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On Kropina transformation of exponential (α, β)-metrics

Mosayeb Zohrehvand[a], Ghodratallah Fasihi-Ramandi[b*] and Shahroud Azami[b]

[a]Department of Mathematic, Faculty of Mathematical Sciences and Statistics, Malayer University, Malayer, Iran
[b]Department of pure Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran

E-mail: m.zohrehvand@malayeru.ac.ir
E-mail: fasihi@sci.ikiu.ac.ir
E-mail: azami@sci.ikiu.ac.ir

Abstract. In this paper, we study the Kropina transformation of exponential (α, β)-metric $F = \alpha \exp(s),\ s := \beta/\alpha$. We characterize the conditions under which this class of (α, β)-metric is locally projectively flat, locally dually flat, and Douglas metric. Based on, we show that the Kropina transformation of an exponential (α, β)-metric is locally projectively flat, locally dually flat and Douglas metric if and only if the exponential (α, β)-metric is locally projectively flat, locally dually flat and Douglas metric, respectively.

Keywords: Locally projectively flat, Locally dually flat, Douglas metric, (α, β)-metric.

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*Corresponding Author
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1. **Introduction**

In 1991 M. Matsumoto introduced the concept of (α, β)-metrics [15]. They form an important and rich class of Finsler metrics that appear on many applications of mathematics in physics, biology, etc (see [3]). (α, β)-metrics are defined by a Riemannian metric $\alpha := \sqrt{a_{ij}y^i y^j}$ and a 1-form $\beta := b_i(x)y^i$. They have been widely studied by many authors partly because they are computable. Also, the research on (α, β)-metrics enrich Finsler geometry and suggest many references for further studies.

The Kropina metric $F = \alpha^2/\beta$ is an (α, β)-metric which was first introduced by Berwald and was investigated by V.K. Kropina [12]. This metric is very interesting because it appears when the general dynamical system is represented by a Lagrangian function [4]. As a geometrical motivation, let us denote an open sea by a Riemannian manifold (M, h) where a wind $W = W^i \frac{\partial}{\partial x^i}$ blows. If $h(W, W) = 1$, then the paths minimizing time of travel of a ship are the geodesics of a Kropina metric [28].

For any Finsler metric F and a non-zero 1-form β, one can consider the β-transformation

$$F(x, y) \to \bar{F}(x, y) := f(F, \beta),$$

where $f(F, \beta)$ is a positively homogeneous function of β and F. In this paper, we consider the β-transformation $\bar{F}(x, y) := \frac{F^2(x,y)}{\beta(x,y)}$, named Kropina transformation of F. It is easy to see that $\bar{F}$ is reduced to the Kropina metric when F is reduced to the Riemannian metric α.

The (α, β)-metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is called exponential metric and studied by many authors [20, 22, 27, 30]. This metric is interesting because the exponential metric

$$F = \alpha \exp\Big(\int_0^s \frac{q\sqrt{b^2 - t^2}}{1 + qt\sqrt{b^2 - t^2}} dt\Big),$$

is an almost regular unicorn metric, where $b := \|\beta\|_\alpha$ and q is a constant. A unicorn metric is a Landsberg metric that is not Berwaldian [23].

This paper is devoted to the study of the Kropina transformation of exponential (α, β)-metric $F = \alpha \exp(s)$, $s := \beta/\alpha$.

Projectively flat Finsler metrics are the smooth solutions of the Hilbert fourth problem, in regular cases. (α, β)-metrics of projectively flat type have been studied by many authors [5, 6, 15, 18, 20, 21, 24, 30]. Locally projectively flat Kropina metrics are studied in [5]. Exponential (α, β)-metrics of locally projectively flat type are studied in [30] and it is proved that an exponential (α, β)-metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α.

Now, we obtain the necessary and sufficient conditions under which the Kropina transformation of exponential (α, β)-metric be locally projectively flat.

Theorem 1.1. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be an (α, β)-metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero 1-form. Then $\bar{F}$ is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α.*

From Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α, β)-metric $F = \alpha \exp(s)$. Then $\bar{F}$ is locally projectively flat if and only if F is locally projectively flat.*

Remarkably, Z. Shen studied locally projectively flat regular (α, β)-metrics of non-Randers type [20]. It is easy to see that $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ is singular at zero. Thus, this class of (α, β)-metrics is not included in the Shen's paper.

Douglas curvature is one of the non-Riemannian quantities which has closely related to projectively flat Finsler metrics. A Finsler metric is of projectively flat type if and only if its Douglas curvature and its Weyl curvature vanish. A Finsler metric with zero Douglas curvature is called Douglas metric. (α, β)-metrics of Douglas type have been considered by many authors [5, 6, 14, 16, 30]. An exponential (α, β)-metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is a Douglas metric if and only if β is parallel with respect to α [30].

Here, we study Kropina transformation of exponential (α, β)-metrics of Douglas type.

Theorem 1.3. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be an (α, β)-metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero one form. Then $\bar{F}$ is a Douglas metric if and only if β is parallel with respect to α.*

From Theorem 1.3, we have the following corollary.

Corollary 1.4. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α, β)-metric $F = \alpha \exp(s)$. Then $\bar{F}$ is a Douglas metric if and only if F is a Douglas metric.*

The notion of locally dually flat metric was introduced by S. I. Amari and H. Nagaoka when they were studying the information geometry on Riemannian manifolds [1, 2].

This notion was extended to Finsler spaces by Z. Shen in [19] and the locally dually flat Finsler metrics are studied. Finsler metrics of locally dually flat type have interesting applications in the study of flat Finsler information structure [7, 8]. Locally dually flat (α, β)-metrics have been mentioned by many authors [17, 25, 27, 29].

Here, we obtain the necessary and sufficient conditions under which $F = \alpha \exp(2s)/s$, is locally dually flat.

Theorem 1.5. *Let $\bar{F} = \alpha\exp(2s)/s$, $s := \beta/\alpha$ be an (α,β)-metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero one form. Then $\bar{F}$ is a locally dually flat metric if and only if*

a) $r_{00} = \frac{2}{3}(\beta\theta - \alpha^2\theta_l b^l)$,
b) $G^i_\alpha = \frac{1}{3}(2\theta y^i + \alpha^2\theta^i)$,
c) $s_{i0} = \frac{1}{3}(\beta\theta_i - \theta b_i)$,

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\theta^l := a^{li}\theta_i$.

A large class of (α,β)-metrics of locally dually flat type is considered in [27] and it is proved that the exponential (α,β)-metric $F = \alpha\exp(s)$, $s := \beta/\alpha$, is locally dually flat if and only if G^i_α, r_{ij}, and s_{ij} satisfy in above conditions. Therefore, we have the following corollary.

Corollary 1.6. *Let $\bar{F} = \alpha\exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α,β)-metric $F = \alpha\exp(s)$. Then $\bar{F}$ is locally dually flat if and only if F is locally dually flat.*

2. **Preliminaries**

For a given Finsler metric $F = F(x,y)$, the geodesic of F satisfies the following system of differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0,$$

where $G^i = G^i(x,y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}.$$

An (α,β)-metric is a Finsler metric expressed in the form, $F = \alpha\phi(s)$, $s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_0$, $x \in M$, and $\phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \leq b < b_0. \tag{2.1}$$

In this case, the fundamental form of the metric tensor induced by F is positive definite [9].

Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji})$$

where b_{ij} means the coefficients of the covariant derivative of β with respect to α. Clearly, β is closed if and only if $s_{ij} = 0$, and we say that β is parallel with respect to α if $r_{ij} = s_{ij} = 0$. Furthermore, we denote

$$r^i_{\ j} := a^{ik} r_{kj}, \qquad r_{i0} := r_{ij} y^j,$$
$$r_{00} := r_{ij} y^i y^j, \qquad r := r_{ij} b^i b^j,$$
$$s^i_{\ j} := a^{ik} s_{kj}, \qquad s_{i0} := s_{ij} y^j,$$
$$s_i := b_j s^j_{\ i}, \qquad s_0 := s_i y^i.$$

The geodesic coefficients G^i of F and geodesic coefficients G^i_α of α are related as follows

$$G^i = G^i_\alpha + \alpha Q s^i_{\ 0} + \{-2Q\alpha s_0 + r_{00}\}\Big\{\Psi b^i + \Theta \alpha^{-1} y^i\Big\} \tag{2.2}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$
$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

A Finsler metric is said to be locally projectively flat if at any point there exists a local coordinate system such that the geodesics are straight lines as point sets. Hamel [11] found a system of PDEs that characterized projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$.

Theorem 2.1. [11] *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if*

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

Using Theorem 2.1, the following lemma can be obtained.

Lemma 2.2. [21] *An (α, β)-metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is projectively flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \tag{2.3}$$

where $y_m := a_{ml} y^l$.

The Douglas tensor $\mathbf{D}$ of a Finsler metric F is defined by $\mathbf{D}_y := D^i_{\ jkl}(x, y) dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$, where

$$D^i_{\ jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) \tag{2.4}$$

Douglas tensor is a non-Riemannian quantity, i.e. it vanishes for Riemannian metrics and it is invariant under the projective transformations.

In [10] the Douglas tensor of an (α,β)-metric is determined by

$$D^i{}_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m} y^i\right) \tag{2.5}$$

where

$$T^i = \alpha Q s^i{}_0 + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i \tag{2.6}$$

and

$$T^m_{y^m} = Q' s_0 + \Psi' \alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi\left[r_0 - Q'(b^2 - s^2)s_0 - Qss_0\right]. \tag{2.7}$$

A Douglas metric is a Finsler metric with a vanishing Douglas tensor. Let $F = \alpha\phi(s)$, $s := \beta/\alpha$ be a Douglas (α,β)- metric, From (2.4) and (2.5) we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1} T^m_{y^m} y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} := H^i_{jk}(x)$ such that

$$T^i - \frac{1}{n+1} T^m_{y^m} y^i = H^i_{00}, \tag{2.8}$$

where $H^i_{00} := H^i_{jk}(x) y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations (2.6) and (2.7), respectively.

A Finsler metric is called locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients of F are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x,y)$ is a local scalar function on the tangent bundle TM of M. Such a coordinate system is called an adapted coordinate system. A system of PDEs that characterized dually flat Finsler metrics on an open subset in $\mathbb{R}^n$, can be found in [19]. In fact, we have the following theorem.

Theorem 2.3. [19] *A Finsler metric $F = F(x,y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is dually flat if and only if the following equation holds:*

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0.$$

In this case

$$H = \frac{1}{6}[F^2]_{x^m} y^m.$$

In [26], an equation is obtained that characterizes dually flat (α,β)-metrics.

Lemma 2.4. [26] *An (α, β)-metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is dually flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if*

$$3\alpha^2 a_{ml} G^m_\alpha + \alpha^3 Q(3s_{l0} - r_{l0}) - \alpha^2 \left(\frac{\partial(y_m G^m_\alpha)}{\partial y^l} + \alpha Q \frac{\partial(b_m G^m_\alpha)}{\partial y^l} \right) + \Big\{ 2Q(y_m G^m_\alpha) + \Phi\big(\alpha r_{00} + 2(b_m \alpha - s y_m) G^m_\alpha\big) \Big\} (\alpha b_l - s y_l) + Q\alpha(r_{00} + 2b_m G^m_\alpha) y_l = 0, \quad (2.9)$$

where $r_{i0} := r_{ij} y^j$, $s_{i0} := s_{ij} y^j$, $y_i := a_{ij} y^j$, and

$$\Phi := \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}.$$

3. Kropina transformation of exponential (α, β)-metrics

In this section, we consider the Kropina transformation of exponential (α, β)-metric $F = \alpha \exp(s)$, i.e.

$$\bar{F} = \alpha \exp(2s)/s, \;\; s := \beta/\alpha.$$

Since $\phi(s) = \exp(2s)/s$ must be a positive function, thus $s > 0$. One can see that when $s > 0$, we have the following lemma.

Lemma 3.1. *$\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$, is a Finsler metric, if and only if $0 < \|\beta_x\|_\alpha < 1$*

Proof. Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$, is a Finsler metric, then from (2.1) we have

$$\frac{s^3 + 2b^2 s^2 - 2b^2 s + b^2 - 2s^4}{s^3} > 0.$$

For $s = b$, we get $0 < b < 1$. Thus $0 < \|\beta_x\|_\alpha < 1$. The converse is easy to prove. □

It is easy to see that, the geodesic coefficients $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$, are given by (2.2) with

$$\begin{aligned} Q &:= \frac{2s - 1}{2s(1 - s)}, \\ \Theta &:= \frac{s(5s - 4s^2 - 2)}{2\Big[s^3(1 - 2s) + b^2(2s^2 - 2s + 1)\Big]}, \\ \Psi &:= \frac{2s^2 - 2s + 1}{2\Big[s^3(1 - 2s) + b^2(2s^2 - 2s + 1)\Big]}. \end{aligned} \quad (3.1)$$

3.1. **Proof of Theorem 1.1.** Suppose that $\bar{F}$ is locally projectively flat. From (2.3) we have

$$\begin{aligned}
2\beta(\alpha-\beta)\Big[b^2\alpha^2\big((\alpha-\beta)^2+\beta^2\big)+\beta^3(\alpha-2\beta)\Big](a_{ml}\alpha^2-y_my_l)G^m_\alpha \\
+\alpha^4\Big[(\alpha-\beta)\big[b^2\alpha^2\big(3b^2\beta(\alpha-\beta)-\alpha^2\big)+4\beta^4\big]+(b^2-1)\alpha^2\beta^3\Big]s_{l0} \\
+\Big[\beta\alpha^2\big(2\beta^2(2\alpha-\beta)+\alpha^2(\alpha-3\beta)\big)r_{00}+\alpha^4\big(2\beta^2(3\alpha-2\beta) \\
+\alpha^2(\alpha-4\beta)\big)s_0\Big](\alpha^2b_l-\beta y_l)=0. \quad (3.2)
\end{aligned}$$

From (3.2), we get

$$\begin{aligned}
2\Big[\alpha^2\beta b^2(\alpha^2+\beta^2)+3\beta^2(b^2\alpha^2-\beta^2)\Big](a_{ml}\alpha^2-y_my_l)G^m_\alpha \\
+\alpha^4\Big[4\beta^2(\beta^2-b^2\alpha^2)-\alpha^2b^2(\alpha^2+2\beta^2)\Big]s_{l0} \\
+\alpha^2\Big[\beta(4\beta^2+\alpha^2)r_{00}+\alpha^2(6\beta^2+\alpha^2)s_0\Big](\alpha^2b_l-\beta y_l)=0, \quad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
2\beta^2\big[\beta^2(\alpha^2+2\beta^2)-b^2\alpha^2(3\alpha^2+2\beta^2)\big](a_{ml}\alpha^2-y_my_l)G^m_\alpha \\
+\alpha^4\beta\big[4b^2\alpha^2(\beta^2+\alpha^2)-\beta^2(\alpha^2+4\beta^2)\big]s_{l0} \\
-\beta\alpha^2\big[\beta(2\beta^2+3\alpha^2)r_{00}+4\alpha^2(\beta^2+\alpha^2)s_0\big](\alpha^2b_l-\beta y_l)=0. \quad (3.4)
\end{aligned}$$

From (3.3) one can see that $(s_0b_l-s_{l0}b^2)\alpha^8$ is divisible by β. Thus, there exist scalar functions $\nu:=\nu_l(x)$ on M such that

$$s_0b_l-s_{l0}b^2=\beta\nu_l, \quad (3.5)$$

for any $l:=1,\dots,n$.

Multiplying (3.5) by y^l, we get $\nu_l=s_l$ and then

$$s_{l0}=\frac{1}{b^2}\{s_0b_l-\beta s_l\}. \quad (3.6)$$

Contracting (3.3) by βb^l and (3.4) by b^l, yields

$$\begin{aligned}
2\beta\big[\alpha^2\beta b^2(\alpha^2+\beta^2)+3\beta^2(b^2\alpha^2-\beta^2)\big](\alpha^2b_m-\beta y_m)G^m_\alpha \\
+\alpha^4\beta\big[4\beta^2(\beta^2-\alpha^2b^2)-\alpha^2b^2(\alpha^2+2\beta^2)\big]s_0 \\
+\alpha^2\beta\big[\beta(4\beta^2+\alpha^2)r_{00}+\alpha^2(6\beta^2+\alpha^2)s_0\big](\alpha^2b^2-\beta^2)=0, \quad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
2\beta^2\big[\beta^2(\alpha^2+2\beta^2)-b^2\alpha^2(3\alpha^2+2\beta^2)\big](\alpha^2b_m-\beta y_m)G^m_\alpha \\
+\alpha^4\beta\big[4b^2\alpha^2(\beta^2+\alpha^2)-\beta^2(\alpha^2+4\beta^2)\big]s_0 \\
-\beta\alpha^2\big[\beta(2\beta^2+3\alpha^2)r_{00}+4\alpha^2(\beta^2+\alpha^2)s_0\big](b^2\alpha^2-\beta^2)=0. \quad (3.8)
\end{aligned}$$

(3.7)+(3.8) yields

$$2\beta^2(\beta^2-\alpha^2)\big[(2b^2\alpha^2-\beta^2)(\alpha^2 b_m-\beta y_m)G^m_\alpha+\alpha^2(\alpha^2b^2-\beta^2)r_{00}-\alpha^4\beta s_0\big]=0.$$

Thus

$$(2b^2\alpha^2-\beta^2)(b_m\alpha^2-\beta y_m)G^m_\alpha=\alpha^2\big(\alpha^2\beta s_0-(\alpha^2b^2-\beta^2)r_{00}\big). \tag{3.9}$$

From (3.9) we see that $(\alpha^2 b_m-\beta y_m)G^m_\alpha$ has the factor α^2, i.e. there exists a function $\eta_2:=\eta_2(x,y)$ on TM such that

$$(\alpha^2 b_m-\beta y_m)G^m_\alpha=\alpha^2\eta_2, \tag{3.10}$$

where $\eta_2(x,y)$ is a homogeneous polynomial of degree two with respect to y. By substituting (3.10) in (3.9), we have

$$\alpha^2(2b^2\eta_2+b^2r_{00}-2\beta s_0)=\beta^2(\eta_2+r_{00}).$$

Thus, there exists a scalar function $\gamma:=\gamma(x)$ on M such that

$$2b^2\eta_2+b^2r_{00}-2\beta s_0=\beta^2\gamma, \tag{3.11}$$

and

$$\eta_2+r_{00}=\alpha^2\gamma. \tag{3.12}$$

From (3.11) and (3.12), we get

$$r_{00}=\frac{1}{b^2}\{(2\alpha^2b^2-\beta^2)\gamma-2\beta s_0\}. \tag{3.13}$$

Substituting (3.13) back into (3.9), we have

$$(2\alpha^2-\beta^2)\Big[(\alpha^2 b_m-\beta y_m)G^m_\alpha+\frac{1}{b^2}\alpha^2(\alpha^2b^2-\beta^2)\gamma\Big]=\frac{1}{b^2}\alpha^2\beta(3b^2\alpha^2-2\beta^2)s_0.$$

Thus

$$s_0=0, \tag{3.14}$$

$$(\alpha^2 b_m-\beta y_m)G^m_\alpha=-\frac{\gamma}{b^2}\alpha^2(\alpha^2b^2-\beta^2). \tag{3.15}$$

Using (3.6), (3.13) and (3.14), we infer

$$s_{l0}=0, \tag{3.16}$$

$$r_{00}=\frac{\gamma}{b^2}\{2\alpha^2b^2-\beta^2\}. \tag{3.17}$$

Substituting (3.14), (3.15), (3.16) and (3.17) in (3.7), we deduce

$$\frac{1}{b^2}\alpha^2\beta^4(2\beta^2-\alpha^2)(\alpha^2b^2-\beta^2)\gamma=0. \tag{3.18}$$

Since

$$\frac{1}{b^2}\alpha^2\beta^4(2\beta^2-\alpha^2)(\alpha^2b^2-\beta^2)\neq 0,$$

thus we get

$$\gamma=0$$

and therefore

$$r_{00}=0. \tag{3.19}$$

From (3.16) and (3.19), we obtained that β is parallel with respect to α.

Substituting (3.14), (3.16), and (3.18) into (3.3), we conclude that

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0,$$

therefore α is projectively flat [13].

Now let α be projectively flat and β be parallel with respect to α. From (2.2) one can see that $\bar{F}$ is locally projectively flat. The proof is complete. $\Box$

In the proof of Theorem 1.3, for simplicity, we assume that $\lambda := \frac{1}{n+1}$.

3.2. **Proof of Theorem 1.3.** The proof of sufficiency is obvious. Therefore, we just need to prove the necessity conditions. If $\bar{F}$ be a Douglas metric, then (2.8) holds. Plugging (3.1) into (2.8), we obtain

$$\frac{A^i_{11}\alpha^{11} + A^i_{10}\alpha^{10} + \cdots + A^i_1\alpha + A^i_0}{P_9\alpha^9 + P_8\alpha^8 + \cdots + P_1\alpha + P_0} = H^i_{00}, \tag{3.20}$$

where

$$\begin{aligned}
A^i_{11} &:= b^2(b^2 s^i{}_0 - b^i s_0),\\
A^i_{10} &:= -6b^2\beta(b^2 s^i{}_0 - b^i s_0),\\
A^i_9 &:= 16b^2\beta^2(b^2 s^i{}_0 - b^i s_0) + 2\lambda b^2\beta y^i(r_0 + s_0) - b^2\beta b^i r_{00},\\
A^i_8 &:= \beta^3(1 - 24b^2)(b^2 s^i{}_0 - b^i s_0) - \lambda b^2\beta^2 y^i(10r_0 + 13s_0)\\
&\quad + b^2\beta^2(\beta s^i{}_0 + 5b^i r_{00}),\\
A^i_7 &:= 2\beta^4(10b^2 - 3)(b^2 s^i{}_0 - b^i s_0) + 24\lambda b^2\beta^3 y^i(r_0 + 2s_0)\\
&\quad - 6b^2\beta^3(\beta s^i{}_0 + 2b^i r_{00}),\\
A^i_6 &:= 2\beta^5[(7 - 4b^2)(b^2 s^i{}_0 - b^i s_0) + 7b^2 s^i{}_0] - \beta^3[(1 - 16b^2)\beta b^i\\
&\quad + 3\lambda b^2 y^i]r_{00} + \lambda\beta^4 y^i[(1 - 14b^2)(2r_0 + 5s_0) - 4b^2 r_0],
\end{aligned}$$

$$\begin{aligned}
A^i_5 &:= \beta^6[-16(2b^2 s^i{}_0 - b^i s_0) + s^i{}_0] + \beta^4[(5 - 15b^2)\beta b^i + 15\lambda b^2 y^i]r_{00}\\
&\quad + 2\lambda\beta^5 y^i[-(5r_0 + 14s_0) + 6b^2(2r_0 + 5s_0)],\\
A^i_4 &:= 2\beta^7[4(b^2 s^i{}_0 - b^i s_0) - 3s^i{}_0] + \beta^5 r_{00}[2(2b^2 - 5)\beta b^i\\
&\quad + (3 - 26b^2)\lambda y^i] + 2\lambda\beta^6 y^i[(10r_0 + 29s_0) - 4b^2(r_0 + 3s_0)],\\
A^i_3 &:= -4\lambda\beta^7 y^i(5r_0 + 14s_0) + \lambda\beta^6 y^i r_{00}(22b^2 - 15)\\
&\quad + 2\beta^7(5b^i r_{00} + 6\beta s^i{}_0),\\
A^i_2 &:= 8\lambda\beta^8 y^i(r_0 + 3s_0) + 2\lambda\beta^7 y^i r_{00}(13 - 8b^2) - 4\beta^8(r_{00} + 2\beta s^i{}_0),\\
A^i_1 &:= -22\lambda y^i\beta^8 r_{00},\\
A^i_0 &:= 8\lambda y^i\beta^9 r_{00},
\end{aligned}$$

and

$$\begin{aligned} P_9 &:= -2b^4\beta, & P_8 &:= 10b^4\beta^2,\\ P_7 &:= -24b^4\beta^3, & P_6 &:= -4b^2\beta^4(1-8b^2),\\ P_5 &:= 4b^2\beta^5(5-6b^2), & P_4 &:= -8b^2\beta^6(5-b^2),\\ P_3 &:= -2\beta^7(1-20b^2), & P_2 &:= 2\beta^8(5-8b^2),\\ P_1 &:= -16\beta^9, & P_0 &:= 8\beta^{10}. \end{aligned}$$

Equation(3.20) is equivalent to

$$A^i_{11}\alpha^{11} + A^i_{10}\alpha^{10} + \cdots + A^i_1\alpha + A^i_0 = H^i_{00}(P_9\alpha^9 + P_8\alpha^8 + \cdots + P_1\alpha + P_0). \tag{3.21}$$

Replacing y^i by $-y^i$ in (3.21) yields

$$-A^i_{11}\alpha^{11} + A^i_{10}\alpha^{10} + \cdots - A^i_1\alpha + A^i_0 = H^i_{00}(-P_9\alpha^9 + P_8\alpha^8 + \cdots - P_1\alpha + P_0). \tag{3.22}$$

(3.21)+(3.22) implies that

$$A^i_{10}\alpha^{10} + A^i_8\alpha^8 + \cdots + A^i_2\alpha^2 + A^i_0 = H^i_{00}(P_8\alpha^8 + P_6\alpha^6 + \cdots + P_2\alpha^2 + P_0). \tag{3.23}$$

Also, from (3.21) − (3.22) we have

$$A^i_{11}\alpha^{10} + A^i_9\alpha^8 + \cdots + A^i_3\alpha^2 + A^i_1 = H^i_{00}(P_9\alpha^8 + P_7\alpha^6 + \cdots + P_3\alpha^2 + P_1). \tag{3.24}$$

From (3.23) and (3.24), we get

$$A^i_0 - H^i_{00}P_0 := 8\beta^9(\lambda y^i r_{00} - H^i_{00}\beta),$$

and

$$A^i_1 - H^i_{00}P_1 := -2\beta^8(11\lambda y^i r_{00} + 8\beta H^i_{00}),$$

have the factor α^2. Therefore we obtained that r_{00} and H^i_{00} have the factor α^2. Thus there exist scalar functions $\sigma := \sigma(x)$ and $\eta^i := \eta^i(x)$ on M such that

$$r_{00} = \sigma\alpha^2, \tag{3.25}$$

$$H^i_{00} = \eta^i\alpha^2, \tag{3.26}$$

where $i \in \{1, 2, \ldots, n\}$. From (3.25), we have

$$r_0 = \beta\sigma. \tag{3.27}$$

On the other hand from (3.21), one can see that $A^i_{11}\alpha^{11} = b^2(b^2 {s^i}_0 - b^i s_0)\alpha^{11}$ has the factor β. Therefore there exist scalar functions $\xi := \xi^i(x)$ on M such that

$$b^2 {s^i}_0 - b^i s_0 = \beta\xi^i, \tag{3.28}$$

where $i \in \{1, 2, \ldots, n\}$. Multiplying (3.28) by y_i, we get $\xi^i y_i = s_0$, thus

$$\xi^i = s^i. \tag{3.29}$$

Substituting (3.29) in (3.28), we obtain

$$s_{ij} = \frac{1}{b^2}\{b_i s_j - b_j s_i\}. \tag{3.30}$$

Substituting (3.25), (3.26), (3.27) and (3.30) into (3.21), we get

$$B^i_9\alpha^9 + B^i_8\alpha^8 + \cdots + B^i_1\alpha + B^i_0 = \eta^i(Q_9\alpha^9 + Q_8\alpha^8 + \cdots + Q_1\alpha + Q_0), \tag{3.31}$$

where

$$\begin{aligned}
B^i_9 &:= b^4(s^i + b^i\sigma),\\
B^i_8 &:= -b^4\beta(6s^i + 5b^i\sigma),\\
B^i_7 &:= 4b^4\beta^2(4s^i + 3b^i\sigma) - 2\lambda b^4 y^i(\beta\sigma + s_0),\\
B^i_6 &:= b^2\beta^2 b^i[(1-16b^2)\beta\sigma - s_0] + 2(1-12b^2)b^2\beta^2 s^i\\
&\quad + 13\lambda b^2\beta y^i(s_0 + \beta\sigma),\\
B^i_5 &:= b^2\beta^3 b^i[(12b^2 - 5) + 6s_0] + 4b^2(5b^2-3)\beta^4 s^i\\
&\quad - \lambda b^4\beta^2 y^i(39\beta\sigma + 42s_0),\\
B^i_4 &:= 2b^2\beta^4 b^i[(5-2b^2)\beta\sigma - 7s_0] - 5\lambda b^2\beta^3 y^i(s_0 + \beta\sigma)\\
&\quad + 4b^2\beta^5 s^i(7-2b^2) + 2\lambda b^4\beta^3 y^i(29\beta\sigma + 35s_0),\\
B^i_3 &:= \lambda b^2\beta^4 y^i[4(7-15b^2)s_0 + \beta\sigma(25-46b^2)]\\
&\quad + \beta^5 b^i s_0(1 + 16b^2 - 10\beta b^2\sigma) + (1-32b^2)\beta^6 s^i,\\
B^i_2 &:= 2\lambda b^2\beta^5 y^i[(8b^2-23)\beta\sigma + (12b^2-29)s_0]\\
&\quad + 2\beta^6 b^i[2b^2\beta\sigma + (3-2b^2)s_0] + 2(8b^2-3)\beta^7 s^i,\\
B^i_1 &:= 14\lambda b^2\beta^6 y^i(3\beta\sigma + 4s_0) + 12\beta^7(\beta s^i - b^i s_0),\\
B^i_0 &:= -8\lambda b^2\beta^7 y^i(2\beta\sigma + 3s_0) - 8\beta^8(\beta s^i - b^i s_0),
\end{aligned}$$

and

$$\begin{aligned}
Q_9 &:= 2b^6, & Q_8 &:= -10b^6\beta,\\
Q_7 &:= 24b^6\beta^2, & Q_6 &:= -4b^4\beta^3(8b^2-1),\\
Q_5 &:= 4b^4\beta^4(6b^2-5), & Q_4 &:= -8b^4\beta^5(b^2-5),\\
Q_3 &:= -2b^2\beta^6(20b^2-1), & Q_2 &:= -2b^2\beta^7(8b^2-5),\\
Q_1 &:= 16b^2\beta^8, & Q_0 &:= -8b^2\beta^9.
\end{aligned}$$

Replacing y^i by $-y^i$ into (3.31) yields

$$B^i_9\alpha^9 - B^i_8\alpha^8 + \cdots + B^i_1\alpha - B^i_0 = \eta^i(Q_9\alpha^9 - Q_8\alpha^8 + \cdots + Q_1\alpha - Q_0). \tag{3.32}$$

By adding (3.31) and (3.32), we have

$$B_9^i\alpha^8 + B_7^i\alpha^6 + B_5^i\alpha^4 + B_3\alpha^2 + B_1^i = \eta^i(Q_9\alpha^8 + Q_7\alpha^6 + Q_5\alpha^4 + Q_3\alpha^2 + Q_1). \quad (3.33)$$

From (3.33) one can see that

$$B_1^i - \eta^i Q_1 = \beta^6\big[7\lambda b^2 y^i(8s_0 + 6\beta\sigma) + 12\beta(\beta s^i - b^i s_0) - 16b^2\beta\eta^i\big],$$

that has the factor α^2, i.e. there exist scalar functions $\omega^i := \omega^i(x)$ on M such that

$$7\lambda b^2 y^i(8s_0 + 6\beta\sigma) + 12\beta(\beta s^i - b^i s_0) - 16b^2\beta^2\eta^i = \alpha^2\omega^i, \quad (3.34)$$

where $i \in \{1, 2, \ldots, n\}$. Multiplying (3.34) by b_i and y_i leads to

$$\beta[7\lambda b^2(8s_0 + 6\beta\sigma) - 12b^2 s_0 - 16b^2\beta\eta^i b_i] = \alpha^2\omega^i b_i, \quad (3.35)$$

and

$$\alpha^2[7\lambda b^2(8s_0 + 6\beta\sigma) - \omega^i y_i] = 16b^2\beta^2\eta^i y_i, \quad (3.36)$$

respectively. From (3.35), we deduce

$$\omega^i b_i = 0, \quad (3.37)$$

$$7\lambda b^2(8s_0 + 6\beta\sigma) - 12b^2 s_0 - 16b^2\beta\eta^i b_i = 0, \quad (3.38)$$

and from (3.36), we have

$$7\lambda b^2(8s_0 + 6\beta\sigma) - \omega^i y_i = 0. \quad (3.39)$$

From (3.39), we obtain

$$\eta_i = 0, \quad (3.40)$$

$$7\lambda b^2(8s_i + 6b_i\sigma) = \omega_i, \quad (3.41)$$

where

$$\eta_i := \eta^j a_{ji}, \qquad \omega_i := \omega^j a_{ji}.$$

Multiplying (3.41) by b^i and using (3.37), we find

$$\sigma = 0. \quad (3.42)$$

Thus

$$r_{00} = 0, \quad (3.43)$$

also from (3.40) we conclude that

$$H^i_{00} = 0.$$

Substituting (3.40) and (3.42) into (3.38), we obtain

$$(14\lambda - 12)b^2\beta s_0 = 0.$$

Since $14\lambda - 12 \neq 0$, thus $s_0 = 0$, and then from (3.30), we have

$$s_{ij} = 0. \quad (3.44)$$

From (3.43) and (3.44), we have that β is parallel with respect to α. Thus the proof is complete. □

3.3. **Proof of Theorem 1.5.** Suppose that $\bar{F}$ is locally dually flat. From (2.9) we have

$$A_l\alpha^6 + B_l\alpha^5 + C_l\alpha^4 + D_l\alpha^3 + E_l\alpha^2 + M_l\alpha + N_l = 0, \tag{3.45}$$

where

$$\begin{aligned}
A_l &:= \beta\frac{\partial b_m G^m_\alpha}{\partial y^l} + 6b_m G^m_\alpha b_l + \beta(r_{l0} - 3s_{l0}) + 3r_{00}b_l,\\
B_l &:= -2\beta^2\frac{\partial b_m G^m_\alpha}{\partial y^l} - 16\beta b_m G^m_\alpha b_l - 8\beta r_{00}b_l - 2\beta^2(r_{l0} - 3s_{l0}),\\
C_l &:= -2\beta^2\frac{\partial y_m G^m_\alpha}{\partial y^l} - 8\beta y_m G^m_\alpha b_l + 16\beta^2 b_m G^m_\alpha b_l - 8\beta b_m G^m_\alpha y_l\\
&\quad + 4\beta(2\beta b_l - y_l)r_{00} + 6a_{ml}G^m_\alpha\beta^2,\\
D_l &:= 2\beta^3\frac{\partial y^m G^m_\alpha}{\partial y^l} + 20\beta^2 y_m G^m_\alpha b_l + 20\beta^2 b_m G^m_\alpha y_l - 6\beta^3 a_{ml}G^m_\alpha\\
&\quad + 10\beta^2 r_{00}y_l,\\
E_l &:= -16\beta^3 y_m G^m_\alpha b_l + 8\beta^2 y_m G^m_\alpha y_l - 16\beta^3 b_m G^m_\alpha y_l - 8\beta^3 r_{00}y_l,\\
M_l &:= -20y_m G^m_\alpha\beta^3 y_l,\\
N_l &:= 16y_m G^m_\alpha\beta^4 y_l.
\end{aligned}$$

From (3.45) and by noticing that the sum of odd powers and even powers of α are zero respectively, we have

$$A_l\alpha^6 + C_l\alpha^4 + E_l\alpha^2 + N_l = 0, \tag{3.46}$$

$$B_l\alpha^4 + D_l\alpha^2 + M_l = 0. \tag{3.47}$$

Contracting (3.46) and (3.47) with b^l, we get

$$A\alpha^6 + C\alpha^4 + E\alpha^2 + N = 0, \tag{3.48}$$

$$B\alpha^4 + D\alpha^2 + M = 0. \tag{3.49}$$

where

$$\begin{aligned}
A &:= \beta\frac{\partial b_m G^m_\alpha}{\partial y^l}b^l + 6b^2 b_m G^m_\alpha + \beta(r_0 - 3s_0) + 3b^2 r_{00},\\
B &:= -2\beta^2\frac{\partial b_m G^m_\alpha}{\partial y^l}b^l - 16\beta b^2 b_m G^m_\alpha - 8\beta b^2 r_{00} - 2\beta^2(r_0 - 3s_0),\\
C &:= -2\beta^2\frac{\partial y_m G^m_\alpha}{\partial y^l}b^l - 8b^2\beta y_m G^m_\alpha - 16b^2\beta^2 G^m_\alpha b_m - 2\beta^2 b_m G^m_\alpha\\
&\quad +4(2b^2 - 1)\beta^2 r_{00},
\end{aligned}$$

$$D := 2\beta^3 \frac{\partial y^m G^m_\alpha}{\partial y^l} b^l + 20\beta^2 b^2 y_m G^m_\alpha + 14\beta^3 b_m G^m_\alpha + 10\beta^3 r_{00},$$
$$E := -16b^2\beta^3 y_m G^m_\alpha + 8\beta^3 y_m G^m_\alpha - 16\beta^4 b_m G^m_\alpha - 8\beta^4 r_{00},$$
$$M := -20\beta^4 y_m G^m_\alpha,$$
$$N := 16\beta^5 y_m G^m_\alpha.$$

From (3.48)×5+(3.49)×4β we have

$$\Big[5\beta \frac{\partial b_m G^m_\alpha}{\partial y^l} b^l + 30b^2 b_m G^m_\alpha + 15b^2 r_{00} + 5\beta(r_0 - 3s_0)\Big]\alpha^4 - \Big[10\beta^2 \frac{\partial y_m G^m_\alpha}{\partial y^l} b^l + 8\beta^3 \frac{\partial b_m G^m_\alpha}{\partial y^l} b^l + 40\beta b^2 y_m G^m_\alpha + 2(5-8b^2)\beta^2 b_m G^m_\alpha + 4(5-2b^2)\beta^2 r_{00} + 8\beta^3(r_0 - 3s_0)\Big]\alpha^2 + 8\beta^4 \frac{\partial y_m G^m_\alpha}{\partial y^l} b^l + 40\beta^3 y_m G^m_\alpha - 24\beta^4 b_m G^m_\alpha = 0. \quad (3.50)$$

Rewriting (3.48) and (3.50) yields that

$$\beta\alpha^6 \frac{\partial b_m G^m_\alpha}{\partial y^l} b^l - 2\beta^2\alpha^4 \frac{\partial y_m G^m_\alpha}{\partial y^l} b^l = -\Big[\beta(r_0 - 3s_0) + 3b^2(r_{00} + 2y_m G^m_\alpha)\Big]\alpha^6 + \Big[2(1-8b^2)\beta^2 b_m G^m_\alpha + 8b^2\beta^2 y_m G^m_\alpha + 4(1-2b^2)\beta^2 r_{00}\Big]\alpha^4 + \Big[8\beta^4 r_{00} + 8\beta^3(2b^2-1) y_m G^m_\alpha + 16\beta^4 y_m G^m_\alpha\Big]\alpha^2 - 16\beta^5 y_m G^m_\alpha, \quad (3.51)$$

and

$$\beta\alpha^2\Big[5\alpha^2 - 8\beta^2\Big] \frac{\partial b_m G^m_\alpha}{\partial y^l} b^l - 2\beta^2\Big[5\alpha^2 - 4\beta^2\Big] \frac{\partial y_m G^m_\alpha}{\partial y^l} b^l = -\Big[5\beta(r_0 - 3s_0) + 15b^2(r_{00} + 2b_m G^m_\alpha)\Big]\alpha^4 + \Big[40b^2\beta y_m G^m_\alpha + 2(5-8b^2)\beta^2 b_m G^m_\alpha + 4(5-2b^2)\beta^2 r_{00} + 8\beta^3(r_0 - 3s_0)\Big]\alpha^2 - 40\beta^3 y_m G^m_\alpha + 24\beta^4 b_m G^m_\alpha. \quad (3.52)$$

From (3.51) $\times (5\alpha^2 - 8\beta^2) -$ (3.52) $\times \alpha^4$, we have

$$\beta^2\alpha^4\Big[\frac{\partial y_m G^m_\alpha}{\partial y^l} b^l - 3b_m G^m_\alpha\Big] = \Big[\beta^2\alpha^2(1+8b^2) - b^2\alpha^4 - 8\beta^4\Big] \times \Big(\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha - 2\beta y_m G^m_\alpha\Big). \quad (3.53)$$

From (3.53), one can see that $\beta^2\alpha^2(1+8b^2) - b^2\alpha^4 - 8\beta^4$ can not be divided by α^4, thus $\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha - 2\beta b_m G^m_\alpha$ is divided by α^4, i.e. there exists a scalar function $\eta := \eta(x)$ on M such that

$$\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha - 2\beta y_m G^m_\alpha = \alpha^4\eta. \quad (3.54)$$

On the other hand from (3.48), we have that $r_{00} + 2b_m G^m_\alpha$ is divided by β and therefore from (3.54) we have that $\eta = 0$. Thus

$$\frac{\partial y_m G^m_\alpha}{\partial y^l} b^l = 3b_m G^m_\alpha, \quad (3.55)$$

$$\alpha^2 r_{00} + 2\alpha^2 b_m G^m_\alpha = 2\beta y_m G^m_\alpha. \quad (3.56)$$

From (3.56) we conclude that

$$y_m G^m_\alpha = \alpha^2 \theta, \tag{3.57}$$

where $\theta = \theta_i(x)y^i$ is a one form on M.

Substituting (3.57) in (3.55), yields

$$b_m G^m_\alpha = \frac{1}{3}\Big(2\beta\theta - \alpha^2\theta_l b^l\Big). \tag{3.58}$$

From (3.56) and (3.58), we have

$$r_{00} = \frac{2}{3}\Big(\beta\theta - \alpha^2\theta_l b^l\Big). \tag{3.59}$$

From (3.59) one can see that

$$r_{l0} = \frac{2}{3}\Big\{b_l\theta + \beta\theta_l - 2\theta_k b^k y_l\Big\}, \tag{3.60}$$

By substituting (3.57)-(3.60) in (3.46) and (3.47), we have

$$6\beta a_{ml} G^m_\alpha = \alpha^2 b_l\theta + \alpha^2\beta\theta_l + 3\alpha^2 s_{l0} + 4\beta\theta y_l, \tag{3.61}$$

and

$$6\beta a_{ml} G^m_\alpha = 2\alpha^2 b_l\theta + 6\alpha^2 s_{l0} + 4\beta\theta y_l, \tag{3.62}$$

respectively.

From (3.62) − (3.61), we deduce

$$s_{l0} := \frac{1}{3}\Big\{\beta\theta_l - \theta b_l\Big\}. \tag{3.63}$$

From (3.61), (3.62), and (3.63), we conclude

$$a_{ml} G^m_\alpha = \frac{1}{3}\Big\{\alpha^2\theta_l + 2\theta y_l\Big\},$$

thus

$$G^m_\alpha = \frac{1}{3}\Big\{\alpha^2\theta^m + 2\theta y^l\Big\},$$

therefore sufficient conditions are proved. The converse can be proved by a direct calculation. □

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