonal T, and $\{\lambda_i\}_{i=1}^n$, $\lambda_i > 0$, be its corresponding eigenvalues of T. Consider an orthonormal basis $\{f_i\}_{i=1}^n$ for $\overline{V}$. Now defining $Ae_i = \frac{1}{\sqrt{\lambda_i}}f_i$, we get A as a T-linear isometry from V to $\overline{V}$. $\square$

Proposition 2.14. *Let $T : V \to V$ be a linear transformation on an inner product space (V, g). If T is a T-isometry, then T is identity.*

Proof. By Lemma 2.10, we suppose $\{e_i\}_{i=1}^n$ be the basis that makes diagonal T, and $\{\lambda_i\}_{i=1}^n$, $\lambda_i > 0$, be its corresponding eigenvalues of T. By equation (2.3), $g(T^2e_i, Te_i) = \lambda_i^3 = 1$ for all i, so $\lambda_i = 1$. Therefore T is identity. $\square$

Proposition 2.15. *Let V and $\overline{V}$ have scalar products g and h respectively, and $T : V \to V$ be a linear transformation. Consider $A : V \to \overline{V}$ be a T-linear isometry and W be a subspace of V. Then $AW^{\perp} = (ATW)^{\perp}$ and $AW^{T} = (AT^2W)^{\perp}$.*

Proof. The following holds

$$AW^{\perp} = \{Av|\ g(v,w) = 0\ \forall w \in W\}.$$

By Definition 2.9,

$$AW^{\perp} = \{Av|\ h(Av, ATw) = 0\ \forall w \in W\}.$$

So $AW^{\perp} \subset (ATW)^{\perp}$. By Lemma 2.10, T is a linear isomorphism. Then by Lemma 2.2-(i), these two subspaces have the same dimension, hence they are equal. By Lemma 2.10, T is symmetric relative to g, and so similarly $AW^{T} = (AT^2W)^{\perp}$. □

3. *T*-Weakly Conformal Maps and Their Characterizations

In this section, we talk about T-weakly conformal maps. These involve a smooth (1,1)-tensor field, noted as T. If we have a smooth map ψ from one Riemannian manifold (M,h) to another $(\overline{M},l)$, it's called T-weakly conformal at a point x in M if there is some number $\Lambda(x)$ making the differential $d\psi_x$ equate $\langle d\psi_x(T_xX), d\psi_x(Y)\rangle_l = \Lambda(x)\langle X,Y\rangle_h$ for all tangent vectors X,Y at x. In simple terms, the map $d\psi_x$ changes how we see angles based on T, but in a consistent way. The number $\Lambda(x)$ is known as the T-square conformality factor. A function ψ is T-weakly conformal if it follows this rule at every point of M.

Some key results are then established, including: necessary and sufficient conditions for a map to be T-weakly conformal in terms of injectivity of $d\psi$ and positivity of ΛT; how to recognize T-weakly conformal maps when they are laid out between regular Euclidean spaces; also important Liouville-type theorems explain the overall nature of these types of maps on $\mathbb{R}^n$. This idea takes the classical definition of conformality and weakly conformal maps and remixes it. This provides a framework to study distorted notions of angle preservation and their implications.

Definition 3.1. *Let T be a smooth $(1,1)$–tensor field on Riemannian manifold (M,h), $\psi : (M^n,h) \to (\overline{M}^m, l)$ be a smooth map from (M,h) to a Riemannian manifold $(\overline{M},l)$. We call ψ is T-weakly conformal (T-conformal) at x if there is a number $\Lambda(x)$ ($\Lambda(x) \neq 0$), call T-square conformality factor at x, such that*

$$\Big\langle d\psi(T(X)), d\psi(Y)\Big\rangle_l = \Lambda(x)\,\langle X,Y\rangle_h\ , \tag{3.1}$$

for any $X,Y \in T_xM$. The map ψ is called T-weakly conformal (T-conformal) if equation (3.1) (and $\Lambda(x) \neq 0$) holds for all $x \in M$. In this case, taking the trace in (3.1) shows that $\Lambda : M \to \mathbb{R}$ is smooth function.

Remark 3.2. *Equation*(3.1) *is equivalent to* $d\psi^* d\psi T = \Lambda I$ *and so* $\Lambda = \frac{1}{n}\sum_{i=1}^{n} \langle d\psi(T(e_i)), d\psi(e_i)\rangle_l = \frac{1}{n}\mathrm{tr}_h(d\psi^* d\psi T)$ *where* $d\psi^*$ *is the adjoint operator and* $\{e_i\}_{i=1}^{n}$ *is an orthonormal frame on* M.

Proposition 3.3. *Let* T *be a smooth tensor field on Riemannian manifold* (M, h), $\psi : (M^n, h) \to (\overline{M}^m, l)$ *be a* T*-weakly conformal map. If* T*-square conformality factor* $\Lambda(x) \neq 0$, *then* $T_x : T_xM \to T_xM$ *is invertable, symmetric relative to Riemannian metric* h *and* $\Lambda(x)T_x$ *is positive definite relative to* h. *Especially if* ψ *is a* T*-conformal map, then* T *is invertable, symmetric relative to Riemannian metric* h *and* ΛT *is positive definite relative to* h.

Theorem 3.4. *Let* T *be a smooth tensor field on* $\mathbb{R}^n$, *and* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *be a smooth map from* $\mathbb{R}^n$ *into* $\mathbb{R}^n$. *If* ψ *is a* T*-conformal map with* T*-square conformality factor* Λ *then*

$$\psi(\mathbf{x}) = \int B\sqrt{\Lambda T^{-1}}\mathrm{d}\mathbf{x} + \mathbf{b},$$

for an unique orthogonal tensor field B *on* $\mathbb{R}^n$ *and* $\mathbf{b} \in \mathbb{R}^n$. *Especially if* T *is a constant tensor field and* Λ *is constant function, so if* ψ *is a* T*-conformal map with* T*-square conformality factor* Λ *then*

$$\psi(\mathbf{x}) = B\sqrt{\Lambda T^{-1}}\mathbf{x} + \mathbf{b},$$

for an unique orthogonal matrix $B \in O(n)$ *(note* B *is constant tensor field, since* $\psi(\sqrt{\Lambda^{-1}T}\mathbf{x})$ *is an isometry) and* $\mathbf{b} \in \mathbb{R}^n$. *In addition, if* $n = 2$ *and*

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

then

$$\psi(x_1, x_2) = \begin{pmatrix} \sqrt{\alpha}\cos\theta & \sqrt{\beta}\cos\phi \\ \sqrt{\alpha}\sin\theta & \sqrt{\beta}\sin\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where

$$\alpha = \frac{\Lambda c}{ac - b^2}, \quad \beta = \frac{\Lambda a}{ac - b^2}, \quad \cos(\theta - \phi) = \frac{\Lambda}{|\Lambda|}\frac{-b}{\sqrt{ac}},$$

and b_1 *and* b_2 *are real constants.*

Proposition 3.5. *Let* T *be a smooth tensor field on Riemannian manifold* (M, h), $\psi : (M^n, h) \to (\overline{M}^m, l)$ *be a smooth map from* (M, h) *to a Riemannian manifold* $(\overline{M}, l)$. *Then* ψ *is* T*-weakly conformal at* x *with* T*-square conformality factor* $\Lambda(x)$ *if and only if precisely one of the following holds:*

- $d\psi_x \circ T_x = 0$, *and so* $\Lambda(x) = 0$;
- $d\psi_x : T_xM \to T_{\psi(x)}\overline{M}$ *is injective and* T*-conformal at* x *with* T*-square conformality factor* $\Lambda(x) \neq 0$.

Especially if ψ *is a* T*-conformal map, then* ψ *is an immersion.*

Proof. Let $d\psi_x \circ T_x \neq 0$, and so $\Lambda(x) \neq 0$. If $d\psi_x(X) = 0$ for some $X \in T_xM$, then by equation (3.1) we get that $\Lambda(x)\langle X, X\rangle_h = 0$, and since $\Lambda(x) \neq 0$, $X = 0$. Therefore $d\psi_x$ is an injective linear transformation. The converse is obvious. □

An immediate result of Proposition 3.5 is the following.

Proposition 3.6. *Let T be a smooth tensor field on Riemannian manifold (M, h), $\psi : (M^n, h) \to (\overline{M}^m, l)$ be a T-weakly conformal map. Let $dim(\overline{M}) < dim(M)$. Then $d\psi \circ T = 0$, especially if T is invertable, then ψ is constant on connected components of M.*

Proposition 3.7. *Let T be a smooth tensor field on Riemannian manifold (M, h), $\psi : (M^n, h) \to (\overline{M}^m, l)$ be a T-weakly conformal map. If T is invertable and anti-symmetric relative to Riemannian metric h, then ψ is constant on connected components of M.*

Proposition 3.8. *Let T be a smooth tensor field on Riemannian manifold (M, h), $\psi : (M^n, h) \to (\overline{M}^m, l)$ be a smooth map. Assume T is invertable, then ψ is a T-weakly conformal map with T-square conformality factor Λ if and only if it is a T^t-weakly conformal map with T^t-square conformality factor Λ, and hence it is a $\frac{T+T^t}{2}$-weakly conformal map with $\frac{T+T^t}{2}$-square conformality factor Λ.*

Remark 3.9. *Let $\psi : (M^n, h) \to (\overline{M}^m, l)$ is a non-constant smooth map and T be a invertable and anti-symmetric relative to Riemannian metric h. So ψ is a $\frac{T+T^t}{2}$-weakly conformal map with T-square conformality factor $\Lambda = 0$. But by Proposition 3.7, it is a not a T-weakly conformal map.*

Proposition 3.10. *Let T be a smooth tensor field on Riemannian manifold (M, h), $\psi_1 : (M^n, h) \to (\overline{M}^m, l)$ be a T-weakly conformal map with T-square conformality factor Λ_1 and $\psi_2 : (\overline{M}, l) \to (\overline{\overline{M}}, k)$ be a weakly conformal map with square conformality factor Λ_2. Then $\psi_2 \circ \psi_1$ is a T-weakly conformal map with T-square conformality factor $\Lambda_1(\Lambda_2 \circ \psi_1)$.*

Proposition 3.11. *Let T be a smooth tensor field on Riemannian manifold (M^n, h), the identity map $i : (M, h) \to (M, l)$ be a T-weakly conformal map with T-square conformality factor Λ. Then $\langle TX, Y\rangle_l = \Lambda\langle X, Y\rangle_h$ for any vector fields X, Y on M, and $\Lambda = \frac{1}{n}\mathrm{tr}_l T$. Especially if $h = l$, then $T = \Lambda I$.*

Proposition 3.12. *Let T be a smooth tensor field on one dimensional Riemannian manifold (M^1, h), $\psi : (M^1, h) \to (\overline{M}^m, l)$ be a smooth map. Then ψ is a T-weakly conformal map with T-square conformality factor $\Lambda(x) = T(x)\langle d\psi(X), d\psi(X)\rangle_l$ where $X \in T_xM$ and X is a unit vector.*

Example 3.13. *Let ψ be a smooth function on $\mathbb{R}$ and T be a smooth tensor field on $\mathbb{R}$. Then ψ is T-weakly conformal function with T-square conformality factor $\Lambda(x) = T(x)(\psi'(x))^2$.*

Proposition 3.14. *Let T be a smooth tensor field on Riemann surface M^2, $\psi : M^2 \to (\overline{M}^m, l)$ be a T-weakly conformal map. Then for any complex coordinate z on M, the following holds*

$$\left\langle d\psi(T\frac{\partial}{\partial z}), d\psi(\frac{\partial}{\partial z}) \right\rangle_l = 0.$$

4. The Connection and Curvature Tensor for T–Conformal Maps

The offered section affords a few crucial consequences regarding conformal maps among Riemannian manifolds in the presence of a smooth tensor field T. After introducing key definitions inclusive of T-local isometries, several useful formulas and properties are derived. Lemma 4.1 gives a formula concerning the covariant derivative of the differential of a T-conformal map ψ to quantities related to T, the conformality factor Λ, and the connections on the domain. Other consequences characterize when T-local isometries are determined by their differential at a point, when flows of a vector field are T-local isometries, and a formula relating the curvature tensor of a T-conformal map. Overall, the text develops the theory of conformal geometry in the presence of a tensor subject, which, namely, is likely to be useful in contexts where a tensor field naturally arises.

Lemma 4.1. *Let T be a smooth tensor field on Riemannian manifold (M^n, h), and $\psi : (M^n, h) \to (\overline{M}^m, l)$ be a smooth map. if ψ is a T-conformal map with T-square conformality factor Λ, then we have the following formulae:*

$$\begin{aligned}
\overline{\nabla}_{Z_2} d\psi(Z_1) =& \frac{1}{2} d\psi \Big\{ (\nabla_{Z_2} \ln|\Lambda|) Z_1 + 2\nabla_{Z_2} Z_1 + (\nabla_{Z_1} \ln|\Lambda|) Z_2 \\
&- \left\langle T^{-1} Z_1, Z_2 \right\rangle T(\nabla \ln|\Lambda|) - (\nabla_{Z_1} T) T^{-1} Z_2 - (\nabla_{Z_2} T) T^{-1} Z_1 \\
&- \sum_k \left\langle (\nabla_{e_k} T^{-1}) Z_1, Z_2 \right\rangle T e_k \Big\} + B(Z_1, Z_2),
\end{aligned}$$

where Z_1, Z_2 are vector fields on M, $\{e_i\}_{i=1}^n$ is an orthogonal frame on M and $B(Z_1, Z_2)$ is vertical part of $\overline{\nabla}_{Z_2} d\psi(Z_1)$ which is orthogonal to image of ψ, and B is tensorial in its two arguments and it is symmetric, $B(Z_1, Z_2) = B(Z_2, Z_1)$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame on M such that for every i, j, $\nabla_{e_j} e_i = 0$ at $p \in M$. By Koszul formula we have at p

$$
\begin{aligned}
&\left\langle \overline{\nabla}_{Z_2} d\psi(Z_1), d\psi(Te_k) \right\rangle_l = \\
&\quad \frac{1}{2}\left\{ \overline{\nabla}_{Z_2} \langle d\psi(Z_1), d\psi(Te_k)\rangle_l + \overline{\nabla}_{Z_1} \langle d\psi(Z_2), d\psi(Te_k)\rangle_l \right. \qquad (4.1)\\
&\quad -\overline{\nabla}_{Te_k} \langle d\psi(Z_2), d\psi(Z_1)\rangle_l - \langle d\psi(Z_2), d\psi[Z_1, Te_k]\rangle - \langle d\psi(Z_1), d\psi[Z_2, Te_k]\rangle \\
&\quad \left. + \langle d\psi(Te_k), d\psi[Z_2, Z_1]\rangle \right\} . \qquad (4.2)
\end{aligned}
$$

By Proposition 3.3, T is invertable. Therefore, by equations (3.1), (4.1) and the relation $\langle d\psi(X), d\psi(Y)\rangle_l = \Lambda \left\langle T^{-1}X, Y \right\rangle_h$ for any vector fields X, Y on M, we have at any point p

$$
\begin{aligned}
&\left\langle \overline{\nabla}_{Z_2} d\psi(Z_1), d\psi(Te_k) \right\rangle_l = \\
&\quad \frac{1}{2}\left\{ \nabla_{Z_2}(\Lambda \langle Z_1, e_k\rangle) + \nabla_{Z_1}(\Lambda \langle Z_2, e_k\rangle) - \nabla_{Te_k}\left(\Lambda \left\langle T^{-1}Z_2, Z_1\right\rangle\right) \right.\\
&\quad - \langle d\psi(Z_2), d\psi((\nabla_{Z_1}T)e_k - \nabla_{Te_k}Z_1)\rangle - \langle d\psi(Z_1), d\psi((\nabla_{Z_2}T)e_k - \nabla_{Te_k}Z_2)\rangle \\
&\quad \left. + \langle d\psi(Te_k), d\psi([Z_2, Z_1])\rangle \right\} \\
&\quad = \frac{1}{2}\left\{ (\nabla_{Z_2}\Lambda)\langle Z_1, e_k\rangle + 2\Lambda \langle \nabla_{Z_2}Z_1, e_k\rangle + (\nabla_{Z_1}\Lambda)\langle Z_2, e_k\rangle \right.\\
&\quad - (\nabla_{Te_k}\Lambda)\left\langle T^{-1}Z_2, Z_1\right\rangle - \Lambda\left\langle (\nabla_{Te_k}T^{-1})Z_2, Z_1\right\rangle - \Lambda\left\langle (\nabla_{Z_1}T)T^{-1}Z_2, e_k\right\rangle \\
&\quad \left. -\Lambda\left\langle (\nabla_{Z_2}T)T^{-1}Z_1, e_k\right\rangle \right\} , \qquad (4.3)
\end{aligned}
$$

We have for every vector field Z on M,

$$
d\psi(Z) = \frac{1}{\Lambda}\sum_k \langle d\psi(Z), d\psi(Te_k)\rangle_l \, d\psi(e_k). \qquad (4.4)
$$

At first assume $n = m$, then by Proposition 3.5, ψ is an immersion, and so by equality of dimensions of M and $\overline{M}$, ψ is a submersion. Therefore we can assume $\overline{\nabla}_{Z_2} d\psi(Z_1) = d\psi(Z_3)$ for some vector field Z_3 on M. Now we get by equations (4.3) and (4.4),

$$
\begin{aligned}
\overline{\nabla}_{Z_2} d\psi(Z_1) &= \frac{1}{\Lambda}\sum_k \left\langle \overline{\nabla}_{Z_2} d\psi(Z_1), d\psi(Te_k)\right\rangle_l d\psi(e_k) \\
&= \frac{1}{2\Lambda}\left\{ (\nabla_{Z_2}\Lambda) d\psi(Z_1) + 2\Lambda\, d\psi(\nabla_{Z_2}Z_1) + (\nabla_{Z_1}\Lambda) d\psi(Z_2) \right.\\
&- \left\langle T^{-1}Z_1, Z_2\right\rangle d\psi(T(\nabla\Lambda)) - \Lambda\left\langle (\nabla_{e_k}T^{-1})Z_1, Z_2\right\rangle d\psi(Te_k) \\
&\left. -\Lambda\, d\psi((\nabla_{Z_1}T)T^{-1}Z_2) - \Lambda\, d\psi((\nabla_{Z_2}T)T^{-1}Z_1) \right\} .
\end{aligned}
$$

Now if $n < m$, by considering horizontal and vertical parts of $\overline{\nabla}_{Z_2} d\psi(Z_1)$, we get the result. □

Definition 4.2. *Let T be a smooth tensor field on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$. We call ψ is a T-local isometry provided each differential* *map $d\psi_x : T_xM \to T_{\psi(x)}\overline{M}$ is a T-linear isometry, and in addition if ψ is a diffeomorphism, we call it a T-isometry.*

Remark 4.3. *As we see, each T-local isometry is a T-conformal map with T-square conformality factor $\Lambda = 1$.*

Remark 4.4. *In view of the inverse function theorem, each T-local isometry is locally a T-isometry that is each point x of M has a neighborhood U such that $\psi|_U$ is an T-isometry of U onto a neighborhood of $\psi(x)$ in $\overline{M}$.*

A T-local isometry is uniquely determined by its differential map at a single point provided that $\nabla T = 0$.

Proposition 4.5. *Let T be a smooth tensor field on Riemannian manifold M, $\phi, \psi : M \to \overline{M}$ be T-local isometries of a connected Riemannian manifold M to a Riemannian manifold $\overline{M}$. If $\nabla T = 0$ and there is a point $x \in M$ such that $d\phi_x = d\psi_x$, then $\phi = \psi$.*

Proof. Let $A = \{p \in M : d\phi_p = d\psi_p\}$. By continuity, A is closed in M. Since A is nonempty it suffices to show that A is open. We assert that if $p \in A$ then any normal neighborhood U of p is contained in A . If $r \in U$ there is a vector $v \in T_pM$ such that $\gamma_v(1) = \exp_p(v) = r$. Since $\nabla T = 0$ and $\Lambda = 1$, by Lemmas 2.10 and 4.1, we get ϕ and ψ preserve Levi-Civita connections. Thus geodesics in M are carried to geodesics in N by ϕ, ψ. Hence

$$\phi(r) = \phi(\gamma_v(1)) = \gamma_{d\phi v}(1) = \gamma_{d\psi v}(1) = \psi(\gamma_v(1)) = \psi(r).$$

Therefore $\phi = \psi$ on U and so $d\phi_q = d\psi_q$ for all $q \in U$. □

Proposition 4.6. *Let T be a smooth tensor field on Riemannian manifold (M, h) and V be a smooth vector field on M. If all local flows of V are T-local isometries, then T is identity.*

Proof. If v, w are tangent vectors at a point in the domain of the flow, by hypothesis we have $\langle d\psi_t(Tv), d\psi_t(w)\rangle = \langle v, w\rangle$ where $\{\psi_t\}$ is a local flow of V. By putting $t = 0$ and nondegeneracy of the metric we get $T = I$. □

Lemma 4.7. *Let T be a smooth tensor field on Riemannian manifold (M^n, h), and $\psi : (M^n, h) \to (\overline{M}^n, l)$ be a smooth map from M into $\overline{M}$. If ψ is a T-conformal map with T-square conformality factor Λ and $\nabla T = 0$, then*

$$\begin{aligned}
&\bar{R}(Z_3, Z_2)d\psi(Z_1) = \\
&\quad \frac{1}{4}d\psi \{4R(Z_3, Z_2)Z_1 + 2\langle \nabla_{Z_3}\nabla \ln|\Lambda|, Z_1\rangle Z_2 - 2\langle \nabla_{Z_2}\nabla \ln|\Lambda|, Z_1\rangle Z_3 \\
&\quad + (\nabla_{Z_1}\ln|\Lambda|)(\nabla_{Z_2}\ln|\Lambda|)Z_3 - (\nabla_{Z_3}\ln|\Lambda|)(\nabla_{Z_1}\ln|\Lambda|)Z_2 \\
&\quad - 2\left\langle T^{-1}Z_1, Z_2\right\rangle T(\nabla_{Z_3}\nabla \ln|\Lambda|) + 2\left\langle T^{-1}Z_1, Z_3\right\rangle T(\nabla_{Z_2}\nabla \ln|\Lambda|) \\
&\quad - \left\langle T^{-1}Z_1, Z_2\right\rangle (\nabla_{T(\nabla \ln|\Lambda|)}\ln|\Lambda|)Z_3 + \left\langle T^{-1}Z_1, Z_3\right\rangle (\nabla_{T(\nabla \ln|\Lambda|)}\ln|\Lambda|)Z_2 \\
&\quad + \left(\left\langle T^{-1}Z_1, Z_2\right\rangle (\nabla_{Z_3}\ln|\Lambda|) - \left\langle T^{-1}Z_1, Z_3\right\rangle (\nabla_{Z_2}\ln|\Lambda|)\right) T(\nabla \ln|\Lambda|)\} ,
\end{aligned}$$

where Z_1, Z_2, Z_3 are vector fields on M, and R and $\bar{R}$ denote the curvature tensors of M and $\overline{M}$, respectively.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame on M such that for every i, j, $\nabla_{e_j}e_i = 0$ at $p \in M$. By Proposition 3.3, T is invertable. We have $\nabla T = 0$ and so $\nabla T^{-1} = 0$, then we get by Lemma 4.1 at p,

$$\begin{aligned}
&\overline{\nabla}_{e_k}\overline{\nabla}_{e_i}d\psi(e_j) = \\
&\quad \frac{1}{2}\overline{\nabla}_{e_k}d\psi\Big\{(\nabla_{e_i}\ln|\Lambda|)e_j + 2\nabla_{e_i}e_j + (\nabla_{e_j}\ln|\Lambda|)e_i - \left\langle T^{-1}e_j, e_i\right\rangle T(\nabla \ln|\Lambda|)\Big\}.
\end{aligned} \tag{4.5}$$

Denote

$$Z_1 = (\nabla_{e_i}\ln|\Lambda|)e_j + 2\nabla_{e_i}e_j + (\nabla_{e_j}\ln|\Lambda|)e_i - \left\langle T^{-1}e_j, e_i\right\rangle T(\nabla \ln|\Lambda|). \tag{4.6}$$

Therefore by Lemma 4.1, equations (4.5) and (4.6) at p,

$$\begin{aligned}
\overline{\nabla}_{e_k}\overline{\nabla}_{e_i}d\psi(e_j) &= \frac{1}{2}\overline{\nabla}_{e_k}d\psi(Z_1) \\
&= \frac{1}{4}d\psi \{(\nabla_{e_k}\ln|\Lambda|)Z_1 + 2\nabla_{e_k}Z_1 + (\nabla_{Z_1}\ln|\Lambda|)e_k \\
&- \left\langle T^{-1}Z_1, e_k\right\rangle T(\nabla \ln|\Lambda|)\} \qquad (4.7)\\
&= \frac{1}{4}d\psi \{(\nabla_{e_k}\ln|\Lambda|)(\nabla_{e_i}\ln|\Lambda|)e_j + (\nabla_{e_k}\ln|\Lambda|)(\nabla_{e_j}\ln|\Lambda|)e_i \\
&+ 2(\nabla_{e_k}\nabla_{e_i}\ln|\Lambda|)e_j + 4\nabla_{e_k}\nabla_{e_i}e_j \\
&+ 2(\nabla_{e_k}\nabla_{e_j}\ln|\Lambda|)e_i - 2\left\langle T^{-1}e_j, e_i\right\rangle T(\nabla_{e_k}\nabla \ln|\Lambda|) \\
&+ (\nabla_{e_i}\ln|\Lambda|)(\nabla_{e_j}\ln|\Lambda|)e_k + (\nabla_{e_j}\ln|\Lambda|)(\nabla_{e_i}\ln|\Lambda|)e_k \\
&- \left\langle T^{-1}e_j, e_i\right\rangle (\nabla_{T(\nabla \ln|\Lambda|)}\ln|\Lambda|)e_k - \left\langle T^{-1}e_j, e_k\right\rangle (\nabla_{e_i}\ln|\Lambda|)T(\nabla \ln|\Lambda|) \\
&- \left\langle T^{-1}e_i, e_k\right\rangle (\nabla_{e_j}\ln|\Lambda|)T(\nabla \ln|\Lambda|)\}. \qquad (4.8)
\end{aligned}$$

So by equation (4.7), we have at p,

$$\begin{aligned}
&\bar{R}(e_k, e_i)d\psi(e_j) = \overline{\nabla}_{e_k}\overline{\nabla}_{e_i}d\psi(e_j) - \overline{\nabla}_{e_i}\overline{\nabla}_{e_k}d\psi(e_j) \\
&\quad = \frac{1}{4}d\psi\,\{4R(e_k, e_i)e_j + (\nabla_{e_j}\ln|\Lambda|)(\nabla_{e_i}\ln|\Lambda|)e_k - (\nabla_{e_k}\ln|\Lambda|)(\nabla_{e_j}\ln|\Lambda|)e_i \\
&\quad + 2(\nabla_{e_k}\nabla_{e_j}\ln|\Lambda|)e_i - 2(\nabla_{e_i}\nabla_{e_j}\ln|\Lambda|)e_k - 2\left\langle T^{-1}e_j, e_i\right\rangle T(\nabla_{e_k}\nabla\ln|\Lambda|) \\
&\quad + 2\left\langle T^{-1}e_j, e_k\right\rangle T(\nabla_{e_i}\nabla\ln|\Lambda|) \\
&\quad - \left\langle T^{-1}e_j, e_i\right\rangle(\nabla_{T(\nabla\ln|\Lambda|)}\ln|\Lambda|)e_k + \left\langle T^{-1}e_j, e_k\right\rangle(\nabla_{T(\nabla\ln|\Lambda|)}\ln|\Lambda|)e_i \\
&\quad - \left\langle T^{-1}e_j, e_k\right\rangle(\nabla_{e_i}\ln|\Lambda|)T(\nabla\ln|\Lambda|) + \left\langle T^{-1}e_j, e_i\right\rangle(\nabla_{e_k}\ln|\Lambda|)T(\nabla\ln|\Lambda|)\}\,,
\end{aligned}$$

Now, by noting at p, $(\nabla_{e_k}\nabla_{e_j}\ln|\Lambda|)e_i = \langle\nabla_{e_k}\nabla\ln|\Lambda|, e_j\rangle$ and using the tensorial properties, we get the result. □

5. PDE Systems for T–Conformal Maps

Let $R^n(c)$ be the simply connected Riemannian space form of constant sectional curvature c which is the Euclidean space $\mathbb{R}^n$, for $c = 0$, and the Hyperbolic space $\mathbb{H}^n$, for $c = -1$, and the Euclidean sphere $\mathbb{S}^n$, for $c = +1$.

The following theorem is an exploitation of proof of "Liouville's theorem for conformal maps" for T-conformal map on $R^n(c)$ when $\nabla T = 0$.

Theorem 5.1. *Let T be a smooth tensor field on $R^n(c)$, and $\psi : U \to R^n(\bar{c})$ be a smooth map from an open subset U of $(R^n(c), h)$ into $(R^n(\bar{c}), l)$. If ψ is a T-conformal map with T-square conformality factor Λ and $\nabla T = 0$, then Λ satisfy the following system of PDEs on U:*

$$\begin{aligned}
&\bar{c}\,\Lambda\left(h_{ks}\sum_r T^{-1}_{rj}h_{ri} - h_{is}\sum_r T^{-1}_{rj}h_{rk}\right) - c\,(h_{ji}h_{ks} - h_{jk}h_{is}) \\
&= 2h_{is}\sum_{l,t}\left(\frac{\partial h^{lt}}{\partial x_k}\frac{\partial\ln|\Lambda|}{\partial x_l} + h^{lt}\frac{\partial^2\ln|\Lambda|}{\partial x_k\partial x_l} + \sum_r h^{lr}\frac{\partial\ln|\Lambda|}{\partial x_l}\Gamma^t_{kr}\right)h_{tj} \\
&- 2h_{ks}\sum_{l,t}\left(\frac{\partial h^{lt}}{\partial x_i}\frac{\partial\ln|\Lambda|}{\partial x_l} + h^{lt}\frac{\partial^2\ln|\Lambda|}{\partial x_i\partial x_l} + \sum_r h^{lr}\frac{\partial\ln|\Lambda|}{\partial x_l}\Gamma^t_{ir}\right)h_{tj} \\
&+ \frac{\partial\ln|\Lambda|}{\partial x_j}\frac{\partial\ln|\Lambda|}{\partial x_i}h_{ks} - \frac{\partial\ln|\Lambda|}{\partial x_k}\frac{\partial\ln|\Lambda|}{\partial x_j}h_{is} \\
&- 2\sum_{l,t,r',s'}\left(\frac{\partial h^{lt}}{\partial x_k}\frac{\partial\ln|\Lambda|}{\partial x_l} + h^{lt}\frac{\partial^2\ln|\Lambda|}{\partial x_k\partial x_l} + \sum_r h^{lr}\frac{\partial\ln|\Lambda|}{\partial x_l}\Gamma^t_{kr}\right)T_{s't}T^{-1}_{r'j}h_{r'i}h_{s's} \\
&+ 2\sum_{l,t,r',s'}\left(\frac{\partial h^{lt}}{\partial x_i}\frac{\partial\ln|\Lambda|}{\partial x_l} + h^{lt}\frac{\partial^2\ln|\Lambda|}{\partial x_i\partial x_l} + \sum_r h^{lr}\frac{\partial\ln|\Lambda|}{\partial x_l}\Gamma^t_{ir}\right)T_{s't}T^{-1}_{r'j}h_{r'k}h_{s's}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l,t,r',s'} \frac{\partial \ln|\Lambda|}{\partial x_l} \frac{\partial \ln|\Lambda|}{\partial x_{s'}} T_{s't} T^{-1}_{r'j} h_{r'i} h_{ks} h^{lt} \\
&+ \sum_{l,t,r',s'} \frac{\partial \ln|\Lambda|}{\partial x_l} \frac{\partial \ln|\Lambda|}{\partial x_{s'}} T_{s't} T^{-1}_{r'j} h_{r'k} h_{is} h^{lt} \\
&+ \sum_{l,t,r'} \left(h_{r'i} \frac{\partial \ln|\Lambda|}{\partial x_k} - h_{r'k} \frac{\partial \ln|\Lambda|}{\partial x_i} \right) \frac{\partial \ln|\Lambda|}{\partial x_l} T^{-1}_{r'j} T_{tr} h_{ts} h^{lr},
\end{aligned}
$$

for every $i, j, k, s = 1, \ldots, n$, where $x_1, \ldots, x_n$ is a local coordinate system on U, $T_{ij} = \sum_k \langle T \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \rangle h^{ki}$ and Γ^k_{ij} are Christoffel symbols of Levi-Civita connection of h. Especially if $c = \bar{c} = 0$, then Λ satisfy the following system of PDEs on U:

$$
\begin{aligned}
&2 \frac{\partial^2 \ln|\Lambda|}{\partial x_k \partial x_j} \delta_{is} - 2 \frac{\partial^2 \ln|\Lambda|}{\partial x_i \partial x_j} \delta_{ks} + \frac{\partial \ln|\Lambda|}{\partial x_j} \frac{\partial \ln|\Lambda|}{\partial x_i} \delta_{ks} - \frac{\partial \ln|\Lambda|}{\partial x_k} \frac{\partial \ln|\Lambda|}{\partial x_j} \delta_{is} \\
&+ 2 \sum_{l=1}^{n} T_{ls} \left(T^{-1}_{jk} \frac{\partial^2 \ln|\Lambda|}{\partial x_i \partial x_l} - T^{-1}_{ji} \frac{\partial^2 \ln|\Lambda|}{\partial x_k \partial x_l} \right) \\
&+ \left(T^{-1}_{jk} \delta_{is} - T^{-1}_{ji} \delta_{ks} \right) \sum_{l,r=1}^{n} T_{lr} \frac{\partial \ln|\Lambda|}{\partial x_l} \frac{\partial \ln|\Lambda|}{\partial x_r} \\
&+ \left(T^{-1}_{ji} \frac{\partial \ln|\Lambda|}{\partial x_k} - T^{-1}_{jk} \frac{\partial \ln|\Lambda|}{\partial x_i} \right) \sum_{l=1}^{n} T_{ls} \frac{\partial \ln|\Lambda|}{\partial x_l} = 0 \,,
\end{aligned}
$$

for every $i, j, k, s = 1, \ldots, n$.

Proof. By Proposition 3.5, ψ is an immersion, and so locally diffeomorphism. The curvature tensor of $R^n(c)$ is

$$R(X, Y)Z = c \left\{ \langle Z, Y \rangle X - \langle Z, X \rangle Y \right\},$$

so by Lemma 4.7, and straightforward computations we get the result. □

In continuation, consider a constant tensor field T on $\mathbb{R}^n$ (so as a matrix) which is symmetric and definite, and two types of maps of Euclidean space $\mathbb{R}^n$ which we use them later.

(1) *T-homotheties.* Let be $\psi : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\psi(\mathbf{x}) = B\sqrt{\Lambda T^{-1}}\mathbf{x} + \mathbf{b},$$

for some orthogonal matrix B, non-zero number Λ such that ΛT is positive definite, and vector $\mathbf{b} \in \mathbb{R}^n$.

(2) *T-inversions in a sphere.* T-inversion in a sphere with radius r and center $\mathbf{a} \in \mathbb{R}^n$, defined as $\varphi : \mathbb{R}^n \setminus \{\mathbf{a}\} \to \mathbb{R}^n \setminus \{\mathbf{a}\}$,

$$\varphi(\mathbf{x}) = \frac{r^2 (\mathbf{x} - \mathbf{a})}{\langle T(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a} \rangle} + \mathbf{a},$$

as we see $\varphi^{-1}(\mathbf{x}) = \varphi(\mathbf{x})$.

Let T be a constant tensor field on $\mathbb{R}^n$, and $|T| = T$ if T is positive definite, $|T| = -T$ if T is negative definite, and $\psi : U \to \mathbb{R}^n$ be a smooth map from an open subset U of $\mathbb{R}^n$ into $\mathbb{R}^n$. If ψ is a T-conformal map and T is positive definite then $\psi(\sqrt{|T|}\mathbf{x})$ is a conformal map. Therefore every T-conformal map is a precomposition of conformal map with the $\sqrt{|T|^{-1}}\mathrm{id}_{\mathbb{R}^n}$ resticted on an open set. For $n \geq 3$, every conformal map is the restriction of an elemet of the Möbius group, i.e., it is the composition of homotheties and inversions (Proposition 2.3.14 of [3]). Therefore for $n \geq 3$, every T-conformal map is a precomposition of element of the Möbius group with the $\sqrt{|T|^{-1}}\mathrm{id}_{\mathbb{R}^n}$ restricted on an open set. Here for more detailed study, we state its direct proof in the following theorem.

Theorem 5.2 (Liouville's theorem for T-conformal maps)**.** *Let T be a constant tensor field on $\mathbb{R}^n$, and $\psi : U \to \mathbb{R}^n$ be a smooth map from an open subset U of $\mathbb{R}^n$ into $\mathbb{R}^n$ ($n \geq 3$). If ψ is a T-conformal map then it is a T-homothety or the precomposition of a T-homothety with a T^{-1}-inversion restricted on U, and so it is a precomposition of element of the Möbius group with the $\sqrt{|T|^{-1}}\mathrm{id}_{\mathbb{R}^n}|_U$.*

Proof. If Λ is constant, then by Theorem 3.4, ψ is a T-homothety, so suppose that Λ is non-constant. Because of similarity, we assume that T is positive definite constant tensor, and so $\Lambda = \lambda^2$ for some smooth positive function λ on U. Writing $\lambda_i = \frac{\partial \lambda}{\partial x_i}$, $\lambda_{ij} = \frac{\partial^2 \lambda}{\partial x_j \partial x_i}$, and using Theorem 5.1, we have for every $i, j, k, s = 1, \ldots, n$,

$$\begin{aligned}
&(\lambda_{jk}\lambda - \lambda_k\lambda_j)\delta_{is} - (\lambda_{ji}\lambda - \lambda_i\lambda_j)\delta_{ks} + \lambda_j\lambda_i\delta_{ks} - \lambda_k\lambda_j\delta_{is} \\
&+ \sum_l T_{ls}\left(T^{-1}_{jk}(\lambda_{il}\lambda - \lambda_i\lambda_l) - T^{-1}_{ji}(\lambda_{kl}\lambda - \lambda_k\lambda_l)\right) \\
&+ \left(T^{-1}_{jk}\delta_{is} - T^{-1}_{ji}\delta_{ks}\right)\sum_{l,r} T_{lr}\lambda_l\lambda_r \\
&+ \left(T^{-1}_{ji}\lambda_k - T^{-1}_{jk}\lambda_i\right)\sum_l T_{ls}\lambda_l = 0.
\end{aligned} \tag{5.1}$$

Set $u = \frac{1}{\lambda}$. On putting $i = j$ into equation (5.1), with i, k, s distinct (this is possible since $n \geq 3$), we obtain

$$T^{-1}_{ii}[Tu'']_{sk} = T^{-1}_{ik}[Tu'']_{si}, \tag{5.2}$$

where $u'' = [\frac{\partial^2 u}{\partial x_r \partial x_l}]$ is the Hessian matrix of u. Since T is positive definite, we have for every r, l distinct, $T^{-1}_{rr} > 0$ and $T^{-1}_{rr}T^{-1}_{ll} - (T^{-1}_{rl})^2 \neq 0$. By equation (5.2),

$$[Tu'']_{sk} = \frac{T^{-1}_{ik}[Tu'']_{si}}{T^{-1}_{ii}} = \frac{(T^{-1}_{ik})^2[Tu'']_{sk}}{T^{-1}_{ii}T^{-1}_{kk}},$$

so $[Tu'']_{sk} = 0$, and therefore Tu'' is diagonal. Now put $i = j$, $k = s$, with i, k distinct. This yields

$$u\Big(\frac{1}{T_{ii}^{-1}}\frac{\partial^2 u}{\partial x_i^2} + [Tu'']_{kk}\Big) = \langle Tu', u'\rangle\,, \tag{5.3}$$

where

$$u' = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})^t$$

is the gradient of u. From equation (5.3), we deduce $[Tu'']_{ii} = [Tu'']_{kk}$ $(i, k = 1, \ldots, n)$, and therefore Tu'' is coefficient of identity matrix. Setting $\rho = [Tu'']_{ii}$, from the equation (5.3) we obtain $\langle Tu', u'\rangle = 2\rho u$. Differentiation of this equation shows that ρ must be constant. We thus obtain the system of equations:

$$\begin{cases} u'' = \rho\, T^{-1}, \\ \langle Tu', u'\rangle = 2\rho u. \end{cases} \tag{5.4}$$

If $\rho \equiv 0$, then u is constant and ψ is a T-homothety. Otherwise, one can see that the system (5.4) has general solution

$$u = \frac{\rho}{2}\sum_{i=1}^{n} T_{ii}^{-1}(x_i - a_i)^2 = \frac{\rho}{2}\left\langle T^{-1}(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a}\right\rangle, \tag{5.5}$$

where $\mathbf{a} = (a_1, \ldots, a_n)^t$ is a constant vector. Consider T^{-1}-inversion

$$\varphi(\mathbf{x}) = \frac{2\,(\mathbf{x} - \mathbf{a})}{\rho\,\langle T^{-1}(\mathbf{x} - \mathbf{a}), \mathbf{x} - \mathbf{a}\rangle} + \mathbf{a},$$

and compose it with ψ, we get $\psi \circ \varphi$ is a T-isometry. It follows that ψ is the precomposition of a T-homothety with a T^{-1}-inversion. □

If the map is globally defined on $\mathbb{R}^n$, T-inversions cannot occur and so we have the following description.

Theorem 5.3 (T-conformal transformations of $\mathbb{R}^n$)**.** *Let T be a constant tensor field on $\mathbb{R}^n$, and $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $n \geq 3$ be a T-conformal map. Then it is a T-homothety.*

References

1. M. Aminian, *Introduction of T–harmonic maps.* Pure. Appl. Math, **30**(2) (2023), 109–129.
2. A. Antonopoulos, *The conformal mapping as a limiting case of mapping.* Boll. Geod. Scienze. Affini, **65**(1) (2006), 25–34.
3. P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds.* Number 29. Oxford University Press, 2003.
4. E. Bedford, *What is... a biholomorphic mapping.* Notices. American. Math. Soc, **59**(2012), 812–818.
5. D. E. Blair, *Inversion theory and conformal mapping*, volume 9. American Mathematical Soc., 2000.

6. L. A. Cabral, A. C. Lucizani, and A. J. S. Capistrano, *Graded-index medium profiles associated with curvature and conformal maps*, 2022 Conference on Lasers and Electro-Optics (CLEO), (2022), 1–2.
7. W. P. Calixto, B. Alvarenga, J. C. da Mota, L. d. C. Brito, M. Wu, A. J. Alves, L. M. Neto, C. F. Antunes, et al, *Electromagnetic problems solving by conformal mapping: a mathematical operator for optimization.* mathematical problems in engineering, 2010.
8. T. Cheng, H. Yang, and S. Yang, *Beltrami system and 1-quasiconformal embeddings in higher dimensions.* arXiv preprint arXiv:1709.07324, 2017.
9. S. R. Das and A. Jevicki, *Large-N collective fields and holography.* Phys. Rev. D, **68**(2003), 044011.
10. J. W. Dettman, *Mathematical methods in physics and engineering.* Courier Corporation, 2013.
11. R. R. L. Heinen, *Conformal mappings and the Schwarz-Christoffel transformation.* PhD thesis, University of Missouri–Columbia, 2017.
12. A. E. Jones, *A plane map from a sphere by double projection.* Eos, Trans. American. Geophys. Union, **25**(1944), 219–220.
13. M. Kapranov, *Conformal maps in higher dimensions and derived geometry.* arXiv preprint arXiv:2102.11507, 2021.
14. D. W. Koon and C. J. Knickerbocker, *What do you measure when you measure resistivity.* Review of Scientific Instruments, **63**(1992), 207–210.
15. U. Leonhardt, *Optical conformal mapping.* Science, **312**(5781) (2006):, 777–1780.
16. D. Ntalampekos, *Rigidity and continuous extension for conformal maps of circle domains.* Trans. American. Math. Soc., 2023.
17. B. Olea and F. J. Palomo, *Lorentzian metrics null-projectively related to semi-Riemannian metrics.* J. Geom. Analysis, **33**(2):71, 2023.
18. P. J. Olver, *Complex analysis and conformal mapping.* University of Minnesota, 806, 2017.
19. B. O'Neill, *Semi-Riemannian Geometry: with applications to relativity. pure and applied Mathematics*, Acad. Press, 1983.
20. O. Philips' Gloeilampenfabrieken, *A method of measuring specific resistivity and hall effect of discs of arbitrary shape*, Philips Res. Rep, **13**(1) (1958), 1–9.
21. M. S. Selig and M. D. Maughmer, *Multipoint inverse airfoil design method based on conformal mapping.* AIAA Journal, **30**(5) (1992), 1162–1170.

Received: 30.01.2024
Accepted: 02.04.2024