

On Kropina transformation of exponential (α, β) -metrics

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Abstract. In this paper, we study the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$. We characterize the conditions under which this class of (α, β) -metric is locally projectively flat, locally dually flat, and Douglas metric. Based on, we show that the Kropina transformation of an exponential (α, β) -metric is locally projectively flat, locally dually flat and Douglas metric if and only if the exponential (α, β) -metric is locally projectively flat, locally dually flat and Douglas metric, respectively.

Keywords: Locally projectively flat, Locally dually flat, Douglas metric, (α, β) -metric.

1. Introduction

In 1991 M. Matsumoto introduced the concept of (α, β) -metrics [15]. They form an important and rich class of Finsler metrics that appear on many applications of mathematics in physics, biology, etc (see [3]). (α, β) -metrics are defined by a Riemannian metric $\alpha := \sqrt{a_{ij}y^i y^j}$ and a 1-form $\beta := b_i(x)y^i$. They have been widely studied by many authors partly because they are computable. Also, the research on (α, β) -metrics enrich Finsler geometry and suggest many references for further studies.

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The Kropina metric $F = \alpha^2/\beta$ is an (α, β) -metric which was first introduced by Berwald and was investigated by V.K. Kropina [12]. This metric is very interesting because it appears when the general dynamical system is represented by a Lagrangian function [4]. As a geometrical motivation, let us denote an open sea by a Riemannian manifold (M, h) where a wind $W = W^i \frac{\partial}{\partial x^i}$ blows. If $h(W, W) = 1$, then the paths minimizing time of travel of a ship are the geodesics of a Kropina metric [28].

For any Finsler metric F and a non-zero 1-form β , one can consider the β -transformation

$$F(x, y) \rightarrow \bar{F}(x, y) := f(F, \beta),$$

where $f(F, \beta)$ is a positively homogeneous function of β and F . In this paper, we consider the β -transformation $\bar{F}(x, y) := \frac{F^2(x, y)}{\beta(x, y)}$, named Kropina transformation of F . It is easy to see that \bar{F} is reduced to the Kropina metric when F is reduced to the Riemannian metric α .

The (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is called exponential metric and studied by many authors [20, 22, 27, 30]. This metric is interesting because the exponential metric

$$F = \alpha \exp\left(\int_0^s \frac{q\sqrt{b^2 - t^2}}{1 + qt\sqrt{b^2 - t^2}} dt\right),$$

is an almost regular unicorn metric, where $b := \|\beta\|_\alpha$ and q is a constant. A unicorn metric is a Landsberg metric that is not Berwaldian [23].

This paper is devoted to the study of the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$.

Projectively flat Finsler metrics are the smooth solutions of the Hilbert fourth problem, in regular cases. (α, β) -metrics of projectively flat type have been studied by many authors [5, 6, 15, 18, 20, 21, 24, 30]. Locally projectively flat Kropina metrics are studied in [5]. Exponential (α, β) -metrics of locally projectively flat type are studied in [30] and it is proved that an exponential (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α .

Now, we obtain the necessary and sufficient conditions under which the Kropina transformation of exponential (α, β) -metric be locally projectively flat.

Theorem 1.1. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be an (α, β) -metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero 1-form. Then \bar{F} is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α .*

From Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$. Then \bar{F} is locally projectively flat if and only if F is locally projectively flat.*

Remarkably, Z. Shen studied locally projectively flat regular (α, β) -metrics of non-Randers type [20]. It is easy to see that $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ is singular at zero. Thus, this class of (α, β) -metrics is not included in the Shen's paper.

Douglas curvature is one of the non-Riemannian quantities which has closely related to projectively flat Finsler metrics. A Finsler metric is of projectively flat type if and only if its Douglas curvature and its Weyl curvature vanish. A Finsler metric with zero Douglas curvature is called Douglas metric. (α, β) -metrics of Douglas type have been considered by many authors [5, 6, 14, 16, 30]. An exponential (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is a Douglas metric if and only if β is parallel with respect to α [30].

Here, we study Kropina transformation of exponential (α, β) -metrics of Douglas type.

Theorem 1.3. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be an (α, β) -metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero one form. Then \bar{F} is a Douglas metric if and only if β is parallel with respect to α .*

From Theorem 1.3, we have the following corollary.

Corollary 1.4. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$. Then \bar{F} is a Douglas metric if and only if F is a Douglas metric.*

The notion of locally dually flat metric was introduced by S. I. Amari and H. Nagaoka when they were studying the information geometry on Riemannian manifolds [1, 2].

This notion was extended to Finsler spaces by Z. Shen in [19] and the locally dually flat Finsler metrics are studied. Finsler metrics of locally dually flat type have interesting applications in the study of flat Finsler information structure [7, 8]. Locally dually flat (α, β) -metrics have been mentioned by many authors [17, 25, 27, 29].

Here, we obtain the necessary and sufficient conditions under which $F = \alpha \exp(2s)/s$, is locally dually flat.

Theorem 1.5. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be an (α, β) -metric on a manifold M with dimension $n \geq 3$, where α is a Riemannian metric and β is a nonzero one form. Then \bar{F} is a locally dually flat metric if and only if*

- a) $r_{00} = \frac{2}{3}(\beta\theta - \alpha^2\theta_l b^l)$,
- b) $G_\alpha^i = \frac{1}{3}(2\theta y^i + \alpha^2\theta^i)$,

$$\text{c) } s_{i0} = \frac{1}{3}(\beta\theta_i - \theta b_i),$$

where $\theta := \theta_i(x)y^i$ is a 1-form on M and $\theta^l := a^{li}\theta_i$.

A large class of (α, β) -metrics of locally dually flat type is considered in [27] and it is proved that the exponential (α, β) -metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is locally dually flat if and only if G_α^i , r_{ij} , and s_{ij} satisfy in above conditions. Therefore, we have the following corollary.

Corollary 1.6. *Let $\bar{F} = \alpha \exp(2s)/s$, $s := \beta/\alpha$ be the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$. Then \bar{F} is locally dually flat if and only if F is locally dually flat.*

2. Preliminaries

For a given Finsler metric $F = F(x, y)$, the geodesic of F satisfies the following system of differential equations:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4}g^{il} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}.$$

An (α, β) -metric is a Finsler metric expressed in the form, $F = \alpha\phi(s)$, $s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < b_0$, $x \in M$, and $\phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \quad (2.1)$$

In this case, the fundamental form of the metric tensor induced by F is positive definite [9].

Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji})$$

where b_{ij} means the coefficients of the covariant derivative of β with respect to α . Clearly, β is closed if and only if $s_{ij} = 0$, and we say that β is parallel with respect to α if $r_{ij} = s_{ij} = 0$. Furthermore, we denote

$$\begin{aligned} r^i_j &:= a^{ik} r_{kj}, & r_{i0} &:= r_{ij} y^j, \\ r_{00} &:= r_{ij} y^i y^j, & r &:= r_{ij} b^i b^j, \\ s^i_j &:= a^{ik} s_{kj}, & s_{i0} &:= s_{ij} y^j, \\ s_i &:= b_j s^j_i, & s_0 &:= s_i y^i. \end{aligned}$$

The geodesic coefficients G^i of F and geodesic coefficients G_α^i of α are related as follows

$$G^i = G_\alpha^i + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\} \left\{ \Psi b^i + \Theta \alpha^{-1} y^i \right\} \quad (2.2)$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \\ \Psi &= \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

A Finsler metric is said to be locally projectively flat if at any point there exists a local coordinate system such that the geodesics are straight lines as point sets. Hamel [11] found a system of PDEs that characterized projectively flat Finsler metrics on an open subset in \mathbb{R}^n .

Theorem 2.1. [11] *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if*

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

Using Theorem 2.1, the following lemma can be obtained.

Lemma 2.2. [21] *An (α, β) -metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is projectively flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \quad (2.3)$$

where $y_m := a_{ml}y^l$.

The Douglas tensor \mathbf{D} of a Finsler metric F is defined by $\mathbf{D}_y := D^i_{jkl}(x, y)dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$, where

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right) \quad (2.4)$$

Douglas tensor is a non-Riemannian quantity, i.e. it vanishes for Riemannian metrics and it is invariant under the projective transformations.

In [10] the Douglas tensor of an (α, β) -metric is determined by

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \quad (2.5)$$

where

$$T^i = \alpha Q s_0^i + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i \quad (2.6)$$

and

$$T_{y^m}^m = Q' s_0 + \Psi' \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q\alpha s_0] + 2\Psi [r_0 - Q'(b^2 - s^2)s_0 - Q s s_0]. \quad (2.7)$$

A Douglas metric is a Finsler metric with a vanishing Douglas tensor. Let $F = \alpha\phi(s)$, $s := \beta/\alpha$ be a Douglas (α, β) - metric, From (2.4) and (2.5) we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} T_{y^m}^m y^i \right) = 0.$$

Then there exists a class of scalar functions $H_{jk}^i := H_{jk}^i(x)$ such that

$$T^i - \frac{1}{n+1} T_{y^m}^m y^i = H_{00}^i, \quad (2.8)$$

where $H_{00}^i := H_{jk}^i(x) y^j y^k$, T^i and $T_{y^m}^m$ are given by the relations (2.6) and (2.7), respectively.

A Finsler metric is called locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients of F are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a local scalar function on the tangent bundle TM of M . Such a coordinate system is called an adapted coordinate system. A system of PDEs that characterized dually flat Finsler metrics on an open subset in \mathbb{R}^n , can be found in [19]. In fact, we have the following theorem.

Theorem 2.3. [19] *A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is dually flat if and only if the following equation holds:*

$$[F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0.$$

In this case

$$H = \frac{1}{6} [F^2]_{x^m} y^m.$$

In [26], an equation is obtained that characterizes dually flat (α, β) -metrics.

Lemma 2.4. [26] *An (α, β) -metric $F = \alpha\phi(s)$, $s := \beta/\alpha$ is dually flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if*

$$\begin{aligned} & 3\alpha^2 a_{ml} G_\alpha^m + \alpha^3 Q(3s_{l0} - r_{l0}) - \alpha^2 \left(\frac{\partial(y_m G_\alpha^m)}{\partial y^l} + \alpha Q \frac{\partial(b_m G_\alpha^m)}{\partial y^l} \right) + \left\{ 2Q(y_m G_\alpha^m) \right. \\ & \left. + \Phi(\alpha r_{00} + 2(b_m \alpha - s y_m) G_\alpha^m) \right\} (\alpha b_l - s y_l) + Q\alpha(r_{00} + 2b_m G_\alpha^m) y_l = 0, \quad (2.9) \end{aligned}$$

where $r_{i0} := r_{ij} y^j$, $s_{i0} := s_{ij} y^j$, $y_i := a_{ij} y^j$, and

$$\Phi := \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}.$$

3. Kropina transformation of exponential (α, β) -metrics

In this section, we consider the Kropina transformation of exponential (α, β) -metric $F = \alpha \exp(s)$, i.e.

$$\bar{F} = \alpha \exp(2s)/s, \quad s := \beta/\alpha.$$

Since $\phi(s) = \exp(2s)/s$ must be a positive function, thus $s > 0$. One can see that when $s > 0$, we have the following lemma.

Lemma 3.1. $\bar{F} = \alpha \exp(2s)/s, \quad s := \beta/\alpha$, is a Finsler metric, if and only if $0 < \|\beta_x\|_\alpha < 1$

Proof. Let $\bar{F} = \alpha \exp(2s)/s, \quad s := \beta/\alpha$, is a Finsler metric, then from (2.1) we have

$$\frac{s^3 + 2b^2s^2 - 2b^2s + b^2 - 2s^4}{s^3} > 0.$$

For $s = b$, we get $0 < b < 1$. Thus $0 < \|\beta_x\|_\alpha < 1$. The converse is easy to prove. \square

It is easy to see that, the geodesic coefficients $\bar{F} = \alpha \exp(2s)/s, \quad s := \beta/\alpha$, are given by (2.2) with

$$\begin{aligned} Q &:= \frac{2s - 1}{2s(1 - s)}, \\ \Theta &:= \frac{s(5s - 4s^2 - 2)}{2[s^3(1 - 2s) + b^2(2s^2 - 2s + 1)]}, \\ \Psi &:= \frac{2s^2 - 2s + 1}{2[s^3(1 - 2s) + b^2(2s^2 - 2s + 1)]}. \end{aligned} \quad (3.1)$$

3.1. Proof of Theorem 1.1. Suppose that \bar{F} is locally projectively flat. From (2.3) we have

$$\begin{aligned} &2\beta(\alpha - \beta) \left[b^2\alpha^2((\alpha - \beta)^2 + \beta^2) + \beta^3(\alpha - 2\beta) \right] (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m \\ &+ \alpha^4 \left[(\alpha - \beta) \left[b^2\alpha^2(3b^2\beta(\alpha - \beta) - \alpha^2) + 4\beta^4 \right] + (b^2 - 1)\alpha^2\beta^3 \right] s_{l0} \\ &+ \left[\beta\alpha^2(2\beta^2(2\alpha - \beta) + \alpha^2(\alpha - 3\beta))r_{00} + \alpha^4(2\beta^2(3\alpha - 2\beta) \right. \\ &\quad \left. + \alpha^2(\alpha - 4\beta))s_0 \right] (\alpha^2 b_l - \beta y_l) = 0. \end{aligned} \quad (3.2)$$

From (3.2), we get

$$\begin{aligned} &2 \left[\alpha^2\beta b^2(\alpha^2 + \beta^2) + 3\beta^2(b^2\alpha^2 - \beta^2) \right] (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m \\ &+ \alpha^4 \left[4\beta^2(\beta^2 - b^2\alpha^2) - \alpha^2 b^2(\alpha^2 + 2\beta^2) \right] s_{l0} \\ &+ \alpha^2 \left[\beta(4\beta^2 + \alpha^2)r_{00} + \alpha^2(6\beta^2 + \alpha^2)s_0 \right] (\alpha^2 b_l - \beta y_l) = 0, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} & 2\beta^2[\beta^2(\alpha^2 + 2\beta^2) - b^2\alpha^2(3\alpha^2 + 2\beta^2)](a_{ml}\alpha^2 - y_my_l)G_\alpha^m \\ & \quad + \alpha^4\beta[4b^2\alpha^2(\beta^2 + \alpha^2) - \beta^2(\alpha^2 + 4\beta^2)]s_{l0} \\ & \quad - \beta\alpha^2[\beta(2\beta^2 + 3\alpha^2)r_{00} + 4\alpha^2(\beta^2 + \alpha^2)s_0](\alpha^2b_l - \beta y_l) = 0. \end{aligned} \quad (3.4)$$

From (3.3) one can see that $(s_0b_l - s_{l0}b^2)\alpha^8$ is divisible by β . Thus, there exist scalar functions $\nu := \nu_l(x)$ on M such that

$$s_0b_l - s_{l0}b^2 = \beta\nu_l, \quad (3.5)$$

for any $l := 1, \dots, n$.

Multiplying (3.5) by y^l , we get $\nu_l = s_l$ and then

$$s_{l0} = \frac{1}{b^2}\{s_0b_l - \beta s_l\}. \quad (3.6)$$

Contracting (3.3) by βb^l and (3.4) by b^l , yields

$$\begin{aligned} & 2\beta[\alpha^2\beta b^2(\alpha^2 + \beta^2) + 3\beta^2(b^2\alpha^2 - \beta^2)](\alpha^2b_m - \beta y_m)G_\alpha^m \\ & \quad + \alpha^4\beta[4\beta^2(\beta^2 - \alpha^2b^2) - \alpha^2b^2(\alpha^2 + 2\beta^2)]s_0 \\ & \quad + \alpha^2\beta[\beta(4\beta^2 + \alpha^2)r_{00} + \alpha^2(6\beta^2 + \alpha^2)s_0](\alpha^2b^2 - \beta^2) = 0, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} & 2\beta^2[\beta^2(\alpha^2 + 2\beta^2) - b^2\alpha^2(3\alpha^2 + 2\beta^2)](\alpha^2b_m - \beta y_m)G_\alpha^m \\ & \quad + \alpha^4\beta[4b^2\alpha^2(\beta^2 + \alpha^2) - \beta^2(\alpha^2 + 4\beta^2)]s_0 \\ & \quad - \beta\alpha^2[\beta(2\beta^2 + 3\alpha^2)r_{00} + 4\alpha^2(\beta^2 + \alpha^2)s_0](b^2\alpha^2 - \beta^2) = 0. \end{aligned} \quad (3.8)$$

(3.7)+(3.8) yields

$$2\beta^2(\beta^2 - \alpha^2)[(2b^2\alpha^2 - \beta^2)(\alpha^2b_m - \beta y_m)G_\alpha^m + \alpha^2(\alpha^2b^2 - \beta^2)r_{00} - \alpha^4\beta s_0] = 0.$$

Thus

$$(2b^2\alpha^2 - \beta^2)(b_m\alpha^2 - \beta y_m)G_\alpha^m = \alpha^2(\alpha^2\beta s_0 - (\alpha^2b^2 - \beta^2)r_{00}). \quad (3.9)$$

From (3.9) we see that $(\alpha^2b_m - \beta y_m)G_\alpha^m$ has the factor α^2 , i.e. there exists a function $\eta_2 := \eta_2(x, y)$ on TM such that

$$(\alpha^2b_m - \beta y_m)G_\alpha^m = \alpha^2\eta_2, \quad (3.10)$$

where $\eta_2(x, y)$ is a homogeneous polynomial of degree two with respect to y .

By substituting (3.10) in (3.9), we have

$$\alpha^2(2b^2\eta_2 + b^2r_{00} - 2\beta s_0) = \beta^2(\eta_2 + r_{00}).$$

Thus, there exists a scalar function $\gamma := \gamma(x)$ on M such that

$$2b^2\eta_2 + b^2r_{00} - 2\beta s_0 = \beta^2\gamma, \quad (3.11)$$

and

$$\eta_2 + r_{00} = \alpha^2\gamma. \quad (3.12)$$

From (3.11) and (3.12), we get

$$r_{00} = \frac{1}{b^2} \{ (2\alpha^2 b^2 - \beta^2) \gamma - 2\beta s_0 \}. \quad (3.13)$$

Substituting (3.13) back into (3.9), we have

$$(2\alpha^2 - \beta^2) \left[(\alpha^2 b_m - \beta y_m) G_\alpha^m + \frac{1}{b^2} \alpha^2 (\alpha^2 b^2 - \beta^2) \gamma \right] = \frac{1}{b^2} \alpha^2 \beta (3b^2 \alpha^2 - 2\beta^2) s_0.$$

Thus

$$s_0 = 0, \quad (3.14)$$

$$(\alpha^2 b_m - \beta y_m) G_\alpha^m = -\frac{\gamma}{b^2} \alpha^2 (\alpha^2 b^2 - \beta^2). \quad (3.15)$$

Using (3.6), (3.13) and (3.14), we infer

$$s_{10} = 0, \quad (3.16)$$

$$r_{00} = \frac{\gamma}{b^2} \{ 2\alpha^2 b^2 - \beta^2 \}. \quad (3.17)$$

Substituting (3.14), (3.15), (3.16) and (3.17) in (3.7), we deduce

$$\frac{1}{b^2} \alpha^2 \beta^4 (2\beta^2 - \alpha^2) (\alpha^2 b^2 - \beta^2) \gamma = 0. \quad (3.18)$$

Since

$$\frac{1}{b^2} \alpha^2 \beta^4 (2\beta^2 - \alpha^2) (\alpha^2 b^2 - \beta^2) \neq 0,$$

thus we get

$$\gamma = 0$$

and therefore

$$r_{00} = 0. \quad (3.19)$$

From (3.16) and (3.19), we obtained that β is parallel with respect to α .

Substituting (3.14), (3.16), and (3.18) into (3.3), we conclude that

$$(a_{ml} \alpha^2 - y_m y_l) G_\alpha^m = 0,$$

therefore α is projectively flat [13].

Now let α be projectively flat and β be parallel with respect to α . From (2.2) one can see that \bar{F} is locally projectively flat. The proof is complete. \square

In the proof of Theorem 1.3, for simplicity, we assume that $\lambda := \frac{1}{n+1}$.

3.2. Proof of Theorem 1.3. The proof of sufficiency is obvious. Therefore, we just need to prove the necessity conditions. If \bar{F} be a Douglas metric, then (2.8) holds. Plugging (3.1) into (2.8), we obtain

$$\frac{A_{11}^i \alpha^{11} + A_{10}^i \alpha^{10} + \cdots + A_1^i \alpha + A_0^i}{P_9 \alpha^9 + P_8 \alpha^8 + \cdots + P_1 \alpha + P_0} = H_{00}^i, \quad (3.20)$$

where

$$\begin{aligned} A_{11}^i &:= b^2(b^2 s_0^i - b^i s_0), \\ A_{10}^i &:= -6b^2 \beta (b^2 s_0^i - b^i s_0), \\ A_9^i &:= 16b^2 \beta^2 (b^2 s_0^i - b^i s_0) + 2\lambda b^2 \beta y^i (r_0 + s_0) - b^2 \beta b^i r_{00}, \\ A_8^i &:= \beta^3 (1 - 24b^2) (b^2 s_0^i - b^i s_0) - \lambda b^2 \beta^2 y^i (10r_0 + 13s_0) \\ &\quad + b^2 \beta^2 (\beta s_0^i + 5b^i r_{00}), \\ A_7^i &:= 2\beta^4 (10b^2 - 3) (b^2 s_0^i - b^i s_0) + 24\lambda b^2 \beta^3 y^i (r_0 + 2s_0) \\ &\quad - 6b^2 \beta^3 (\beta s_0^i + 2b^i r_{00}), \\ A_6^i &:= 2\beta^5 [(7 - 4b^2) (b^2 s_0^i - b^i s_0) + 7b^2 s_0^i] - \beta^3 [(1 - 16b^2) \beta b^i \\ &\quad + 3\lambda b^2 y^i] r_{00} + \lambda \beta^4 y^i [(1 - 14b^2) (2r_0 + 5s_0) - 4b^2 r_0], \end{aligned}$$

$$\begin{aligned} A_5^i &:= \beta^6 [-16(2b^2 s_0^i - b^i s_0) + s_0^i] + \beta^4 [(5 - 15b^2) \beta b^i + 15\lambda b^2 y^i] r_{00} \\ &\quad + 2\lambda \beta^5 y^i [-(5r_0 + 14s_0) + 6b^2 (2r_0 + 5s_0)], \\ A_4^i &:= 2\beta^7 [4(b^2 s_0^i - b^i s_0) - 3s_0^i] + \beta^5 r_{00} [2(2b^2 - 5) \beta b^i \\ &\quad + (3 - 26b^2) \lambda y^i] + 2\lambda \beta^6 y^i [(10r_0 + 29s_0) - 4b^2 (r_0 + 3s_0)], \\ A_3^i &:= -4\lambda \beta^7 y^i (5r_0 + 14s_0) + \lambda \beta^6 y^i r_{00} (22b^2 - 15) \\ &\quad + 2\beta^7 (5b^i r_{00} + 6\beta s_0^i), \\ A_2^i &:= 8\lambda \beta^8 y^i (r_0 + 3s_0) + 2\lambda \beta^7 y^i r_{00} (13 - 8b^2) - 4\beta^8 (r_{00} + 2\beta s_0^i), \\ A_1^i &:= -22\lambda y^i \beta^8 r_{00}, \\ A_0^i &:= 8\lambda y^i \beta^9 r_{00}, \end{aligned}$$

and

$$\begin{aligned} P_9 &:= -2b^4 \beta, & P_8 &:= 10b^4 \beta^2, \\ P_7 &:= -24b^4 \beta^3, & P_6 &:= -4b^2 \beta^4 (1 - 8b^2), \\ P_5 &:= 4b^2 \beta^5 (5 - 6b^2), & P_4 &:= -8b^2 \beta^6 (5 - b^2), \\ P_3 &:= -2\beta^7 (1 - 20b^2), & P_2 &:= 2\beta^8 (5 - 8b^2), \\ P_1 &:= -16\beta^9, & P_0 &:= 8\beta^{10}. \end{aligned}$$

Equation(3.20) is equivalent to

$$\begin{aligned} A_{11}^i \alpha^{11} + A_{10}^i \alpha^{10} + \cdots + A_1^i \alpha + A_0^i &= \\ H_{00}^i (P_9 \alpha^9 + P_8 \alpha^8 + \cdots + P_1 \alpha + P_0). \end{aligned} \quad (3.21)$$

Replacing y^i by $-y^i$ in (3.21) yields

$$\begin{aligned} -A_{11}^i \alpha^{11} + A_{10}^i \alpha^{10} + \cdots - A_1^i \alpha + A_0^i &= \\ H_{00}^i (-P_9 \alpha^9 + P_8 \alpha^8 + \cdots - P_1 \alpha + P_0). \end{aligned} \quad (3.22)$$

(3.21)+(3.22) implies that

$$\begin{aligned} A_{10}^i \alpha^{10} + A_8^i \alpha^8 + \cdots + A_2^i \alpha^2 + A_0^i &= \\ H_{00}^i (P_8 \alpha^8 + P_6 \alpha^6 + \cdots + P_2 \alpha^2 + P_0). \end{aligned} \quad (3.23)$$

Also, from (3.21) - (3.22) we have

$$\begin{aligned} A_{11}^i \alpha^{10} + A_9^i \alpha^8 + \cdots + A_3^i \alpha^2 + A_1^i &= \\ H_{00}^i (P_9 \alpha^8 + P_7 \alpha^6 + \cdots + P_3 \alpha^2 + P_1). \end{aligned} \quad (3.24)$$

From (3.23) and (3.24), we get

$$A_0^i - H_{00}^i P_0 := 8\beta^9 (\lambda y^i r_{00} - H_{00}^i \beta),$$

and

$$A_1^i - H_{00}^i P_1 := -2\beta^8 (11\lambda y^i r_{00} + 8\beta H_{00}^i),$$

have the factor α^2 . Therefore we obtained that r_{00} and H_{00}^i have the factor α^2 . Thus there exist scalar functions $\sigma := \sigma(x)$ and $\eta^i := \eta^i(x)$ on M such that

$$r_{00} = \sigma \alpha^2, \quad (3.25)$$

$$H_{00}^i = \eta^i \alpha^2, \quad (3.26)$$

where $i \in \{1, 2, \dots, n\}$. From (3.25), we have

$$r_0 = \beta \sigma. \quad (3.27)$$

On the other hand from (3.21), one can see that $A_{11}^i \alpha^{11} = b^2 (b^2 s_0^i - b^i s_0) \alpha^{11}$ has the factor β . Therefore there exist scalar functions $\xi := \xi^i(x)$ on M such that

$$b^2 s_0^i - b^i s_0 = \beta \xi^i, \quad (3.28)$$

where $i \in \{1, 2, \dots, n\}$. Multiplying (3.28) by y_i , we get $\xi^i y_i = s_0$, thus

$$\xi^i = s^i. \quad (3.29)$$

Substituting (3.29) in (3.28), we obtain

$$s_{ij} = \frac{1}{b^2} \{b_i s_j - b_j s_i\}. \quad (3.30)$$

Substituting (3.25), (3.26), (3.27) and (3.30) into (3.21), we get

$$B_9^i \alpha^9 + B_8^i \alpha^8 + \cdots + B_1^i \alpha + B_0^i = \eta^i (Q_9 \alpha^9 + Q_8 \alpha^8 + \cdots + Q_1 \alpha + Q_0), \quad (3.31)$$

where

$$\begin{aligned}
B_9^i &:= b^4(s^i + b^i\sigma), \\
B_8^i &:= -b^4\beta(6s^i + 5b^i\sigma), \\
B_7^i &:= 4b^4\beta^2(4s^i + 3b^i\sigma) - 2\lambda b^4y^i(\beta\sigma + s_0), \\
B_6^i &:= b^2\beta^2b^i[(1 - 16b^2)\beta\sigma - s_0] + 2(1 - 12b^2)b^2\beta^2s^i \\
&\quad + 13\lambda b^2\beta y^i(s_0 + \beta\sigma), \\
B_5^i &:= b^2\beta^3b^i[(12b^2 - 5) + 6s_0] + 4b^2(5b^2 - 3)\beta^4s^i \\
&\quad - \lambda b^4\beta^2y^i(39\beta\sigma + 42s_0), \\
B_4^i &:= 2b^2\beta^4b^i[(5 - 2b^2)\beta\sigma - 7s_0] - 5\lambda b^2\beta^3y^i(s_0 + \beta\sigma) \\
&\quad + 4b^2\beta^5s^i(7 - 2b^2) + 2\lambda b^4\beta^3y^i(29\beta\sigma + 35s_0), \\
B_3^i &:= \lambda b^2\beta^4y^i[4(7 - 15b^2)s_0 + \beta\sigma(25 - 46b^2)] \\
&\quad + \beta^5b^i s_0(1 + 16b^2 - 10\beta b^2\sigma) + (1 - 32b^2)\beta^6s^i, \\
B_2^i &:= 2\lambda b^2\beta^5y^i[(8b^2 - 23)\beta\sigma + (12b^2 - 29)s_0] \\
&\quad + 2\beta^6b^i[2b^2\beta\sigma + (3 - 2b^2)s_0] + 2(8b^2 - 3)\beta^7s^i, \\
B_1^i &:= 14\lambda b^2\beta^6y^i(3\beta\sigma + 4s_0) + 12\beta^7(\beta s^i - b^i s_0), \\
B_0^i &:= -8\lambda b^2\beta^7y^i(2\beta\sigma + 3s_0) - 8\beta^8(\beta s^i - b^i s_0),
\end{aligned}$$

and

$$\begin{aligned}
Q_9 &:= 2b^6, & Q_8 &:= -10b^6\beta, \\
Q_7 &:= 24b^6\beta^2, & Q_6 &:= -4b^4\beta^3(8b^2 - 1), \\
Q_5 &:= 4b^4\beta^4(6b^2 - 5), & Q_4 &:= -8b^4\beta^5(b^2 - 5), \\
Q_3 &:= -2b^2\beta^6(20b^2 - 1), & Q_2 &:= -2b^2\beta^7(8b^2 - 5), \\
Q_1 &:= 16b^2\beta^8, & Q_0 &:= -8b^2\beta^9.
\end{aligned}$$

Replacing y^i by $-y^i$ into (3.31) yields

$$B_9^i\alpha^9 - B_8^i\alpha^8 + \cdots + B_1^i\alpha - B_0^i = \eta^i(Q_9\alpha^9 - Q_8\alpha^8 + \cdots + Q_1\alpha - Q_0). \quad (3.32)$$

By adding (3.31) and (3.32), we have

$$\begin{aligned}
B_9^i\alpha^8 + B_7^i\alpha^6 + B_5^i\alpha^4 + B_3\alpha^2 + B_1^i = \\
\eta^i(Q_9\alpha^8 + Q_7\alpha^6 + Q_5\alpha^4 + Q_3\alpha^2 + Q_1). \quad (3.33)
\end{aligned}$$

From (3.33) one can see that

$$B_1^i - \eta^i Q_1 = \beta^6[7\lambda b^2y^i(8s_0 + 6\beta\sigma) + 12\beta(\beta s^i - b^i s_0) - 16b^2\beta\eta^i],$$

that has the factor α^2 , i.e. there exist scalar functions $\omega^i := \omega^i(x)$ on M such that

$$7\lambda b^2y^i(8s_0 + 6\beta\sigma) + 12\beta(\beta s^i - b^i s_0) - 16b^2\beta^2\eta^i = \alpha^2\omega^i, \quad (3.34)$$

where $i \in \{1, 2, \dots, n\}$. Multiplying (3.34) by b_i and y_i leads to

$$\beta[7\lambda b^2(8s_0 + 6\beta\sigma) - 12b^2s_0 - 16b^2\beta\eta^i b_i] = \alpha^2\omega^i b_i, \quad (3.35)$$

and

$$\alpha^2[7\lambda b^2(8s_0 + 6\beta\sigma) - \omega^i y_i] = 16b^2\beta^2\eta^i y_i, \quad (3.36)$$

respectively. From (3.35), we deduce

$$\omega^i b_i = 0, \quad (3.37)$$

$$7\lambda b^2(8s_0 + 6\beta\sigma) - 12b^2s_0 - 16b^2\beta\eta^i b_i = 0, \quad (3.38)$$

and from (3.36), we have

$$7\lambda b^2(8s_0 + 6\beta\sigma) - \omega^i y_i = 0. \quad (3.39)$$

From (3.39), we obtain

$$\eta_i = 0, \quad (3.40)$$

$$7\lambda b^2(8s_i + 6b_i\sigma) = \omega_i, \quad (3.41)$$

where

$$\eta_i := \eta^j a_{ji}, \quad \omega_i := \omega^j a_{ji}.$$

Multiplying (3.41) by b^i and using (3.37), we find

$$\sigma = 0. \quad (3.42)$$

Thus

$$r_{00} = 0, \quad (3.43)$$

also from (3.40) we conclude that

$$H_{00}^i = 0.$$

Substituting (3.40) and (3.42) into (3.38), we obtain

$$(14\lambda - 12)b^2\beta s_0 = 0.$$

Since $14\lambda - 12 \neq 0$, thus $s_0 = 0$, and then from (3.30), we have

$$s_{ij} = 0. \quad (3.44)$$

From (3.43) and (3.44), we have that β is parallel with respect to α . Thus the proof is complete. \square

3.3. Proof of Theorem 1.5. Suppose that \bar{F} is locally dually flat. From (2.9) we have

$$A_l \alpha^6 + B_l \alpha^5 + C_l \alpha^4 + D_l \alpha^3 + E_l \alpha^2 + M_l \alpha + N_l = 0, \quad (3.45)$$

where

$$\begin{aligned} A_l &:= \beta \frac{\partial b_m G_\alpha^m}{\partial y^l} + 6b_m G_\alpha^m b_l + \beta(r_{l0} - 3s_{l0}) + 3r_{00} b_l, \\ B_l &:= -2\beta^2 \frac{\partial b_m G_\alpha^m}{\partial y^l} - 16\beta b_m G_\alpha^m b_l - 8\beta r_{00} b_l - 2\beta^2(r_{l0} - 3s_{l0}), \\ C_l &:= -2\beta^2 \frac{\partial y_m G_\alpha^m}{\partial y^l} - 8\beta y_m G_\alpha^m b_l + 16\beta^2 b_m G_\alpha^m b_l - 8\beta b_m G_\alpha^m y_l \\ &\quad + 4\beta(2\beta b_l - y_l)r_{00} + 6a_{ml} G_\alpha^m \beta^2, \\ D_l &:= 2\beta^3 \frac{\partial y_m G_\alpha^m}{\partial y^l} + 20\beta^2 y_m G_\alpha^m b_l + 20\beta^2 b_m G_\alpha^m y_l - 6\beta^3 a_{ml} G_\alpha^m \\ &\quad + 10\beta^2 r_{00} y_l, \\ E_l &:= -16\beta^3 y_m G_\alpha^m b_l + 8\beta^2 y_m G_\alpha^m y_l - 16\beta^3 b_m G_\alpha^m y_l - 8\beta^3 r_{00} y_l, \\ M_l &:= -20y_m G_\alpha^m \beta^3 y_l, \\ N_l &:= 16y_m G_\alpha^m \beta^4 y_l. \end{aligned}$$

From (3.45) and by noticing that the sum of odd powers and even powers of α are zero respectively, we have

$$A_l \alpha^6 + C_l \alpha^4 + E_l \alpha^2 + N_l = 0, \quad (3.46)$$

$$B_l \alpha^4 + D_l \alpha^2 + M_l = 0. \quad (3.47)$$

Contracting (3.46) and (3.47) with b^l , we get

$$A\alpha^6 + C\alpha^4 + E\alpha^2 + N = 0, \quad (3.48)$$

$$B\alpha^4 + D\alpha^2 + M = 0. \quad (3.49)$$

where

$$\begin{aligned} A &:= \beta \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l + 6b^2 b_m G_\alpha^m + \beta(r_0 - 3s_0) + 3b^2 r_{00}, \\ B &:= -2\beta^2 \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l - 16\beta b^2 b_m G_\alpha^m - 8\beta b^2 r_{00} - 2\beta^2(r_0 - 3s_0), \\ C &:= -2\beta^2 \frac{\partial y_m G_\alpha^m}{\partial y^l} b^l - 8b^2 \beta y_m G_\alpha^m - 16b^2 \beta^2 G_\alpha^m b_m - 2\beta^2 b_m G_\alpha^m \\ &\quad + 4(2b^2 - 1)\beta^2 r_{00}, \end{aligned}$$

$$\begin{aligned}
D &:= 2\beta^3 \frac{\partial y^m G_\alpha^m}{\partial y^l} b^l + 20\beta^2 b^2 y_m G_\alpha^m + 14\beta^3 b_m G_\alpha^m + 10\beta^3 r_{00}, \\
E &:= -16b^2 \beta^3 y_m G_\alpha^m + 8\beta^3 y_m G_\alpha^m - 16\beta^4 b_m G_\alpha^m - 8\beta^4 r_{00}, \\
M &:= -20\beta^4 y_m G_\alpha^m, \\
N &:= 16\beta^5 y_m G_\alpha^m.
\end{aligned}$$

From (3.48) $\times 5 + (3.49) \times 4\beta$ we have

$$\begin{aligned}
&\left[5\beta \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l + 30b^2 b_m G_\alpha^m + 15b^2 r_{00} + 5\beta(r_0 - 3s_0) \right] \alpha^4 - \left[10\beta^2 \frac{\partial y_m G_\alpha^m}{\partial y^l} b^l \right. \\
&\quad + 8\beta^3 \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l + 40\beta b^2 y_m G_\alpha^m + 2(5 - 8b^2)\beta^2 b_m G_\alpha^m + 4(5 - 2b^2)\beta^2 r_{00} \\
&\quad \left. + 8\beta^3(r_0 - 3s_0) \right] \alpha^2 + 8\beta^4 \frac{\partial y_m G_\alpha^m}{\partial y^l} b^l + 40\beta^3 y_m G_\alpha^m - 24\beta^4 b_m G_\alpha^m = 0. \quad (3.50)
\end{aligned}$$

Rewriting (3.48) and (3.50) yields that

$$\begin{aligned}
\beta \alpha^6 \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l - 2\beta^2 \alpha^4 \frac{\partial y_m G_\alpha^m}{\partial y^l} b^l &= - \left[\beta(r_0 - 3s_0) + 3b^2(r_{00} + 2y_m G_\alpha^m) \right] \alpha^6 \\
&\quad + \left[2(1 - 8b^2)\beta^2 b_m G_\alpha^m + 8b^2 \beta^2 y_m G_\alpha^m + 4(1 - 2b^2)\beta^2 r_{00} \right] \alpha^4 \\
&\quad + \left[8\beta^4 r_{00} + 8\beta^3(2b^2 - 1)y_m G_\alpha^m + 16\beta^4 y_m G_\alpha^m \right] \alpha^2 - 16\beta^5 y_m G_\alpha^m, \quad (3.51)
\end{aligned}$$

and

$$\begin{aligned}
\beta \alpha^2 \left[5\alpha^2 - 8\beta^2 \right] \frac{\partial b_m G_\alpha^m}{\partial y^l} b^l - 2\beta^2 \left[5\alpha^2 - 4\beta^2 \right] \frac{\partial y_m G_\alpha^m}{\partial y^l} b^l &= - \left[5\beta(r_0 - 3s_0) \right. \\
&\quad \left. + 15b^2(r_{00} + 2b_m G_\alpha^m) \right] \alpha^4 + \left[40b^2 \beta y_m G_\alpha^m + 2(5 - 8b^2)\beta^2 b_m G_\alpha^m \right. \\
&\quad \left. + 4(5 - 2b^2)\beta^2 r_{00} + 8\beta^3(r_0 - 3s_0) \right] \alpha^2 - 40\beta^3 y_m G_\alpha^m + 24\beta^4 b_m G_\alpha^m. \quad (3.52)
\end{aligned}$$

From (3.51) $\times (5\alpha^2 - 8\beta^2) - (3.52) \times \alpha^4$, we have

$$\begin{aligned}
\beta^2 \alpha^4 \left[\frac{\partial y_m G_\alpha^m}{\partial y^l} b^l - 3b_m G_\alpha^m \right] &= \left[\beta^2 \alpha^2(1 + 8b^2) - b^2 \alpha^4 - 8\beta^4 \right] \\
&\quad \times \left(\alpha^2 r_{00} + 2\alpha^2 b_m G_\alpha^m - 2\beta y_m G_\alpha^m \right). \quad (3.53)
\end{aligned}$$

From (3.53), one can see that $\beta^2 \alpha^2(1 + 8b^2) - b^2 \alpha^4 - 8\beta^4$ can not be divided by α^4 , thus $\alpha^2 r_{00} + 2\alpha^2 b_m G_\alpha^m - 2\beta y_m G_\alpha^m$ is divided by α^4 , i.e. there exists a scalar function $\eta := \eta(x)$ on M such that

$$\alpha^2 r_{00} + 2\alpha^2 b_m G_\alpha^m - 2\beta y_m G_\alpha^m = \alpha^4 \eta. \quad (3.54)$$

On the other hand from (3.48), we have that $r_{00} + 2b_m G_\alpha^m$ is divided by β and therefore from (3.54) we have that $\eta = 0$. Thus

$$\frac{\partial y_m G_\alpha^m}{\partial y^l} b^l = 3b_m G_\alpha^m, \quad (3.55)$$

$$\alpha^2 r_{00} + 2\alpha^2 b_m G_\alpha^m = 2\beta y_m G_\alpha^m. \quad (3.56)$$

From (3.56) we conclude that

$$y_m G_\alpha^m = \alpha^2 \theta, \quad (3.57)$$

where $\theta = \theta_i(x)y^i$ is a one form on M .

Substituting (3.57) in (3.55), yields

$$b_m G_\alpha^m = \frac{1}{3} (2\beta\theta - \alpha^2 \theta_l b^l). \quad (3.58)$$

From (3.56) and (3.58), we have

$$r_{00} = \frac{2}{3} (\beta\theta - \alpha^2 \theta_l b^l). \quad (3.59)$$

From (3.59) one can see that

$$r_{10} = \frac{2}{3} \{ b_l \theta + \beta \theta_l - 2\theta_k b^k y_l \}, \quad (3.60)$$

By substituting (3.57)-(3.60) in (3.46) and (3.47), we have

$$6\beta a_{ml} G_\alpha^m = \alpha^2 b_l \theta + \alpha^2 \beta \theta_l + 3\alpha^2 s_{10} + 4\beta \theta y_l, \quad (3.61)$$

and

$$6\beta a_{ml} G_\alpha^m = 2\alpha^2 b_l \theta + 6\alpha^2 s_{10} + 4\beta \theta y_l, \quad (3.62)$$

respectively.

From (3.62) – (3.61), we deduce

$$s_{10} := \frac{1}{3} \{ \beta \theta_l - \theta b_l \}. \quad (3.63)$$

From (3.61), (3.62), and (3.63), we conclude

$$a_{ml} G_\alpha^m = \frac{1}{3} \{ \alpha^2 \theta_l + 2\theta y_l \},$$

thus

$$G_\alpha^m = \frac{1}{3} \{ \alpha^2 \theta^m + 2\theta y^l \},$$

therefore sufficient conditions are proved. The converse can be proved by a direct calculation. \square

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