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Research Paper

MATHEMATICAL MODEL FOR BINGHAM FLOW PROPERTIES OF BLOOD IN UNIFORM TAPERED TUBE

Arun Kumar Pandey and V. K. Chaubey

ABSTRACT. The stenosis and non-Newtonian property of the fluid in the blood flow represent the behavior of Herschel- Buckley fluid. In a tapered tube model all the vessels which carry blood towards the tissues are considered as long and its one end slowly tapering cones rather than cylinders. Since the blood flow consist of two regions in which one is central region, consist of concentrated blood cells and its behavior is non-Newtonian and other region is peripheral layer of plasma which represent the Newtonian behavior of fluid motion. In present paper, we have considered the Bingham fluid model and study the flow of blood in a uniform tapered tube and obtained conditions and its variation in various graphs for shear stress and pressure gradient.

Key Words: Newtonian Fluid; Blood Flow; wall shear stress; tapered vessel; Stenosis; Bingham Model.

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∗Address correspondence to V. K. Chaubey, E-mail: vkchaubey@outlook.com.

1. Introduction

Womersley [14] introduced the concept of a tapered tube model in which all the vessels which carry blood towards the tissues are considered as long and its one end slowly tapering cones rather than cylinders. Further Charm and Kurland [4] prove that if the flow of blood in a non-uniform capillary tubes then the experimental values are equivalent to anticipated value and if the flow of blood in cylindrical tubes then the experimental and anticipated values diverge unless it is validated in straight tubes.

Oka [9] considered non-Newtonian flow in tapered tubes and obtained the pressure gradient for Power law, Bingham body and Casson fluids. Further authors of the paper [5, 8] studied the case of uniform laminar flow of blood in tapered tube and obtained the analytical expressions for shear stress, pressure gradient, angular and axial velocity. Further various authors [1, 2, 3, 6, 7, 10, 11, 12, 13] studied the role of plasma peripheral layer in capillaries for the flow of blood.

In present paper we have considered an anomalous behavior of blood flow in uniform tapered tubes and studied its complex rheological characteristics. Further we considered Bingham blood model and obtained the analytical expressions for the shear stress and pressure gradient. Further in various graphs we represent the variation of shear stress and pressure gradient.

2. The Mathematical Model

In present paper we considered a incompressible viscous non-Newtonian fluid model in a tapered tube of circular cross-section for laminar flow which satisfy the following assumptions:
(i) The tapered angle is very small.
(ii) The flow of the motion is in z-direction and steady axisymmetric.
(iii) No forces act over the entire volume of the fluid.
(iv) Inertia term can be neglected because the motion is very slow.
(v) Pressure gradient is a function of axial co-ordinates only.
Further a geometrical representation of tapered vessel given in following figure:

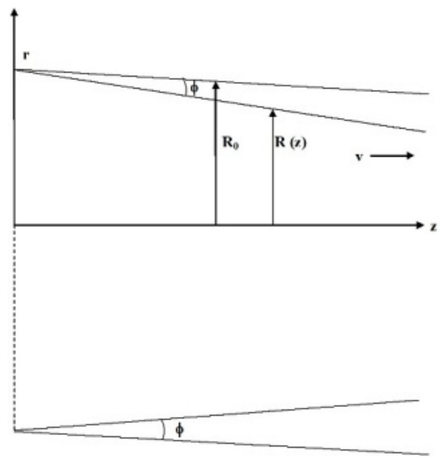

Figure 1: Geometry of the Vessel

The radius $R(z)$ of the tapered tube is given by

$$R(z) = R_\theta - z\tan\phi$$

where z is the axis of the tapered tube, ϕ is the tapered angle and R_θ is radius of the tube at $z = 0$.

The Governing equations

The laminar flow problem of an incompressible fluid in a cylindrical co-ordinate system (r, θ, z) are given by

Continuity equation

$$\frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{2.1}$$

Momentum equations

$$\rho\frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}(2\mu\frac{\partial v_z}{\partial z}) + \frac{1}{r}\frac{\partial}{\partial \theta}[\mu(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z})] + \frac{\partial}{\partial r}[\mu(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r})] + \frac{\mu}{r}(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}) \tag{2.2}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + v_z\frac{\partial}{\partial z} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}$ and $'p'$ is the pressure.

By using second assumptions of the model, we have

$$\frac{\partial}{\partial t} = 0, \quad \frac{\partial}{\partial \theta} = 0, \quad v_r = v_\theta = 0, \quad v_z = v(r) \tag{2.3}$$

where v is the axial velocity

Mathematical Analysis

Using equation (3) in equation (2), the equation of motion and continuity in a steady viscous incompressible laminar flow under any force acting on the entire volume of fluid is given

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{2.4}$$

$$0 = \frac{\partial p}{\partial r} \tag{2.5}$$

$$\frac{\partial v}{\partial z} = 0 \tag{2.6}$$

where $\tau_{rz} = (\mu\frac{\partial v}{\partial r})$ the shear stress in the z-direction and normal to $'r'$.

Constitutive equation:

The constitutive equation for the shear stress τ and strain rate $\dot{\gamma}$ is given by

$$\tau = \tau_0 + \mu\dot{\gamma}; \quad \tau \geq \tau_0, \qquad and \quad \dot{\gamma} = 0; \quad \tau \leq \tau_0 \tag{2.7}$$

where τ_0 is the yield stress, μ the coefficient of viscosity and $\dot{\gamma}$ the shear strain rate.

The Boundary conditions

The appropriate boundary conditions are given by

$$v = 0 \;\; at \quad r = R(z) \tag{2.8}$$

$$\tau_{rz} = \tau_w \quad at \quad r = R(z) \tag{2.9}$$

$$v = v_p \quad at \quad r = R_p \tag{2.10}$$

$$\tau_{rz} \;\; is \;\; finite \;\; at \;\; r = 0. \tag{2.11}$$

where R_p and v_p is the plug radius and plug velocity respectively.

The Solution for velocities, Volume flow rate and wall shear stress

Velocities

Now integrating equation (4) and using the condition (11), we have

$$\tau_{rz} = \frac{r}{2}\frac{\partial p}{\partial z} \tag{2.12}$$

Again using equation (7) and (12), we have

$$\frac{dv}{dr} = \frac{1}{\mu}(\frac{\partial p}{\partial z}\frac{r}{2} - \tau_0); \quad R_p \le r \le R(z) \tag{2.13}$$

$$\frac{dv_p}{dr} = 0; \quad 0 \le r \le R_p \tag{2.14}$$

Thus we say that, the plug flow exists whenever the shear stress does not exceed. So from equations (13), (14) and the conditions from (8) to (10), we have

$$v = \frac{\tau_w(z)}{2\mu}R(z)[1 - \frac{r^2}{R^2(z)} - 2\beta(1 - \frac{r}{R(z)})] \tag{2.15}$$

$$v_p = \frac{\tau_w(z)}{2\mu}R(z)(1-\beta)^2 \tag{2.16}$$

where $\beta = \frac{\tau_0}{\tau_w(z)}$.

Volume flow rate and wall shear stress

If V be the volume flow rate, then it is given by

$$V = V_1 + V_2 \tag{2.17}$$

where V_1 and V_2 are given by

$$V_1 = \int_0^{R_p} 2\pi v_p r dr = \pi v_p R_p^2 \tag{2.18}$$

$$V_2 = \int_{R_p}^{R(z)} 2\pi v r dr \tag{2.19}$$

Now substituting the values of v_p and v from equations (15) and (16) in equations (18) and (19), we obtain

$$V_1 = \frac{\pi}{2}\frac{\tau_w(z)}{\mu}R^3(z)(1-\beta)^2 \tag{2.20}$$

$$V_2 = \frac{\pi}{2\mu}\tau_w(z)R^3(z)[\frac{1}{2} - \frac{2}{3}\beta - \beta^2 + 2\beta^3 - \frac{5}{6}\beta^4] \tag{2.21}$$

Using equations (20) and (21) in equation (17), we have

$$V = \frac{\pi}{4\mu}\tau_w(z)R^3(z)(1 - \frac{4}{3}\beta) \tag{2.22}$$

higher order of β are neglected.
By using equation (12), (22) and (9), the pressure gradient is given by

$$\frac{\partial p}{\partial z} = \frac{8\mu V}{\pi R^4(z)}(1 + \frac{4}{3}\beta) \tag{2.23}$$

From equation (22) we have the shear stress at the wall as

$$\tau_w(z) = \frac{4\mu V}{\pi R^3(z)}(1 + \frac{4}{3}\beta) \tag{2.24}$$

Now using equations (23) and (24), we have

$$\tau_w(z) = \frac{R(z)}{2}\frac{\partial p}{\partial z} \tag{2.25}$$

From the equations (23) and (24) it is clear that if pressure and the shear stress at the wall will be increases whenever radius R_z of the tapered tube is decreases.

3. Results and Discussion:

The pressure gradient and shear stress on the wall are given by equations (23) and (25) respectively, from which we conclude that whenever the radius of the tapered tube decreases the pressure gradient and shear stress on the wall will increase. Thus the pressure gradient is not considered constant.

To represent the variation of pressure gradient and shear stress at the wall we considered the radius of tapered vessel $R_\theta = 100\mu m$ and by use of equations (23) and (24) we obtained both the variation for the flow rate 0.02 to 0.10 cc/sec in various tapered angles ($1^0 \leq \theta \leq 2^0$) and the suspension concentrations 20%, 30% and 40% .

Figures 2, 3 and 4 shows the variation of flow rate V for the pressure gradient for various values of suspension concentration, tapered angle and axial distance. From the variation we can conclude that the pressure gradient will be increases with increase in suspension concentration, tapered angle and axial distance.

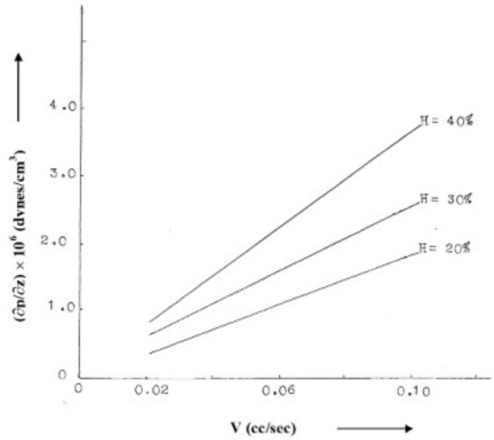

Figure 2: Variation of the pressure gradient with flow rate for various suspension concentrations for fixed value of z=0.10 cm, $\phi = 1.4^0$, R_θ=0.01 cm.

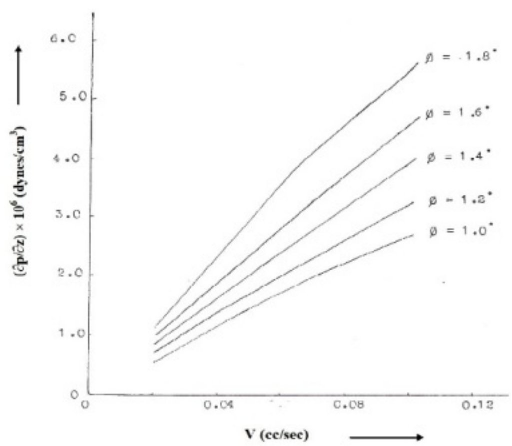

Figure 3: Variation of flow rate for the pressure gradient various tapered angles for fixed value of H=40%, $R_\theta = 0.01cm$, z=0.10 cm.

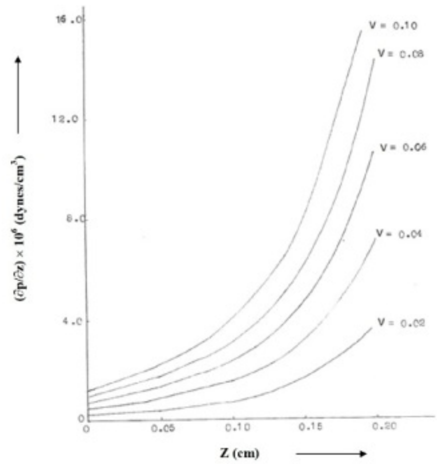

Figure 4 : Variation of the pressure gradient with axial distance for different flow rate for fixed value of $R_\theta = 0.01cm$, H=40%, ϕ=1.4^0.

Now the Figures 5, 6 and 7 show the variation of flow rate V for the shear stress at the wall with respect to various values of tapered angles, axial distance and suspension concentration.

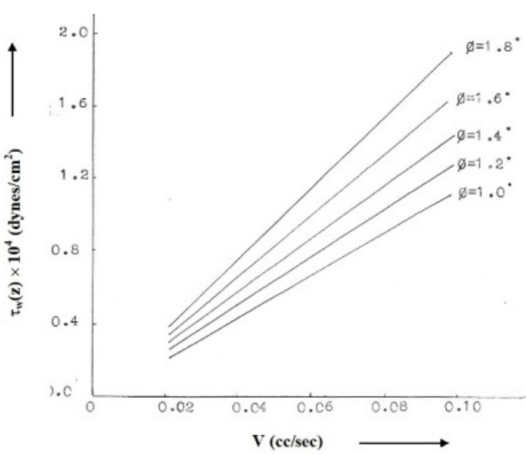

Figure 5: Variation of flow rate for the wall shear stress with various tapered angles for fixed value of H=40%, $R_\theta = 0.01cm$, z=0.10 cm.

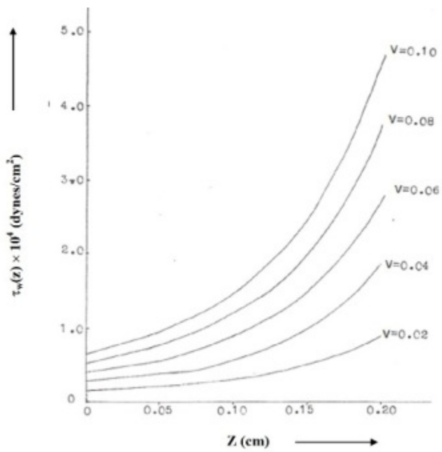

Figure 6: Variation of wall shear stress with various axial distance for different flow rate for fixed value of H=40%, $\phi = 1.4^0$, $R_\theta = 0.01cm$.

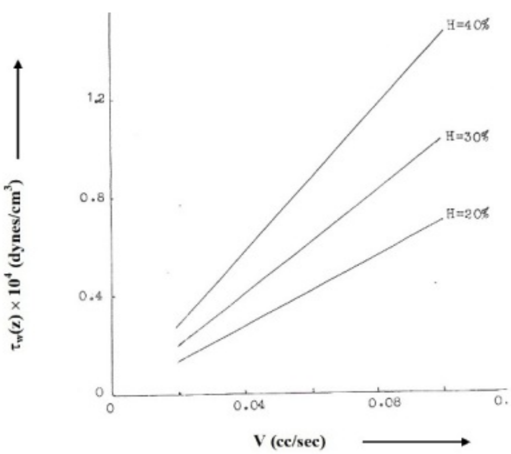

Figure 7 : Variation of flow rate for wall shear stress with various concentration for fixed value of ϕ=1.4^0, z=0.1 cm, $R_\theta = 0.01cm$.

From above variation we can conclude that the wall shear stress will be increases with suspension concentration and tapered angles. Again $\tau_w z$ is an increasing function for the axial distance, we can conclude that the shear stress at any point of the tapered tube can be calculated for any given axial distance. These results are very useful for understanding the flow in vascular fluid mechanics.

Now from figures 8, 9 and 10 represents variation of flow rate V for the pressure gradient in case of Newtonian fluid $\beta = 0$ for various values of suspension concentration, tapered angle and axial distance. Since we know that in Bingham Fluid Model, the values of pressure gradient are less. From these Figures we can conclude that the same trends for pressure gradient are obtained in Bingham Fluids.

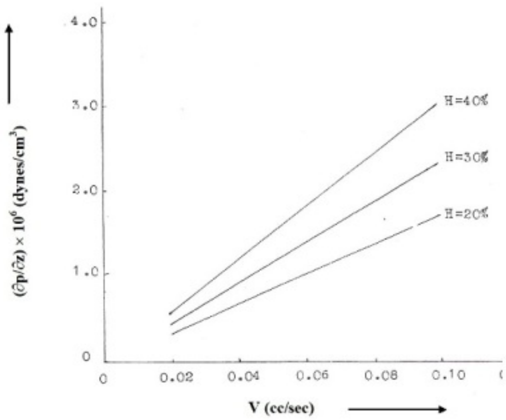

Figure 8: Variation of flow rate for the pressure gradient with various suspension concentration in Newtonian fluid for fixed value of $\phi = 1.4^0$, z=0.10 cm, $R_\theta = 0.01cm$, $\beta = 0$.

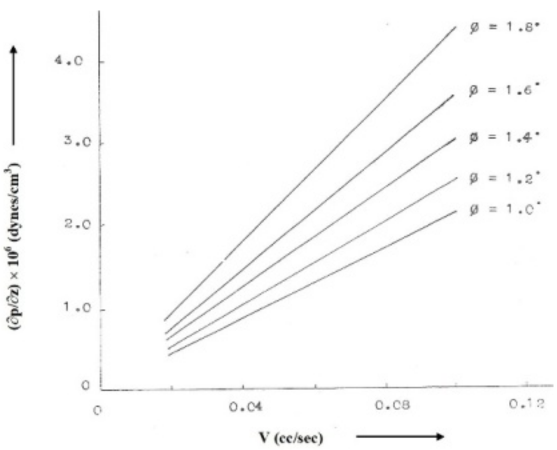

Figure 9: Variation of flow rate for the pressure gradient with various tapered angles in Newtonian fluid for fixed value of H=40%, $R_\theta = 0.01cm$, z=0.10 cm, $\beta = 0$.

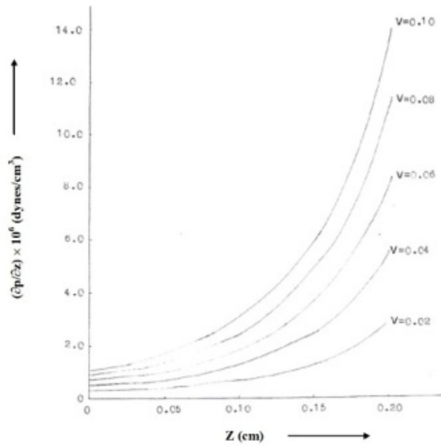

Figure 10 : Variation of flow rate for the pressure gradient with various axial distance in Newtonian fluid for fixed value of H=40%, $R_\theta = 0.01cm$, $\phi = 1.4^0$, $\beta = 0$.

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Arun Kumar Chaubey
Department of Mathematics, Buddha Institute of Technology, GIDA, Gorakhpur (U.P.)-273209, India
Email: pandeyarun151281@gmail.com

V. K. Chaubey
Department of Mathematics, North Eastern Hill University, Shillong-793022, India
Email: vkchaubey@outlook.com