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On quintic (α, β)-metrics in Finsler geometry

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Abstract. In this paper, we study the class of quintic (α, β)-metrics. We show that every weakly Landsberg 5-th root (α, β)-metrics has vanishing S-curvature. Using it, we prove that a quintic (α, β)-metric is a weakly Landsberg metric if and only if it is a Berwald metric. Then, we show that a quintic (α, β)-metric satisfies $\Xi = 0$ if and only if $\mathbf{S} = 0$.

Keywords: Weakly Landsberg metric, Landsberg metric, Berwald metric, (α, β)-metric, S-curvature, Ξ-curvature.

1. **Introduction**

Let $F = F(x, y)$ be a Finsler metric on tangent bundle TM defined as $F = \sqrt[m]{A}$, where $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ and $a_{i_1 \dots i_m}$ are symmetric in all its indices. Then, F is called an m-th root Finsler metric on the manifold M. The class of m-th root Finsler metrics has been developed by Shimada in [10],

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and applied to biology as an ecological metric by Antonelli in [1]. The fifth root metrics $F = \sqrt[5]{a_{ijklp}(x)y^i y^j y^k y^l y^p}$ are called the quintic metrics.

In order to understand the structure of quintic root metrics, one can study the non-Riemannian curvatures of these metrics [11][12][13]. Among these quantities, the mean Landsberg curvature $\mathbf{J}$ and the S-curvature $\mathbf{S}$ have important and deep relation with each other. Let us give a brief explanation of their relations. The distortion $\tau = \tau(x, y)$ is a non-Riemannian quantity that is determined by the Busemann-Hausdorff volume form. The vertical and horizontal derivations of distortion τ on each tangent space give rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^s} dx^s$ and S-curvature $\mathbf{S} = \tau_{|t} y^t$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|s} y^s$. Finsler metrics with $\mathbf{J} = 0$ are called weakly Landsberg metrics. The mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_eM_0$. It has been shown that on a weakly Landsberg manifold, the volume function $V = Vol(x)$ is a constant [3]. The constancy of the volume function is required to establish a Gauss-Bonnet theorem for Finsler manifolds [2]. In [7], Shen showed that if $\mathbf{J} = 0$, then all the slit tangent spaces T_eM_0 are minimal in TM_0. Some rigidity problems also lead to weakly Landsberg manifolds. For example, for a closed Finsler manifold with non-positive flag curvature, if the S-curvature is a constant, then it is weakly Landsbergian [8]. We remark that, S-curvature is constructed by Shen for the given comparison theorems on Finsler manifolds. Apparently, the S-curvature and mean Landsberg curvature deserve further investigation.

There is a relation between an m-th root metric and an (α, β)-metric. In [4], Matsumoto-Numata studied the class of cubic (α, β)-metrics and found a complete form of these Finsler metrics on a manifold of dimension $n \geq 3$. Inspired by their results, we characterize 5-th root (α, β)-metrics and investigate the explicit form of these metrics (Lemma 3.1). Then, we show that every weakly Landsberg 5-th root (α, β)-metric has vanishing S-curvature (Theorem 3.3). Using it, we prove that weakly Landsberg 5-th root (α, β)-metrics are Berwaldian.

Theorem 1.1. *Let $F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5}$ be a 5-th root (α, β)-metric on a manifold M. Then, F is a weakly Landsberg metric if and only if it is a Berwald metric.*

A Finsler metric F on a manifold M is called relatively isotropic mean Landsberg metric if $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. From Theorem 1.1, we obtain the following.

Corollary 1.2. *Every 5-th root (α, β)-metric has relatively isotropic mean Landsberg curvature if and only if it is a Berwald metric.*

The Ξ-curvature $\mathbf{\Xi} = \Xi_j dx^j$ on the tangent bundle TM is defined by $\Xi_j := \mathbf{S}_{.j|m} y^m - \mathbf{S}_{|j}$, where "." and "|" denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F, respectively [9]. It is obvious that $\mathbf{S} = 0$ implies $\Xi = 0$. We show that for quintic (α, β)-metrics, the converse is true.

Theorem 1.3. *Let $F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5}$ be a 5-th root (α, β)-metric on a manifold M. Then $\Xi = 0$ if and only if $\mathbf{S} = 0$.*

2. **Preliminaries**

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{e\in M} T_eM$ be the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_eM is called the fundamental tensor

$$\mathbf{g}_y(v, u) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}\Big[F^2(y + tv + su)\Big]|_{t=s=0}, \quad u, v \in T_eM.$$

Let $e \in M$ and $F := F|_{T_eM}$. To measure the non-Euclidean feature of F_e, one can define $\mathbf{C}_y : T_eM \times T_eM \times T_eM \longrightarrow \mathbb{R}$ by

$$\mathbf{C}_y(w, v, u) := \frac{1}{2}\frac{d}{ds}[\mathbf{g}_{y+su}(w, v)]_{s=0} = \frac{1}{4}\frac{\partial^3}{\partial r \partial t \partial s}[F^2(y + rw + tv + su)]_{r=t=s=0},$$

where $w, v, u \in T_eM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_eM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion.

For $y \in TM_0$, define $\mathbf{I}_y : T_eM \longrightarrow \mathbb{R}$ by

$$\mathbf{I}_y(w) = \sum_{i=1}^{n} g^{mt}(y)\mathbf{C}_y(w, \partial_m, \partial_t),$$

where $g^{mt} = (g_{mt})^{-1}$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in T_eM_0}$ is called the mean Cartan torsion.

For a Finsler manifold (M, F) of dimension n, F induced spray $\mathbf{G}$ on $TM_0 := TM - \{0\}$, in local coordinates in TM_0,it is given by

$$\mathbf{G} = y^t\frac{\partial}{\partial x^t} - 2G^t\frac{\partial}{\partial y^t},$$

where $G^i = G^i(x, y)$ are local functions on TM_0 expressed by

$$G^i := \frac{1}{4}g^{is}\Big\{\frac{\partial^2[F^2]}{\partial x^t \partial y^s}y^t - \frac{\partial[F^2]}{\partial x^s}\Big\}, \qquad y \in T_eM.$$

$\mathbf{G}$ is called the associated spray to (M, F).

For a Finsler manifold (M, F), the Busemann-Hausdorf volume form $dV_F = \sigma_F(x)dx^1...dx^n$ is defined as follows:

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\big\{(y^t) \in \mathbb{R}^n | F(y^t\frac{\partial}{\partial x^t}|_x) < 1\big\}}.$$

Then, for $y = y^m \partial/\partial x^m|_e \in T_eM$, the S-curvature is defined by

$$\mathbf{S}(y) := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\Big[ln\sigma_F(x)\Big]. \tag{2.1}$$

The S-curvature has been introduced by Shen for the formulation of a comparison theorem on Finsler manifolds .

Let (M, F) be an n-dimensional Finsler manifold. The non-Riemannian quantity Ξ-curvature $\mathbf{\Xi} = \Xi_j dx^j$ on the tangent bundle TM is defined by

$$\Xi_j := \mathbf{S}_{.j|m}y^m - \mathbf{S}_{|j},$$

where "." and "|" denote the vertical and horizontal covariant derivatives with respect to the Berwald connection of F, respectively. F is said to be of almost vanishing Ξ-curvature if

$$\Xi_j = -(n+1)F^2\left(\frac{\theta}{F}\right)_{y^j},$$

where $\theta := t_s(x)y^s$ is a 1-form on M

For a non-zero vector $y \in T_eM$, define $\mathbf{B}_y : T_eM \times T_eM \times T_eM \to T_eM$ by $\mathbf{B}_y(v, u, w) = B^m_{ijl}v^iu^jw^l\frac{\partial}{\partial x^m}|_l$, where

$$B^m_{ijl} := \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature, and F represents a Berwald metric if $\mathbf{B} = 0$.

The mean of Berwald curvature is defined by $\mathbf{E}_y : T_eM \times T_eM \to \mathbb{R}$, were

$$\mathbf{E}_y(v, w) = \sum_{i=1}^{n} g^{ij}(y)g_y\mathbf{B}_y(v, w, e_i, e_j).$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y\in T_eM_0}$ is called the mean Berwald curvature or E-curvature. In local coordinates, $\mathbf{E}_y(u, v) := E_{sl}(x, y)v^su^l$, were

$$E_{sl} = \frac{1}{2}\mathbf{S}_{y^sy^l}(x, y) = \frac{1}{2}B^m_{\ ijm},$$

If $\mathbf{E} = 0$, then F is a weakly Berwald metric. By (**??**), one can get the following equation

$$\mathbf{S}_{y^sy^l} = [G^m]_{y^sy^ly^m} = E_{sl}.$$

Thus $\mathbf{S} = 0$ implies that $\mathbf{E} = 0$.

To measure the changes of the Cartan torsion $\mathbf{C}$ along geodesics, we define $\mathbf{L}_y : T_eM \otimes T_eM \otimes T_eM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := \frac{d}{ds}\Big[\mathbf{C}_{\dot{c}(s)}(U(s), V(s), W(s))\Big]\Big|_{s=0},$$

where $c(s)$ is a geodesic and $U(s), V(s), W(s)$ are parallel vector fields along $c(s)$ with $\dot{c}(0) = y, U(0) = u, V(0) = v, W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM\setminus\{0\}}$

is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_eM$ define $\mathbf{J}_y : T_eM \longrightarrow \mathbb{R}$ by $\mathbf{J}_y(u) := J_t(y)u^t$, where $J_t := I_{t|s}y^s$. $\mathbf{J}$ is called the mean Landsberg curvature or $\mathbf{J}$-curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$.

3. **Proof of the Theorem 1.1**

In this section, we are going to prove the Theorem 1.1. In order to prove it, we need to note some necessary facts. In [5], Matsumoto-Numata studied the class of cubic metrics and found the explicit form of a cubic (α, β)-metric. Here, we prove the following results.

Lemma 3.1. *Let $F = \sqrt[5]{A}$ be a 5-th Finsler metric on a manifold M. Then, we have:*

- **(1):** *Let $dim(M) = 2$. In this case, by choosing a suitable quadratic form $\alpha = \sqrt{a_{jt}(x)y^jy^t}$ and one form $\beta = b_j(x)y^j$, F is always written in the form*
$$F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3},$$
where c_1 and c_2 are real constants and α^2 may be degenerate.
- **(2):** *If $dim(M) \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{\alpha_{jt}(x)y^jy^t}$ and a 1-form $\beta = \beta_j(x)y^j$, then it is written in the following form*
$$F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5},$$
where c_1, c_2 and c_3 are real constants.

Proof. By the same argument used by Matsumoto-Numata to obtain the explicit form of a cubic (α, β)-metric in [5], we get the proof. □

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{jt}(x)y^jy^t}$ is a Riemannian metric and $\beta = b_j(x)y^j$ is a 1-form over the manifold M. For an (α, β)-metric, let us define $b_{j;k}$ by $b_{j;k}\theta^k := db_j - b_k\theta_j^k$, where $\theta^j := dx^j$ and $\theta_j^k := \Gamma^k_{js}dx^s$ denote the Levi-Civita connection form of α. Let

$$r_{it} := \frac{1}{2}(b_{i;t} + b_{t;i}), \qquad s_{it} := \frac{1}{2}(b_{i;t} - b_{t;i}), \qquad r_{i0} := r_{it}y^t,$$
$$r_{00} := r_{it}y^iy^t, \qquad r_t := b^ir_{it}, \qquad s_{i0} := s_{it}y^t, \qquad s_t := b^is_{it},$$
$$s^i{}_t = a^{is}s_{st}, \qquad s^i{}_0 = s^i{}_ty^t, \qquad r_0 := r_ty^t, \qquad s_0 := s_ty^t.$$

where $a^{it} = (a_{it})^{-1}$ and $b^i := a^{it}b_t$. Put

$$\begin{aligned} Q :=& \frac{\phi'}{\phi - s\phi'}, \\ \Theta :=& \frac{\phi\phi' - s(\phi'\phi' + \phi\phi'')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi''}, \\ \Psi :=& \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \end{aligned} \tag{3.1}$$

where $B := ||\beta||^2_\alpha$. Let $G^t = G^t(x, y)$ and $G^t_\alpha = G^t_\alpha(x, y)$ denote the coefficients of F and α, respectively, in the same coordinate system. By definition, we have

$$G^t = G^t_\alpha + \alpha Q s^t{}_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^t + \Psi b^t). \tag{3.2}$$

where

$$P := \Big[- 2Q\alpha s_0 + r_{00} \Big]\Theta\alpha^{-1}, \quad Q^t := \Psi\Big[r_{00} - 2\alpha Q s_0\Big]b^t + \alpha Q s^t{}_0.$$

Clearly, if β is parallel with respect to α, that is $r_{ij} = 0$ and $s_{ij} = 0$, then $P = 0$ and $Q^i = 0$. In this case, $G^i = G^i_\alpha$ are quadratic in y. In this case, F is a Berwald metric. Put

$$\Phi : = (sQ' - Q)\{n\Delta + sQ + 1\} - (B - s^2)(sQ + 1)Q''.$$

By direct computation, we can obtain a formula for the mean Cartan torsion of (α, β)- metrics as follows

$$I_j = -\frac{(\phi - s\phi')\Phi}{2\Delta\phi\alpha^2}(\alpha b_j - s y_j). \tag{3.3}$$

Thus $\mathbf{I} = 0$ if and only if $\Phi = 0$.

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Then the S-curvature of F is given by

$$\mathbf{S} = \Big[2\Psi - \frac{f'(b)}{bf(b)}\Big](s_0 + r_0) - \frac{\Phi}{2\Delta^2\alpha}(r_{00} - 2Q\alpha s_0),$$

where

$$\begin{aligned} f(b) : =& \frac{\int_0^\pi \sin^{n-2} t\ T(b\cos t)dt}{\int_0^\pi \sin^{n-2} t dt}, \\ T(s) : =& \phi(\phi - s\phi')^{n-2}\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]. \end{aligned}$$

Here, we calculate the S-curvature of 5-th root (α, β)-metric and obtain the following.

Lemma 3.2. *The S-curvature of* 5*-th root* (α,β)*-metric is given by*

$$\begin{aligned}
\mathbf{S} \;=\;& \frac{1}{2s^2\varphi\mu}\Big\{3c_2^2s^2+13c_2c_3s^4+10c_3^2s^6+3c_1c_2+10c_1c_3s^2-\frac{f'(b)}{bf(b)}\Big\}\big(s_0\\
&+r_0\big)-\frac{1}{4\alpha\varphi^2\mu^2s^3}\Big\{8c_2c_3^3s^{12}-3c_2^4b^2s^4-20c_3^4b^2s^{12}-60nc_1c_2^2c_3s^6\\
&-112nc_2c_1c_3^2s^8+36nc_1c_2^4b^2s^6+120nc_1c_2c_3^3b^2s^{12}+640nc_1c_3^4b^2s^{14}\\
&+348nc_2^4c_3b^2s^{10}+120nc_2^3c_3^2b^2s^{12}+2456nb^2c_2^2c_3^3s^{14}+2240nb^2c_2c_3^4s^{16}\\
&-56nc_1^2c_2^2c_3s^8-128nc_1^2c_2c_3^2s^{10}-304nc_1c_2^3c_3s^{10}-94nc_1c_2^2c_3^2s^{12}\\
&-120nc_1c_2c_3^3s^{14}-6c_2^4s^6-20nc_1^2c_2c_3s^4-58c_1c_3b^2c_2^2s^4-136c_1c_3^2c_2s^6\\
&-42nc_2^3c_3s^8-104nc_2^2c_3^2s^{10}-108nc_2c_3^3s^{12}-64nc_3^3c_1s^{10}-34b^2c_1^2c_2c_3s^2\\
&+24c_1^2c_2c_3s^4+800nc_3^5b^2s^{18}-24nc_1^2c_3^2s^6+36nb^2c_2^5s^8+4c_1c_2^3s^4\\
&-800nc_3^5s^{20}+4c_1^2c_2^2s^2-8nc_1^2c_2^3s^6-10nc_2^3c_1s^4-2456nc_2^2c_3^3s^{16}\\
&-36nc_1c_2^4s^8-96nc_1^2c_3^3s^{12}-640nc_1c_3^4s^{16}-348nc_2^4c_3s^{12}-62c_2c_3^3b^2s^{10}\\
&-2240nc_2c_3^4s^{18}+28c_1c_2^2c_3s^6+80c_1c_2c_3^2s^8+8c_2^2c_3^2s^{10}-9c_1c_2^3b^2s^2\\
&-21c_2^3c_3b^2s^6-60b^2c_2^2c_3^2s^8-80c_1c_3^3b^2s^8-60c_1^2c_3^2b^2s^4-40c_3^4s^{14}\\
&-130nc_2^3c_3^2s^{14}-36c_2^5s^{10}+48c_1c_3^3s^{10}+48c_1^2c_3^2s^6+56b^2c_1^2c_2^2c_3s^6\\
&+128nc_1^2c_2c_3^2b^2s^8-4nc_1^2c_2^2s^2+8nb^2c_1^2c_2^3s^4+944nb^2c_1c_2^2c_3^2s^{10}\\
&+304nb^2c_1c_2^3c_3s^8+96nc_1^2c_3^3b^2s^{10}\Big\}\Big(r_{00}+(c_1+3c_2s^2+5c_3s^4)s_0\Big),
\end{aligned}$$

where

$$\begin{aligned}
\varphi :=& -c_2s^2-c_3s^4+2c_1c_2b^2s^2+4c_1c_3b^2s^4+6c_2^2b^2s^4+22c_2c_3b^2s^6+20c_3^2b^2s^8\\
&-2c_1c_2s^4-4c_1c_3s^6-6c_2^2s^6-22c_2c_3s^8-20c_3^2s^{10},\\
\mu :=& \; c_2+2c_3s^2,\\
T :=& \;(c_1s+c_2s^3+c_3s^5)\big(c_1s+c_2s^3+c_3s^5-s(c_1+3c_2s^2+5c_3s^4)\big)^{n-2}\Big\{c_1s\\
&+c_2s^3+c_3s^5-s(c_1+3c_2s^2+5c_3s^4)+(b^2-s^2)(6c_2s+20c_3s^3)\Big\}.
\end{aligned}$$

Now, we study weakly Landsberg 5-th root (α,β)-metrics and prove the following.

Theorem 3.3. *Every weakly Landsberg* 5*-th root* (α,β)*-metric has vanishing* S*-curvature.*

Proof. For an (α,β)-metric $F=\alpha\phi(s)$, the mean Landsberg curvature is given by

$$J_t = -\frac{1}{2\Delta\alpha^4}\Bigg[\frac{2\alpha^2}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](r_0+s_0)h_t + \frac{\alpha}{b^2-s^2}(\Psi_1+s\frac{\Phi}{\Delta})(r_{00}-2\alpha Qs_0)h_t+\alpha\Big[-\alpha Q's_0h_t+\alpha Q(\alpha^2 s_t-y_ts_0) + \alpha^2\Delta s_{i0}+\alpha^2(r_{i0}-2\alpha Qs_t)-(r_{00}-2\alpha Qs_0)y_t\Big]\frac{\Phi}{\Delta}\Bigg], \tag{3.4}$$

where

$$\Psi_1 := \sqrt{b^2-s^2}\Big[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]'\Delta^{\frac{1}{2}},\quad h_t:=b_t-\alpha^{-1}sy_t,\quad B=b^2.$$

contracting (3.4) with b^t and simplifying it, we have $\mathbf{J}=b^tJ_t=0$. It is equal to following

$$d_7\alpha^7+d_5\alpha^5+d_4\alpha^4+d_3\alpha^3+d_2\alpha^2+d_1\alpha+d_0=0, \tag{3.5}$$

where

$$\begin{aligned}
d_0 :=& \Big(-7\beta^6b^2c_1^3c_2^4+33\beta^6b^4c_1^2c_2^5+4\beta^6nc_1^4c_2^3+52\beta^6b^4c_1^3c_2^3c_3+150\beta^6b^6c_1^3c_2^5\\
&-128\beta^6b^4c_1^4c_2c_3^2+78\beta^6c_1^4c_2^2c_3b^2-48\beta^6b^8c_1^4c_2^5-2\beta^6b^2nc_1^3c_2^4\\
&-96\beta^6b^4c_1^4c_2^4+32\beta^6b^4nc_1^4c_2^4+348\beta^6b^6c_1^4c_2^3c_3-12b^4\beta^5c_1^3c_2^4-4\beta^6c_1^4c_2^3\\
&-28\beta^6b^2nc_1^4c_2^2c_3\Big)\Big((2\beta^5c_3+\beta^3c_2\alpha^2)r_{00}-(\alpha^6c_1+3\alpha^4c_2\beta^2+5\alpha^2c_3\beta^4)s_0\Big),\\
d_1 :=& 8b^4c_1^2\beta^5\big(-15b^2c_2^2-56b^2c_1c_3+14c_1c_2+21b^4c_1c_2^2-2nc_1c_2\big)\Big((2\beta^5c_3\\
&+\beta^3c_2\alpha^2)r_{00}-(\alpha^6c_1+3\alpha^4c_2\beta^2+5\alpha^2c_3\beta^4)s_0\Big),\\
d_2 :=& \Big(1620\beta^6b^4c_1^3c_2^3c_3+1424\beta^6b^4c_1^4c_2c_3^2-386\beta^6c_1^4c_2^2c_3b^2+108\beta^6b^2nc_1^3c_2^4\\
&-140c_1^4c_2^3c_3\beta^6+112b^4\beta^5c_1^3c_2^4-135\beta^6b^2c_1^3c_2^4+198\beta^6b^4c_1^2c_2^5-12\beta^6nc_1^4c_2^3\\
&+180c_1^4c_2^4\beta^6+200\beta^6b^2nc_1^4c_2^2c_3+12\beta^6c_1^4c_2^3-45\beta^6b^6c_1c_2^6+342\beta^6b^8c_1^2c_2^6\\
&-120\beta^5b^6c_2^5c_1^2+168\beta^5b^8c_1^3c_2^5+496\beta^6b^6nc_1^4c_2^3c_3-272\beta^6b^4nc_1^4c_2c_3^2\\
&-48\beta^6b^8nc_1^4c_2^5-2352\beta^6b^6c_1^3c_2^2c_3^2-90\beta^6b^4nc_1^2c_2^5+1688\beta^6b^8c_1^4c_2^2c_3^2\\
&-144\beta^6b^4nc_1^4c_2^4-714\beta^6b^6c_1^3c_2^5-600\beta^6b^6c_1^4c_3^3-504\beta^6b^4nc_1^3c_2^3c_3\\
&+2436\beta^6b^8c_1^3c_2^4c_3+312nc_1^3c_2^5\beta^6-873\beta^6b^6c_1^2c_2^4c_3-448\beta^5b^6c_1^3c_2^3c_3\\
&-16\beta^5nc_1^3c_2^4b^4\Big)\Big(b^2s_0c_1^4c_2^3+3b^4r_0c_1^3c_2^4-4ns_0c_1^4c_2^3b^2-46b^4s_0c_1^4c_2^2c_3\\
&+27b^6s_0c_1^4c_2^4-9b^4s_0c_1^3c_2^4\Big)\beta^4,
\end{aligned}$$

$$
\begin{aligned}
d_3 :=\ & 396b^8\beta^4 s_0 c_1^4 c_2^3 c_3 - 132b^6\beta^4 s_0 c_1^4 c_2^4 + 186b^4\beta^4 s_0 c_1^3 c_2^4 - 624\beta^4 s_0 c_1^3 c_2^3 c_3 \\
& - 24b^4 r_0 \beta^4 c_1^3 c_2^4 - 42b^2\beta^4 s_0 c_1^4 c_2^3 - 100b^4 n\beta^4 s_0 c_1^4 c_2^2 c_3 + 48b^6 n\beta^4 s_0 c_1^4 c_2^4 \\
& - 520c_1^4 c_2 c_3^2 \beta^4 s_0 + 198c_1^3 c_2^5 \beta^4 s_0 + 436b^4\beta^4 s_0 c_1^4 c_2^2 c_3 + 24n\beta^4 s_0 c_1^4 c_2^3 b^2 \\
& - 48\beta^6 b^6 c_1^3 c_2^5 r_0 \alpha^2 - 48\beta^3 b^6 c_1^4 c_2^4 s_0 \alpha^6 - 144\beta^5 b^6 c_1^3 c_2^5 s_0 \alpha^4 \\
& - 240\alpha^2\beta^7 b^6 c_1^3 c_2^4 c_3 s_0 - 54b^4 n\beta^4 s_0 c_1^3 c_2^4 - 81b^6\beta^4 c_1^2 c_2^5 s_0 - 96\beta^8 b^6 c_1^3 c_2^4 r_0 c_3, \\
d_4 :=\ & -6b^4 c_1^4 c_2^3 \beta^2 s_0, \\
d_5 :=\ & 3b^4 c_1^3 c_2^2 \beta^2 \big(12b^4 c_1 c_2^2 - 4nc_1 c_2 - 21b^2 c_2^2 - 54b^2 c_1 c_3 + 16c_1 c_2\big) s_0,
\end{aligned}
$$

$$
d_7 : \quad = -18b^6 c_1^4 c_2^3 s_0. \tag{3.6}
$$

(3.5) implies that

$$
d_7\alpha^6 + d_5\alpha^4 + d_3\alpha^2 + d_1 = 0, \tag{3.7}
$$

$$
d_4\alpha^4 + d_2\alpha^2 + d_0 = 0. \tag{3.8}
$$

By (3.7), we find that there exists a non-zero function $\gamma = \gamma(x, y)$ such that

$$
r_{00} = \frac{c_1\alpha^6 + 3c_2\alpha^4\beta^2 + 5c_3\alpha^2\beta^4}{2c_3\beta^5 + c_2\beta^3\alpha^2} s_0 + \gamma\alpha^2. \tag{3.9}
$$

Similarly, (3.8) implies that there exists a non-zero function $\delta = \delta(x, y)$ such that

$$
r_{00} = \frac{c_1\alpha^6 + 3c_2\alpha^4\beta^2 + 5c_3\alpha^2\beta^4}{2c_3\beta^5 + c_2\beta^3\alpha^2} s_0 + \delta\alpha^2. \tag{3.10}
$$

Since $\gamma \neq \delta$ and also γ is not a multiple of δ, then by (3.9) and (3.10) we get

$$
r_{00} = \Big(\frac{c_1\alpha^6 + 3\alpha^4 c_2\beta^2 + 5\alpha^2 c_3\beta^4}{2\beta^5 c_3 + \beta^3 c_2\alpha^2}\Big) s_0. \tag{3.11}
$$

Taking a vertical derivation of (3.11) give us the following

$$
\begin{aligned}
r_{i0} = \ & \Big\{\tfrac{6c_1\alpha^4 y_i + 12c_2\alpha^2 y_i\beta^2 + 6c_2\alpha^4\beta b_i + 10c_3\beta^4 y_i + 20c_3\alpha^2\beta^3 b_i}{2\beta^5 c_3 + \beta^3 c_2\alpha^2} \\
& - \tfrac{(\alpha^6 c_1 + 3\alpha^4 c_2\beta^2 + 5\alpha^2 c_3\beta^4)(10c_3\beta^4 b_i + 3c_2\beta^2\alpha^2 b_i + 2c_2\beta^3 y_i)}{(2\beta^5 c_3 + \beta^3 c_2\alpha^2)^2}\Big\} s_0 \\
& + \Big(\tfrac{\alpha^6 c_1 + 3\alpha^4 c_2\beta^2 + 5\alpha^2 c_3\beta^4}{2\beta^5 c_3 + \beta^3 c_2\alpha^2}\Big) s_i.
\end{aligned} \tag{3.12}
$$

Contracting (3.12) with b^i yields

$$
\begin{aligned}
r_0 = \ & \Big\{\tfrac{6c_1\alpha^4\beta + 12c_2\alpha^2\beta^3 + 6c_2\alpha^4\beta b^2 + 10c_3\beta^5 + 20c_3\alpha^2\beta^3 b^2}{2\beta^5 c_3 + \beta^3 c_2\alpha^2} \\
& - \tfrac{(\alpha^6 c_1 + 3\alpha^4 c_2\beta^2 + 5\alpha^2 c_3\beta^4)(10c_3\beta^4 b^2 + 3c_2\beta^2\alpha^2 b^2 + 2c_2\beta^4)}{(2\beta^5 c_3 + \beta^3 c_2\alpha^2)^2}\Big\} s_0.
\end{aligned} \tag{3.13}
$$

By putting (3.11) and (3.13) into (3.6) and simplifying the result, we have

$$
\eta(x, y) s_0 = 0. \tag{3.14}
$$

where

$$\eta(x,y) := (15c_3^2b^4 + 5c_3c_1 - 66c_3c_2b^2 + 42c_2^2)\alpha^4 + (-15c_3^2\beta^2b^2 + 22c_3\beta^2c_2)\alpha^2 + 5c_3^2\beta^4.$$

By (3.14), it is obvious that $\eta = 0$ or $s_i = 0$. Let $\eta(x,y) = 0$. One can rewrite $\eta = 0$ as follows

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0, \tag{3.15}$$

where $\theta = \theta(x,y)$, $\gamma = \gamma(x,y)$ and $\varepsilon = \varepsilon(x,y)$ are functions on TM. (3.15) implies that

$$\alpha^2 = (\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\delta}}{2\theta})\beta^2. \tag{3.16}$$

This contradicts with the positive-definiteness of α. Thus $\eta \neq 0$ and $s_i = 0$. Putting it into (3.11) gives $r_{ij} = 0$. By putting these relations in (3.4), we obtain $\mathbf{S} = 0$. □

Proof of Theorem 1.1: In [6] Najafi-Tayebi showed that every weakly Landsberg (α,β)-metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$ is a Berwald metric. By Theorem 1.1, every weakly Landsberg 5-th root metric on M of dimension $n \geq 3$ is a Berwald metric. We consider the class 5-th (α,β)-metrics of dimension $n = 2$. We know that Every 2-dimensional Finsler manifold is C-reducible

$$C_{ijt} = \frac{1}{3}\Big\{h_{ij}I_t + h_{jt}I_i + h_{ti}I_j\Big\}. \tag{3.17}$$

Taking a horizontal derivation of (3.17) along Finslerian geodesic yields

$$L_{ijt} = \frac{1}{3}\Big\{h_{ij}J_t + h_{jt}J_i + h_{ti}J_j\Big\}. \tag{3.18}$$

By putting $\mathbf{J} = 0$ in (3.18) implies that $\mathbf{L} = 0$. On the other hand, the Berwald curvature Finsler manifold of dimensional $n = 2$ can be written as follows

$$B^i{}_{jkt} = -\frac{2}{F}L_{jkt}l^i + \frac{2}{3}\Big\{E_{jk}h^i_t + E_{kt}h^i_j + E_{tj}h^i_k\Big\}. \tag{3.19}$$

By Putting $\mathbf{L} = 0$ and $\mathbf{E} = 0$ in (3.19), we conclude that F is a Berwald metric. The proof is complete. □

Proof of Corollary 1.2: Let $F = \sqrt[5]{A}$ 5-th root metric on manifold M, where $A := a_{ijklm}(x)y^iy^jy^ky^ly^m$, with a_{ijklm} symmetric in all its indices. Put

$$A_j = \frac{\partial A}{\partial y^j}, \quad A_{jt} = \frac{\partial^2 A}{\partial y^j \partial y^t}, \quad A_{x^t} = \frac{\partial A}{\partial x^t}, \quad A_0 = A_{x^t}y^t, \quad A_{0j} = A_{x^ty^j}y^t.$$

Assuming that (A^{jt}) is the inverse of the definite positive tensor (A_{jt}). In this case we have

$$g_{jt} = \frac{1}{25}A^{-\frac{8}{5}}\mathbb{A}, \quad g^{jt} = A^{-\frac{2}{5}}\mathbb{A}^{jt}, \quad y_i = \frac{1}{5}A^{\frac{-3}{5}}A_i,$$

where

$$\mathbb{A}_{jt} := 5AA_{jt} - 3A_jA_t$$

and

$$\mathbb{A}^{jt} := 5AA^{jt} + \frac{3}{4}y^jy^t.$$

The Cartan tensor of F is given by

$$C_{ijs} = \frac{1}{5}A^{-\frac{12}{3}}\mathbb{C}_{ijs}, \tag{3.20}$$

where

$$\mathbb{C}_{ijs} := A^2A_{ijs} + \frac{24}{25}A_iA_jA_s - \frac{3}{5}A\{A_iA_{js} + A_jA_{si} + A_sA_{ij}\}.$$

Thus the mean Cartan torsion is as follows

$$I_s = g^{ij}C_{ijs} = \frac{1}{5}A^{-3}\mathbb{A}^{ij}\mathbb{C}_{ijs}. \tag{3.21}$$

In [14], Yu and You found that the spray coefficients of F are given by

$$G^i = \frac{1}{2}(A_{0s} - A_{x^s})A^{is}. \tag{3.22}$$

It is easy to see that G^i are rational functions in y. Since

$$L_{ijs} = \frac{1}{2}y_tG^t{}_{y^iy^jy^s},$$

then we have

$$L_{ijs} = -\frac{1}{10}A^{-\frac{3}{5}}A_tG^t{}_{y^iy^jy^s}.$$

Therefore, we have

$$J_s = g^{ij}L_{ijs} = -\frac{1}{10}A^{-1}\mathbb{A}^{ij}A_tG^t{}_{y^iy^jy^s}. \tag{3.23}$$

Since F has relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, then by (3.23), (3.21) and $F = \sqrt[5]{A}$, we have

$$A^2A_tG^t{}_{y^iy^jy^s} = -2c\sqrt[5]{A}\mathbb{C}_{ijs}, \tag{3.24}$$

The left hand side of (3.24) is a rational function in y, while the other side is an irrational function in y. So $c = 0$ and F reduces to a weakly Landsberg metric. By Theorem 1.1, we get the proof. □

4. **Proof of Theorem 1.3**

In these section, we will prove a generalized version of Theorem 1.3. Indeed we study 5-th root (α,β)-metrics with almost vanishing Ξ-curvature. More precisely, we prove the following.

Theorem 4.1. *Let* $F=\sqrt[5]{c_1\alpha^4\beta+c_2\alpha^2\beta^3+c_3\beta^5}$ *be a 5-th root* (α,β)*-metric on a manifold* M*. Then* F *has almost vanishing* Ξ*-curvature if and only if* $\mathbf{S}=0$*.*

for proving Theorem 4.1, we calculate the Ξ-curvature of 5-th root (α,β)-metrics. For any (α,β)-metric, the Ξ-curvature is given by

$$\Xi_j := H_{j;t}y^t - H_{;j} - 2H_{.j.t}H^t, \tag{4.1}$$

where ";" denotes the horizontal covariant derivative with respect to α. By calculating the right side of the (4.1) and gaining the following

$$H_{;j} := \quad c_1\tfrac{r_{00;j}}{\alpha} + c_2\tfrac{r_{j0}-s_{j0}}{\alpha} + c_3 s_{0;j} + 2c_4(r_j+s_j) + 2\Psi r_{0;j},$$

where

$$\begin{aligned}
&A := r_{00} - 2\alpha Q s_0,\\
&c_1 := (n+1)(\Psi'+\Theta),\\
&c_2 := \Big\{(n+1)\Theta' + (B-s^2)\Psi'' - 2s\Psi'\Big\}\frac{A}{\alpha} + 2\Psi' r_0\Big\{2(sQ'-Q)\Psi\\
&\quad + - Q'' - sQ - 2(n+1)Q'\Theta - (Q' + 2Q'\Psi' + 2(B-s^2)\Psi Q'')\Big\}s_0,\\
&c_3 := Q' - 2s\Psi Q - 2(n+1)Q\Theta - 2(B-s^2)(Q'\Psi+\Psi'Q),\\
&c_4 := \Psi'\frac{A}{\alpha} - 2Q'\Psi s_0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
H_{.j;t}y^t := \quad & p_{5j}\tfrac{r_{00}}{\alpha} + p_6 s_{j;0} + 2p_{7j}(r_0+s_0) + 2\Psi r_{j;0}\\
&+\Lambda\Big((r_{j0}+s_{j0})\tfrac{1}{\alpha} - \tfrac{r_{00}y_j}{\alpha^3}\Big) + \Lambda_{;t}y^t(\alpha b_j - s y_j)\tfrac{1}{\alpha^2},
\end{aligned}$$

where

$$\begin{aligned}
p_{5j} :=& 2\Psi' r_j + \Big\{Q'' - \Big((B-s^2)Q'' + Q - sQ'\Big)\Big\}s_j\\
&+\Big\{(n+1)\Theta' - 2\Psi' s + \Psi''(B-s^2)\Big\}\Big(\frac{2r_{j0}}{\alpha} - r_{00}\frac{y_j}{\alpha^3} - 2Qs_j\Big),\\
p_6 :=& Q' - (B-s^2)Q' - sQ,\\
p_{7j} :=& \Big(\frac{2r_{j0}}{\alpha} - r_{00}\frac{y_j}{\alpha^3} - 2Qs_j\Big)\Psi' + Q's_j,
\end{aligned}$$

$$\Lambda := \frac{A}{\alpha}\big\{2\Psi' r_0 + (n+1)\Theta' + \Psi''(B-s^2) - 2\Psi' s\big\} + \Big\{Q'' - 2Q\Psi + 2(Q'\Psi - Q\Psi')s - 2(n+1)Q'\Theta - 2\Big(2\Psi'Q' - \Psi Q''\Big)(B-s^2)\Big\}s_0,$$

$$\begin{aligned}
\Lambda_{;t}y^t := & p_{11}r_{00;0} + \big(\frac{1}{\alpha}p_{12} - 2p_{11}Q's_0\big)r_{00} + \big(Q'' - 2(n+1)Q'\Theta - 2\alpha Q p_{11}\big)s_{0;0} \\
& + 2\Psi''\frac{A}{\alpha}(r_0+s_0) + 2\Psi' r_{0;0} - 2s\frac{\Psi'}{\alpha}\big(r_{00;0} - 2Q's_0r_{00} - 2\alpha Q s_{0;0}\big) \\
& + p_{21}\frac{r_{00}}{\alpha} + p_{22}s_{0;0} + 2p_{23}(r_0+s_0) + p_{31}\frac{r_{00}}{\alpha} + p_{32}s_{0;0} - 4Q''\Psi(r_0+s_0),
\end{aligned}$$

$$p_{11} := \frac{1}{\alpha}\big\{(n+1)\Theta' + (B-s^2)\Psi''\big\},$$

$$p_{12} := \frac{A}{\alpha}\big\{(n+1)\Theta'' - 2s\Psi'' + (B-s^2)\Psi'''\big\} - \big\{2(n+1)(\Theta'Q' + Q''\Theta) - Q'''\big\}s_0,$$

$$p_{21} := 2\Psi'' r_0 - 2\big\{3\Psi'Q's + (\Psi' + \Psi''s)Q\big\}s_0 - 2\frac{A}{\alpha}(\Psi' + \Psi''s),$$

$$p_{22} := -2\Psi'Qs - 4(B-s^2)\Psi'Q',$$

$$p_{23} := -4Q'\Psi's_0,$$

$$p_{31} := \big\{2(Q'\Psi' - Q\Psi' + 3Q''\Psi)s - 2(B-s^2)(Q''\Psi' + Q'''\Psi)\big\}s_0,$$

$$p_{32} := 2\big\{sQ' - Q - (B-s^2)Q''\big\}\Psi,$$

instead of $H_{.j.t}H^t$ in (4.1) is given by

$$\begin{aligned}
H_{.j.t}H^t = \; & Q\Big\{c_{1j}s_0 + c_{2j}\alpha + c_{3j}\alpha^2 + \Lambda\Big(-\tfrac{s_0}{\alpha^2}y_j - \tfrac{s s_{j0}}{\alpha}\Big) + \Lambda_{.m}s^m{}_0\Big(b_j - \tfrac{s y_j}{\alpha}\Big)\Big\} \\
& + A\Psi\Big\{\tfrac{c_{1j}(B-s^2)}{\alpha} + \Lambda\Big[(3\tfrac{s^2}{\alpha^3} - \tfrac{B}{\alpha^3})y_j - 2\tfrac{s}{\alpha^2}b_j\Big] + \Lambda_{.m}b^m\big(\tfrac{b_j}{\alpha} - \tfrac{s y_j}{\alpha^2}\big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_{.t}s^t{}_0 := & 2(\alpha p_{11} - 2\Psi's)\Big(\frac{q_{00}}{\alpha} - Qt_0 - \frac{Q'}{\alpha}s_0^2\Big) \\
& + \frac{1}{\alpha}(p_{12} + p_{21} + p_{31})s_0 + (p_{41} + p_{22} + p_{32})t_0 + 2\Psi'q_0, \\
\Lambda_{.t}b^t := & (\alpha p_{11} - 2\Psi's)\Big(2\frac{r_0}{\alpha} - \frac{s r_{00}}{\alpha^2} - 2(B-s^2)Q'\frac{s_0}{\alpha}\Big) \\
& + \frac{(B-s^2)}{\alpha}(p_{12} + p_{21} + p_{31}) + 2\Psi' r,
\end{aligned}$$

$$p_{41} := Q'' - 2(n+1)Q'\Theta,$$

$$\begin{aligned}
c_{1j} := & 2\Psi' r_j + \Big[(n+1)\Theta' + (B-s^2)\Psi'' - 2s\Psi'\Big]\Big(2\frac{r_{j0}}{\alpha} - \frac{r_{00}y_j}{\alpha^3} - 2Qs_j\Big) \\
& - \Big\{2Q'\Big((n+1)\Theta + (B-s^2)\Psi'\Big) - Q'' + \Big(Q + sQ' + Q''(B-s^2) - 2Q's\Big)\Big\}s_j,
\end{aligned}$$

$$c_{2j} := \Big[(B-s^2)\Psi' + (n+1)\Theta\Big]\Big(2\frac{q_{j0}}{\alpha} - 2\frac{q_{00}y_j}{\alpha^3} - \frac{r_{00}y_j}{\alpha^3}s_{j0}\Big),$$

$$c_{3j} := \Big[(B-s^2)\Psi' + (n+1)\Theta\Big]\Big(2\frac{r_j}{\alpha} - 2\frac{r_{j0}}{\alpha^2}s + 3\frac{r_{00}s y_j}{\alpha^4} - 2\frac{r_0 y_j}{\alpha^3} - \frac{r_{00}b_j}{\alpha^3}\Big).$$

Lemma 4.2. *Let $F = F(x, y)$ be a 5-th root (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is of almost vanishing Ξ-curvature. Then F has vanishing Ξ-curvature.*

Proof. Let $F = \sqrt[5]{a_{ijklm}y^iy^jy^ky^ly^m}$ be a 5-th root metric with almost vanishing Ξ-curvature on an n-dimensional manifold M. Then its Ξ-curvature can be expressed as

$$\Xi_j = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^j}. \tag{4.2}$$

where $\theta = t_j(x)y^j$ is a 1-form on M. By Lemma 2.1 in [14], the spray coefficients of an m-th root metric is rational function in y. By definition, the S-curvature and then the Ξ-curvature of F are rational functions in y. It follows that the left side of (4.2) is a rational function in y. while the right side of (4.2) is an irrational function in y. Thus we get $\Xi = 0$. □

Proof of Theorem 4.1: Let $F = \sqrt[5]{c_1\alpha^4\beta + c_2\alpha^2\beta^3 + c_3\beta^5}$ be a 5-th root (α, β)-metric on a manifold M. Suppose that F has almost vanishing Ξ-curvature. Then, by Lemma 4.2, we have $\Xi_j = 0$. Let us define $\Xi := \Xi_j b^j$. So multiplying (4.1) with b^j yields

$$f_7\alpha^7 + f_6\alpha^6 + f_5\alpha^5 + f_4\alpha^4 + f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0 = 0. \tag{4.3}$$

where

$$
\begin{aligned}
f_0 &:= -12960c_3^3\beta^6r_{00}^2,\\
f_1 &:= -5400c_3^3\beta^6r_{00}^2,\\
f_2 &:= 1290c_3^3\beta^5r_{00;0} - 8568c_2c_3^2\beta^4r_{00}^2 + 1260c_3^3\beta^5r_{00;0} + 5610c_3^3\beta^4r_{00}^2\\
&\quad - 1620\beta^5c_3^3r_{00}s_0 + 6370c_3^3\beta^4r_{00}^2 + 100(n+1)c_3^3\beta^4r_{00}^2,\\
f_3 &:= 38880c_3^3\beta^5s_0^2 - 100c_3^3\beta^4r_{00}^2 - 270c_3^3\beta^5r_{00}s_0 + 120c_3^3\beta^5r_{00} + 4200c_3^3\beta^4r_{00}^2\\
&\quad + 200c_3^3B\beta^4r_{00}^2 + 38880c_3^3\beta^5r_0s_0 + 15552c_3^3\beta^5r_{00} - 35640c_2c_3^2\beta^4r_{00}^2\\
f_4 &: = -9020\beta^4c_3^3s_{0;0} + 40500\beta^4c_3^3s_0^2 + 1080\beta^3c_3^3{r_{00}}^2 + 31104c_3^3\beta^5s_0 + 31104\beta^5c_3^3r_0\\
&\quad + 64800(n+1)\beta^4c_3^3s_{0;0} + 12960\beta^4c_3^3r_{00} + 2052\beta^3c_3^3r_{00}s_0\\
&\quad + 1420c_3^3\beta^3r_0r_{00} + 4100\beta^3c_3^3Br_{00;0} - 1080(n+1)c_3^3\beta^3r_{00}^2 - 808c_3^2\beta^3r_{00;0}c_2\\
&\quad - 970\beta^2c_3^3B^2r_{00}^2 - 380\beta^2c_3^2r_{00}^2c_1 + 4184\beta^2c_3^2r_{00}^2c_2 - 254\beta^2c_3r_{00}^2c_2^2\\
&\quad + 5400(n+1)c_3^3\beta^3r_{00}s_0 - 34560(n+1)c_3^3B\beta^3r_{00;0} - 47520c_3^3\beta^3r_0r_{00}+\\
&\quad - 1080c_3^2\beta^3r_{00}s_0c_2 - 55980c_3^3B\beta^2r_{00}^2 + 97452(n+1)c_3^2\beta^2r_{00}^2c_2\\
&\quad + 371088c_3^2B\beta^2r_{00}^2c_2 - 130c_3^3\beta^2Br_{00}^2 + 700\beta^3c_3^3Br_{00}s_0 81216\beta^3c_3^2r_{00;0}c_2,\\
f_5 &= 3600\beta^2c_3^3Br_{00}^2 - 266976\beta^3c_3^2s_0^2c_2 + 858\beta^3c_3^2r_{00}c_2 + 120\beta^3c_3^3Br_{00}s_0\\
&\quad - 560c_3^3\beta^3Br_{00} + 324\beta^3c_3^3r_0r_{00} + 266976\beta^3c_3^2r_0s_0c_2 + 150\beta^2c_3^2Br_{00}^2c_2
\end{aligned}
$$

$$- 430c_3^3\beta^2B^2r_{00}^2 + 1030\beta^3c_3^2r_{00}c_2 - 330\beta^4c_3^3s_0^2 - 640\beta^3c_3^3Br_{00}$$
$$+ 240c_3^2\beta^2r_{00}^2c_2 - 18\beta^3c_3^3Br_0s_0 - 180900\beta^3c_3^2r_{00}s_0c_2 + 181440\beta^3c_3^3Bs_0^2$$
$$+ 23400c_3^3\beta^3r_{00}s_0 - 9200\beta^2c_3^3Br_{00}^2 - 162\beta^2c_3^2{r_{00}}^2c_1 - 105426\beta^2c_3{r_{00}}^2c_2^2$$
$$- 63\beta^2c_3^2{r_{00}}^2c_2 - 9450\beta^3c_3^3r_{00}s_0,$$

$$f_6 = 4516r_{00}^2c_2^3 + 840c_3^3B^3r_{00}^2 + 640\beta^3c_3^3q_0 - 639720c_3^2B^2r_{00}^2c_2 + 120\beta^2c_3^3r_{00}$$
$$+ 20970c_3^3B^2r_{00}^2 + 191160c_3^2r_{00}^2c_1 + 1288c_3r_{00}^2c_2^2 - 45360\beta^3c_3^3s_0 - 220\beta^3c_3^3r_0$$
$$+ 113400\beta^3c_3^3t_0 - 92610\beta^2c_3^3s_0^2 + 51840\beta^2c_3^3r_0^+106350(n+1)c_3^3B^2r_{00}^2$$
$$+ 168480c_3^2Br_{00}^2c_1 - 879912c_3^2Br_{00}^2c_2 + 11720(n+1)c_3^2r_{00}^2c_1 + 279c_3r_{00}^2c_2^2$$
$$- 219024c_3r_{00}^2c_1c_2 + 1099152c_3Br_{00}^2c_2^2 - 32400(n+1)\beta^3c_3^3q_0 - 81\beta^3c_3^3t_0$$
$$+ 32400(n+1)\beta^3c_3^3s_0 + 420\beta^2c_3^3r_0s_0 - 175500\beta^2c_3^3Bs_0^2 + 100\beta^2c_3^3Bq_{00}$$
$$+ 420\beta^2c_3^3rr_{00} + 210\beta^2c_3^3r_0r_{00} + 480\beta^2c_3^3r_00s_0 + 339120\beta^2c_3^3Bs_{0;0}$$
$$+ 607392\beta^2c_3^2s_{0;0}c_2 + 69480c_2c_3^2\beta r_{00}^2 - 120\beta^3c_3^3Br_0 + 42768\beta^2c_3^2s_{0;0}c_2$$
$$- 4320\beta^2c_3^2q_{00}c_2 + 276750\beta^2c_3^2s_0^2c_2 - 14400\beta c_3^3Br_{00}^2 - 37080\beta c_3^3B^2r_{00;0}$$
$$- 380\beta c_3^2r_{00;0}c_1 - 260\beta c_3r_{00;0}c_2^2 - 316(n+1)c_3^2Br_{00}^2c_2 - 129600\beta^3c_3^3Bs_0$$
$$+ 207360\beta^3c_3^2s_0c_2 + 20760\beta^3c_3^2r_0c_2 - 226c_3^3\beta^2Bs_{0;0} - 81\beta^2c_3^3r_{00}s_0$$
$$- 1510(n+1)\beta^2c_3^3r_0s_0 - 210\beta^2c_3^3B(n+1)q_{00} - 430\beta^2c_3^3Br_{00}$$
$$+ 8640(n+1)\beta^2c_3^2q_{00}c_2 + 84672\beta^2c_3^2(n+1)r_{00}c_2 - 121500\beta c_3^3B^2r_{00}s_0$$
$$- 628920\beta c_3^3Br_{00}s_0 + 9000\beta c_3^3Br_{00}^2 - 340\beta c_3^3Br_0r_{00} - 51840\beta c_3^2r_{00}s_0c_1$$
$$+ 1402848\beta c_3^2r_{00}s_0c_2 + 160\beta c_3^2r_{00;0}c_1 + 264\beta c_3^2Br_{00;0}c_2 - 620\beta c_3^2r_{00}^2c_2$$
$$- 3216\beta c_3r_{00}s_0c_2^2 + 103500\beta c_3^3Br_{00}s_0 + 820(n+1)\beta c_3^3Br_0$$
$$+ 15840(n+1)c_3^2\beta r_{00}s_0c_2 - 297072(n+1)\beta c_3^2r_0r_{00}c_2 - 2108c_3^2\beta Br_{00;0}c_2$$
$$+ 2160\beta c_3^3B^2r_{00;0} + 944\beta c_3^2r_{00}c_2 + 220\beta c_3r_{00;0}c_2^2 + 470\beta c_3^2Br_{00}s_0c_2,$$

$$f_7 := 184916c_2^3r_{00}^2 + 39125c_3^3B^3r_{00}^2 + 25800c_3^3B^2r_{00}^2 + 12960c_1c_3^2r_{00}^2$$
$$+ 317250\beta^2c_3^3s_0^2 + 477c_3^3B^2(n+1)r_{00}^2 - 2860c_3^2B^2r_{00}^2c_2 + 729c_3^2Br_{00}^2c_1$$
$$+ 324(n+1)c_3^2r_{00}^2c_1 - 9208c_3(n+1)r_{00}^2c_2^2 - 907c_3r_{00}^2c_1c_2 + 47c_3Br_{00}^2c_2^2$$
$$- 160(n+1)\beta^2c_3^3s_0^2 + 155\beta^2c_3^3Bs_0^2 - 220\beta^2c_3^2s_0^2c_2 + 90\beta c_3^3B^2r_{00}$$
$$+ 380\beta c_3^2r_{00}c_1 - 168\beta c_3^2s_0^2c_1 + 2744\beta c_3r_{00}c_2^2 - 828\beta c_3s_0^2c_2^2 - 90c_3^2Br_{00}^2c_2$$
$$- 2180\beta c_3^3B^2r_{00}s_0 + 351\beta c_3^3B^2r_0s_0 - 490\beta c_3^3Br_{00}s_0 + 108\beta c_3^3B^2r_{00}$$
$$+ 12432\beta c_3^2Bs_0^2c_2 + 46656\beta c_3^2r_{00}c_1 + 1368\beta c_3^2r_0s_0c_1 - 3788\beta c_3^2Br_{00}c_2$$
$$+ 220\beta c_3^2r_0r_{00}c_2 + 17120\beta c_3^2r_{00}s_0c_2 + 8248\beta c_3r_0s_0c_2^2 + 3176\beta c_3r_{00}c_2^2$$
$$+ 24950(n+1)\beta^2c_3^3s_0^2 + 1750(n+1)\beta c_3^3Br_{00}s_0 - 1242432\beta c_3^2Br_0s_0c_2$$
$$+ 811800\beta c_3^2Br_{00}s_0c_2 - 600300\beta c_3^2(n+1)r_{00}s_0c_2 - 4240\beta c_3^2Br_{00}c_2$$
$$+ 807864c_2^2c_3r_{00}^2 - 6260c_3^2Br_{00}^2c_2 + 81\beta^2c_3^3r_0s_0 - 35\beta c_3^3B^2s_0^2$$
$$- 80\beta c_3^2r_{00}s_0c_1 - 5430\beta c_3r_{00}s_0c_2^2 - 820\beta c_3^3Br_0r_{00},$$

By (4.3), we get

$$f_7\alpha^6 + f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \tag{4.4}$$
$$f_6\alpha^6 + f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \tag{4.5}$$

(4.4) implies that there exists a non-zero function $\mu = \mu(x, y)$ such that

$$\beta^6 c_3^3 {r_{00}}^2 = \mu\alpha^2. \tag{4.6}$$

Similarly, (4.5) implies that there exists a non-zero function $\nu = \nu(x, y)$ such that the following holds

$$\beta^6 c_3^3 {r_{00}}^2 = \nu\alpha^2. \tag{4.7}$$

Since $\mu \neq \nu$ and also μ is not a multiplication of ν, then by (4.6) and (4.7) we get

$$\beta^6 c_3^3 {r_{00}}^2 = 0. \tag{4.8}$$

Since $\beta^6 c_3^3 \neq 0$, then $r_{ij} = 0$ which implies that $r_i = 0$. Putting these relations in (4.3) yields

$$g_4\alpha^4 + g_3\alpha^3 + g_2\alpha^2 + g_1\alpha + g_0 = 0. \tag{4.9}$$

where

$$\begin{aligned}
g_0 :&= +1440c_3^2 s_0^2,\\
g_1 :&= 150c_3^2 s_0^2\beta^3 - 152c_3^2 s_0\beta^4 - 240(n+1)c_3^2\beta^3 s_{0;0} + 360c_3^2\beta^3 s_{0;0},\\
g_2 :&= 120c_3^2 s_0^2\beta^3 - 670c_3^2\beta^2 s_0^2 + 988c_2c_3\beta^2 s_0^2,\\
g_3 :&= 300(n+1)c_3^2\beta^2 t_0 - 1200(n+1)c_3^2\beta^2 s_0 - 7680c_2c_3\beta^2 s_0 + 2496c_2c_3\beta s_{0;0}\\
&\quad - 9250(n+1)c_3^2\beta s_0^2 - 4200c_3^2\beta^2 t_0 + 1680c_3^2\beta^2 s_0 + 3430c_3^2\beta s_0^2\\
&\quad + 8400(n+1)c_3^2 B\beta s_{0;0} - 1260c_3^2\beta s_{0;0} + 480c_3^2 B\beta^2 s_0 - 1050c_2c_3\beta s_0^2\\
&\quad + 650c_3^2 B\beta s_0^2 - 140(n+1)c_3\beta s_{0;0}c_2,\\
g_4 :&= 6250(n+1)c_3^2\beta s_0^2 - 460c_2c_3 B s_0^2 - 5625c_3^2 B\beta s_0^2 + 8500c_2c_3\beta s_0^2\\
&\quad - 11750c_3^2\beta s_0^2 + 130c_3^2 B^2 s_0^2 + 514c_1c_3 s_0^2 + 3024c_2^2 s_0^2,
\end{aligned}$$

By (4.9), we get

$$g_3\alpha^2 + g_1 = 0, \tag{4.10}$$
$$g_4\alpha^4 + g_2\alpha^2 + g_0 = 0. \tag{4.11}$$

(4.11) implies that

$$\eta(x, y)s_0^2 = 0, \tag{4.12}$$

where

$$\eta(x,y) = \Big\{620(n+1)c_3^2\beta - 4606c_2c_3B - 5625c_3^2B\beta + 800c_2c_3\beta - 1150c_3^2\beta + 1300c_3^2B^2 + 5184c_1c_3 + 30424c_2^2\Big\}\alpha^4 + \Big\{1250c_3^2\beta^3 - 6720c_3^2B\beta^2 + 988c_2c_3\beta^2\Big\}\alpha^2 + 140c_3^2\beta^4.$$

By (4.12), one can find that $\eta = 0$ or $s_i = 0$. Let $\eta(x,y) = 0$. We rewrite η as follows

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0, \tag{4.13}$$

where $\theta = \theta(x,y)$, $\gamma = \gamma(x,y)$ and $\varepsilon = \varepsilon(x,y)$ are scalar functions on TM. (4.13) give us

$$\alpha^2 = (\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\delta}}{2\theta})\beta^2. \tag{4.14}$$

This contradicts with the positive-definiteness of α. Thus

$$\eta \neq 0.$$

Hence, we get

$$s_i = 0.$$

Putting $r_{ij} = 0$ and $s_i = 0$ in (3.4) imply that $\mathbf{S} = 0$. The converse is trivial. □

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