A NOTE ON QUASI-HEMI SLANT SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLDS

Shamsur Rahman ⓘD and Amit Kumar Rai

ABSTRACT. Here our main objective is to introduce the notion of quasi hemi-slant submanifolds as a generalized case of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds of contact metric manifolds. We mainly focus on quasi hemi-slant submanifold of nearly trans-Sasakian manifold. During this manner, we tend to study and investigate integrability of distributions which are concerned in the definition of quasi hemi-slant submanifold of nearly trans-Sasakian manifold. Moreover, we tend to get necessary and sufficient conditions for quasi hemi-slant submanifold of nearly trans-Sasakian manifold to be totally geodesic for such manifolds.

1. INTRODUCTION

The concept of slant submanifolds of almost Hermitian manifolds has been studied by B.Y. Chen [6], and also studied on natural generalization of holomorphic immersions and totally real immersions and many more [6], [5]. A. Lotta [2] introduced and studied slant immersions of a Riemannian manifold into almost contact metric manifold. The Lorentzian

para-Sasakian manifolds were defined K. Matsumoto [9]. I. Mihai and R. Rosca [7] are also studied.

K. Matsumoto and I. Mihai [10] has defined and studied Lorentzian para-Sasakian manifolds. Later, many articles have appeared exploring the generalization of semi-slant submanifold, pseudo-slant submanifold, bi-slant submanifold and hemi-slant submanifold etc., in known differentiable manifolds [14], [15], [16].

2. Preliminaries

If $\bar{\mathcal{M}}$ is an $(2n + 1)$- dimensional almost contact manifold, endowed with structure $(\phi, \xi, \upsilon, <, >)$, then we obtain

$$(2.1) \qquad \phi^2 = \upsilon \otimes \xi - I, \quad 1 = \upsilon(\xi)$$

$$\phi\xi = 0, \quad 0 = \upsilon(\phi) \quad and \quad rank(\phi) = 2n$$

$$(2.2) \qquad < \phi\mathcal{X}, \phi\mathcal{Y} >=< \mathcal{X}, \mathcal{Y} > -\upsilon(\mathcal{X})\upsilon(\mathcal{Y}),$$

$$(2.3) \qquad \upsilon(\mathcal{X}) =< \mathcal{X}, \xi > \quad and \quad - < \mathcal{X}, \phi\mathcal{Y} >=< \phi\mathcal{X}, \mathcal{Y} >$$

where $\mathcal{X}$ and $\mathcal{Y}$ are vector fields on $\mathcal{M}$ and if the almost complex structure $\mathcal{J}$ on the product manifold $\bar{\mathcal{M}} \times \mathcal{R}$ satisfies

$$(2.4) \qquad \mathcal{J}(\mathcal{X}, fd/dt) = (\phi\mathcal{X} - f\xi, \upsilon(\mathcal{X})d/dt),$$

then the almost contact structure (ϕ, ξ, υ) has said to be normal. For trans-Sasakian manifold, the following conditions are equivalent

$$(\bar{\nabla}_\mathcal{X}\phi)\mathcal{Y} = -\mu\{\upsilon(\mathcal{Y})\mathcal{X} - < \mathcal{X}, \mathcal{Y} > \xi\} - \rho\{\upsilon(\mathcal{Y})\phi\mathcal{X} + < \mathcal{X}, \phi\mathcal{Y} >\}$$

$$(2.5) \qquad \bar{\nabla}_\mathcal{X}\xi = -\mu\phi\mathcal{X} - \rho\phi^2\mathcal{X}$$

$$(\bar{\nabla}_\mathcal{X}\upsilon)\mathcal{Y} = -\mu < \phi\mathcal{X}, \mathcal{Y} > +\rho < \phi\mathcal{X}, \phi\mathcal{Y} > \quad and \quad \bar{\nabla}_\xi\phi = 0.$$

It is nearly trans-Sasakian manifolds if

$$(2.6) \qquad (\bar{\nabla}_\mathcal{Y}\phi)\mathcal{X} + (\bar{\nabla}_\mathcal{X}\phi)\mathcal{Y} \;=\; -\rho\{\upsilon(\mathcal{X})\phi\mathcal{Y} + \upsilon(\mathcal{Y})\phi\mathcal{X}\}$$
$$-\mu\{\upsilon(\mathcal{Y})\mathcal{X} + \upsilon(\mathcal{X})\mathcal{Y}$$
$$-2 < \mathcal{X}, \mathcal{Y} > \xi\},$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ where $\bar{\nabla}$ denotes Riemannian connection with respect to $<, >$.

Now, suppose $\mathcal{M}$ be a submanifold of a contact Lorentzian metric manifold $\bar{\mathcal{M}}$ with the induced metric $<, >$ and ξ be tangent to $\mathcal{M}$. Also suppose ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle

$T\mathcal{M}$ and the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively. Then the Gauss-Weingarten formulas are given by

$$(2.7) \qquad\qquad \bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y}$$

$$(2.8) \qquad\qquad \bar{\nabla}_{\mathcal{X}}\lambda = -\Lambda_{\lambda}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}\lambda,$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and any vector filed λ normal to $\mathcal{M}$, where σ and Λ_{λ} are the second fundamental form and the shape operator for the immersion of $\mathcal{M}$ into $\bar{\mathcal{M}}$. The second fundamental form σ and shape operator Λ_{λ} are related by

$$(2.9) \qquad\qquad < \sigma(\mathcal{X}, \mathcal{Y}), \lambda >=< \Lambda_{\lambda}\mathcal{X}, \mathcal{Y} >,$$

for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$, we can write

$$(2.10) \qquad\qquad \phi\mathcal{X} = T\mathcal{X} + N\mathcal{X}$$

$$(2.11) \qquad\qquad \phi\lambda = t\lambda + s\lambda,$$

where $T\mathcal{X}$ and $t\lambda$ are the tangential components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively, where as $N\mathcal{X}$ and $\mathcal{F}\lambda$ are the normal components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively. Thus by using (2.10) and (2.11), we can obtain

$$(2.12) \qquad\qquad (\nabla_{\mathcal{X}}T)\mathcal{Y} - T(\nabla_{\mathcal{X}}\mathcal{Y}) = (\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y}$$

$$(\nabla^{\perp}_{\mathcal{X}}N)\mathcal{Y} - N(\nabla_{\mathcal{X}}\mathcal{Y}) = (\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y}$$

$$(2.13) \qquad\qquad (\nabla_{\mathcal{X}}t)\lambda - t(\nabla^{\perp}_{\mathcal{X}})\lambda = (\bar{\nabla}_{\mathcal{X}}t)\lambda,$$

$$(\nabla^{\perp}_{\mathcal{X}}s)\lambda - s(\nabla^{\perp}_{\mathcal{X}}\lambda) = (\bar{\nabla}_{\mathcal{X}}s)\lambda$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$. The mean curvature vector σ of $\mathcal{M}$ is given by

$$(2.14) \qquad\qquad \mathcal{H} = \frac{1}{m}trace(\sigma) = \frac{1}{m}\sum_{i=1}^{m}\sigma(\varepsilon_i, \varepsilon_i),$$

where m is the dimension of $\mathcal{M}$ and $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$ is a local orthonormal frame of $\mathcal{M}$. A submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is said to be totally umbilical if

$$(2.15) \qquad\qquad < \mathcal{X}, \mathcal{Y} > \mathcal{H} = \sigma(\mathcal{X}, \mathcal{Y}),$$

where σ is the mean curvature vector. A submanifold $\mathcal{M}$ is said to be totally geodesic, if $\sigma(\mathcal{X}, \mathcal{Y}) = 0$. For all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and $\mathcal{M}$ is said to be minimal if $\mathcal{H} = 0$.

Now, $\mathcal{P}_{\mathcal{X}}\mathcal{Y}$ and $\mathcal{F}_{\mathcal{X}}\mathcal{Y}$ are the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y}$, then we decompose

$$(2.16) \qquad \mathcal{F}_{\mathcal{X}}\mathcal{Y} + \mathcal{P}_{\mathcal{X}}\mathcal{Y} = (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y},$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$. Thus, we can obtain

$$(2.17) \qquad \mathcal{P}_{\mathcal{X}}\mathcal{Y} = -t\sigma(\mathcal{X}, \mathcal{Y}) - \Lambda_{N\mathcal{Y}}\mathcal{X} + (\nabla_{\mathcal{X}}T)\mathcal{Y}$$

and

$$(2.18) \qquad \mathcal{F}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, T\mathcal{Y}) - \mathcal{F}\sigma(\mathcal{X}, \mathcal{Y}) + (\nabla_{\mathcal{X}}N)\mathcal{Y}.$$

Similarly, $\mathcal{P}_{\mathcal{X}}\lambda$ and $\mathcal{F}_{\mathcal{X}}\lambda$ are the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}}\phi)\lambda$, respectively, then we infer

$$(2.19) \qquad \mathcal{P}_{\mathcal{X}}\lambda = (\nabla_{\mathcal{X}}t)\lambda - \Lambda_{\mathcal{F}\lambda}\mathcal{X} + T\Lambda_{\lambda}\mathcal{X}$$

and

$$(2.20) \qquad \mathcal{F}_{\mathcal{X}}\lambda = (\nabla_{\mathcal{X}}\mathcal{F})\lambda + \sigma(t\lambda, \mathcal{X}) + N\Lambda_{\lambda}\mathcal{X},$$

for the vector field λ normal to $\mathcal{M}$. By using (2.6), we deduce

$$(2.21) \quad (\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} \;=\; -(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} - \rho\{v(\mathcal{X})\phi\mathcal{Y} + v(\mathcal{Y})\phi\mathcal{X}\}$$
$$-\mu\{v(\mathcal{Y})\mathcal{X} + v(\mathcal{X})\mathcal{Y} - 2 <\mathcal{X}, \mathcal{Y}> \xi\}$$

and

$$(\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} + (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = -\phi\bar{\nabla}_{\mathcal{Y}}\mathcal{X} + \bar{\nabla}_{\mathcal{Y}}\phi\mathcal{X} - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y} + \bar{\nabla}_{\mathcal{X}}\phi\mathcal{Y}.$$

From (2.7), (2.8), (2.10) and (2.11), we get

$$(\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} \;=\; -\phi(\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y}) - \phi(\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{Y}}\mathcal{X})$$
$$-(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} + \bar{\nabla}_{\mathcal{X}}N\mathcal{Y} + \bar{\nabla}_{\mathcal{Y}}N\mathcal{X} + \bar{\nabla}_{\mathcal{X}}T\mathcal{Y} + \bar{\nabla}_{\mathcal{Y}}T\mathcal{X}$$

$$(2.22) \quad \nabla_{\mathcal{X}}T\mathcal{Y} \;+\; \sigma(\mathcal{X}, T\mathcal{Y}) - \Lambda_{N\mathcal{Y}}\mathcal{X} + \nabla_{\mathcal{X}}^{\perp}N\mathcal{Y}$$
$$-T\nabla_{\mathcal{X}}\mathcal{Y} - N\nabla_{\mathcal{X}}\mathcal{Y} - 2t\sigma(\mathcal{X}, \mathcal{Y})$$
$$-2s\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{Y}}T\mathcal{X} + \sigma(\mathcal{Y}, T\mathcal{X}) - \Lambda_{N\mathcal{X}}\mathcal{Y}$$
$$+\nabla_{\mathcal{Y}}^{\perp}N\mathcal{X} - T\nabla_{\mathcal{Y}}\mathcal{X} - N\nabla_{\mathcal{Y}}\mathcal{X} = 0.$$

Then using (2.21) and (2.22), we deduce

$$(2.23) \quad (\nabla_{\mathcal{Y}}T)\mathcal{X} \;=\; -(\nabla_{\mathcal{X}}T)\mathcal{Y} + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} - 2t\sigma(\mathcal{X}, \mathcal{Y})$$
$$-\mu\{v(\mathcal{X})\mathcal{Y} + v(\mathcal{Y})\mathcal{X} - 2 <\mathcal{X}, \mathcal{Y}> \xi\}$$
$$+\rho\{v(\mathcal{X})T\mathcal{Y} + v(\mathcal{Y})T\mathcal{X}\}$$

$$(2.24) \quad (\nabla_{\mathcal{Y}}N)\mathcal{X} \;=\; -(\nabla_{\mathcal{X}}N)\mathcal{Y} - \sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y})$$
$$+2f\sigma(\mathcal{X}, \mathcal{Y}) + \rho\{v(\mathcal{Y})N\mathcal{X} + v(\mathcal{X})N\mathcal{Y}\}.$$

Take $\mathcal{Y} = \xi$ in (2.6) and by using (2.2), (2.7) and (2.8), we infer

(2.25) $T[\mathcal{X}, \xi] = \rho T\mathcal{X} - \mu\phi^2\mathcal{X} - 2t\sigma(\mathcal{X}, \xi) + (\nabla_\xi T)\mathcal{X} - T\nabla_\xi\mathcal{X} - \Lambda_{N\mathcal{X}}\xi$

(2.26) $N[\mathcal{X}, \xi] = (\nabla_\xi N)\mathcal{X} - N\nabla_\xi\mathcal{X} - 2f\sigma(\mathcal{X}, \xi) + \sigma(T\mathcal{X}, \xi) + \rho N\mathcal{X}.$

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T_\mathcal{X}\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$ and carrying a Riemannian manifold $\mathcal{M}$ isometrically absorbed in an almost contact metric manifold $\bar{\mathcal{M}}$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is anti-invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T_\mathcal{X}^\perp\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$.

If ξ is tangential in $\mathcal{M}$ for a submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ then, the submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is slant for each non zero vector $\mathcal{X}$ tangent to $\mathcal{M}$ at $\mathcal{X} \in M$ such that $\mathcal{X}$ is linearly independent to $\xi_\mathcal{X}$, the angle $\theta(\mathcal{X})$ between $\phi\mathcal{X}$ and $T_\mathcal{X}\mathcal{M}$ is constant i.e. it does not depend on the choice of the point $\mathcal{X} \in M$ and $\mathcal{X} \in T_\mathcal{X}\mathcal{M} - \{\xi\}$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold $\mathcal{M}$ is proper slant submanifold for neither $\theta = 0$ nor $\theta = \pi/2$. Here $T\mathcal{M} = \mathcal{D}_\theta \oplus \{\xi\}$, where $\mathcal{D}_\theta$ is slant distribution with slant angle θ.

If $\theta = 0$, then slant submanifolds is said to be an invariant submanifolds and if $\theta = \pi/2$, then the slant submanifolds is said to be anti-invariant submanifolds.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-invariant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\},$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}^\perp$ is anti -invariant i.e. $\phi\mathcal{D}^\perp \subset (T^\perp\mathcal{M})$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-slant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}_\theta$ on $\mathcal{M}$ such that

$$TM = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \{\xi\},$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}_\theta$ is slant with slant angle θ here the angle θ is called semi-slant angle.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is hemi-slant if there exist two orthogonal complementary distributions $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that

$$TM = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \{\xi\},$$

where $\mathcal{D}^\perp$ is anti- invariant i.e. $\phi\mathcal{D}^\perp \subset (T^\perp\mathcal{M})$ and $\mathcal{D}_\theta$ is slant with slant angle θ here the angle θ is hemi-slant angle.

3. Quasi hemi-slant submanifolds of a nearly trans-Sasakian manifolds

Studying the existence of quasi hemi-slant submanifolds in a nearly trans-Sasakian manifolds is the goal of this section.

We say that $\mathcal{M}$ is quasi hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$, if there exist three orthogonal complementary distributions $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that
(a) TM admits the orthogonal direct decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \{\xi\}, \quad \xi \in \Gamma(\mathcal{D}_\theta)$$

(b) $\phi\mathcal{D} = \mathcal{D}$
(c) $\phi\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$.
(d) The distribution $\mathcal{D}_\theta$ is a slant with slant constant angle θ, where $\theta =$ slant angle.
In this case, θ is said to be quasi hemi- slant angle of $\mathcal{M}$. If the dimension of distributions $\mathcal{D}, \mathcal{D}_\theta$ and $\mathcal{D}^\perp$ are m_1, m_2 and m_3 respectively, then
(a) $\mathcal{M}$ is a hemi-slant submanifold for $m_1 = 0$.
(b) $\mathcal{M}$ is a semi-invariant submanifold for $m_2 = 0$.
(c) $\mathcal{M}$ is a semi-slant submanifold for $m_3 = 0$.
The quasi hemi-slant submanifold $\mathcal{M}$ is proper if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_\theta \neq \{0\}$, $\mathcal{D}^\perp = \{0\}$ and $\theta \neq 0, \pi/2$.

It represents that quasi hemi-slant submanifols is a generalization of invariant, anti-invariant, semi-invarint, slant, hemi-slant, semi-slant submanifolds.

It is clear from definition that if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_\theta \neq \{0\}$ and $\mathcal{D}^\perp = \{0\}$, then $dim\mathcal{D} \geq 2, dim\mathcal{D}_\theta \geq 2$ and $\mathcal{D}^\perp \geq 1$. So for proper quasi hemi slant manifold $\mathcal{M}$, the $dim\mathcal{M} \geq 6$.

Suppose $\mathcal{M}$ be a quasi hemi-slant submanifold of Sasakian manifold $\bar{\mathcal{M}}$ and the projections on $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively, then for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$(3.1) \qquad \mathcal{X} = \mathcal{RX} + \mathcal{QX} + \mathcal{PX} + \upsilon(\mathcal{X})\xi.$$

Now put

$$(3.2) \qquad T\mathcal{X} + N\mathcal{X} = \phi\mathcal{X},$$

where $T\mathcal{X}$ and $N\mathcal{X}$ are tangential and normal part of $\phi\mathcal{X}$ on M. From (3.1) and (3.2), we derive

$$(3.3) \qquad \phi\mathcal{X} = N\mathcal{RX} + T\mathcal{RX} + N\mathcal{QX} + T\mathcal{QX} + N\mathcal{PX} + T\mathcal{PX}.$$

As $\phi\mathcal{D} = \mathcal{D}$ and $\phi\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain $N\mathcal{PX} = 0$, and $T\mathcal{RX} = 0$ and

$$(3.4) \qquad \phi\mathcal{X} = N\mathcal{RX} + N\mathcal{QX} + T\mathcal{QX} + T\mathcal{PX}.$$

For all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$T\mathcal{X} = T\mathcal{PX} + T\mathcal{QX}$$

and

$$N\mathcal{X} = N\mathcal{QX} + N\mathcal{RX}.$$

Using (3.4) we deduce the following decomposition,

$$(3.5) \qquad \phi(T\mathcal{M}) = \mathcal{D} \oplus T\mathcal{D}_\theta \oplus N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp.$$

As $N\mathcal{D}_\theta \subseteq T^\perp\mathcal{M}$ and $N\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain

$$(3.6) \qquad T^\perp\mathcal{M} = N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp \oplus \kappa,$$

where κ denotes the orthogonal component of $N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp$ in $\Gamma(T^\perp\mathcal{M})$ and invariant with respect to ϕ.

For all non-zero vector field λ normal to $\mathcal{M}$, we infer

$$(3.7) \qquad \phi\lambda = t\lambda + s\lambda,$$

where $t\lambda$ tangent to $\mathcal{M}$ and $s\lambda$ normal to $\mathcal{M}$.

Proposition 3.1. For a submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$(3.8) \quad (\nabla_\mathcal{Y} T)\mathcal{X} \;=\; -(\nabla_\mathcal{X} T)\mathcal{Y} + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} + 2t\sigma(\mathcal{X},\mathcal{Y})$$
$$+\mu\{\upsilon(\mathcal{X})\mathcal{Y} + \upsilon(\mathcal{Y})\mathcal{X} - 2 < \mathcal{X},\mathcal{Y} > \xi\}$$
$$-\rho\{\upsilon(\mathcal{Y})T\mathcal{X} + \upsilon(\mathcal{X})T\mathcal{Y}\}$$

$$(3.9) \quad (\nabla_\mathcal{Y} N)\mathcal{X} \;=\; -(\nabla_\mathcal{X} N)\mathcal{Y} + 2s\sigma(\mathcal{X},\mathcal{Y}) - \sigma(\mathcal{X}, T\mathcal{Y})$$
$$-\sigma(\mathcal{Y}, T\mathcal{X}) - \rho\{\upsilon(\mathcal{Y})N\mathcal{X} + \upsilon(\mathcal{X})N\mathcal{Y}\},$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$.

Proposition 3.2. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$(3.10) \qquad T\mathcal{D} = \mathcal{D}, \quad T\mathcal{D}_\theta = \mathcal{D}_\theta, \quad T\mathcal{D}^\perp = \{0\},$$

$$tN\mathcal{D}_\theta = \mathcal{D}_\theta, \quad tN\mathcal{D}_\theta = \mathcal{D}^\perp.$$

From (3.2), (3.7) and $\phi^2 = -I + \upsilon \otimes \xi$, we get

Proposition 3.3. For the endomorphism T and N, t and s of a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ in the tangent bundle of $\mathcal{M}$, we infer
(i) $T^2 + tN = -I + \upsilon \otimes \xi$ on tangent $\mathcal{M}$
(ii) $NT + sN = \{0\}$ on tangent $\mathcal{M}$
(iii) $Nt + s^2 = -I$ on normal $\mathcal{M}$
(iv) $Tt + ts = 0$ on on normal $\mathcal{M}$.

Lemma 3.4. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer
(1) $T^2\mathcal{X} = -(\cos^2 \theta)\mathcal{X}$,
(2) $< T\mathcal{X}, T\mathcal{Y} >= (\cos^2 \theta) < \mathcal{X}, \mathcal{Y} >$
(3) $< N\mathcal{X}, N\mathcal{Y} >= (\sin^2 \theta) < \mathcal{X}, \mathcal{Y} >$
for all $\mathcal{X}, \mathcal{Y} \in D_\theta$.

Proof: The proof is the same as in [11].

Proposition 3.5. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$\begin{aligned}
(\bar{\nabla}_\mathcal{Y} T)\mathcal{X} &= -(\bar{\nabla}_\mathcal{X} T)\mathcal{Y} + 2t\sigma(\mathcal{X}, \mathcal{Y}) + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} \\
&\quad + \mu\{\upsilon(\mathcal{X})\mathcal{Y} + \upsilon(\mathcal{Y})\mathcal{X} - 2 < \mathcal{X}, \mathcal{Y} > \xi\} \\
&\quad - \rho\{\upsilon(\mathcal{X})T\mathcal{Y} + \upsilon(\mathcal{Y})T\mathcal{X}\}
\end{aligned}$$

$$\begin{aligned}
(\bar{\nabla}_\mathcal{Y} N)\mathcal{X} &= -(\bar{\nabla}_\mathcal{X} N)\mathcal{Y} - \rho\{\upsilon(\mathcal{Y})N\mathcal{X} + \upsilon(\mathcal{X})N\mathcal{Y}\} \\
&\quad - \sigma(\mathcal{X}, T\mathcal{Y}) - \sigma(\mathcal{Y}, T\mathcal{X}) + 2s\sigma(\mathcal{X}, \mathcal{Y})
\end{aligned}$$

$$(\bar{\nabla}_\mathcal{X} t)\lambda = -(\bar{\nabla}_\mathcal{Y} t)\lambda + \Lambda_{s\lambda}\mathcal{X} + \Lambda_{s\lambda}\mathcal{Y} - T\Lambda_\lambda\mathcal{X} - T\Lambda_\lambda\mathcal{Y}$$

and
$$(\bar{\nabla}_{\mathcal{X}}s)\lambda = -(\bar{\nabla}_{\mathcal{Y}}s)\lambda - \sigma(\mathcal{X}, t\lambda) + \sigma(\mathcal{Y}, t\lambda) - N\Lambda_\lambda\mathcal{X} - N\Lambda_\lambda\mathcal{Y},$$
for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector fields λ normal to $\mathcal{M}$.

Proposition 3.6. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer
$$\nabla_{\mathcal{X}}\xi = -\mu T\mathcal{X} + \rho\mathcal{X}$$
and
$$\sigma(\mathcal{X}, \xi) = -\mu N\mathcal{X} - \rho v(\mathcal{X})\xi,$$
for all vector fields $\mathcal{X}$ tangent to $\mathcal{M}$.

Lemma 3.7. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer
$$\sigma_{\phi\mathcal{Z}}\mathcal{W} = \sigma_{\phi\mathcal{W}}\mathcal{Z},$$
for all $\mathcal{Z}, \mathcal{W} \in \mathcal{D}^\perp$.

Lemma 3.8. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer
$$< [\mathcal{Y}, \mathcal{X}], \xi > -2\mu < T\mathcal{Y}, \mathcal{X} > +2\rho < \mathcal{Y}, \mathcal{X} >= 0$$
$$< \bar{\nabla}_{\mathcal{Y}}\mathcal{X}, \xi > -\mu < T\mathcal{Y}, \mathcal{X} > +\rho < \mathcal{Y}, \mathcal{X} > -\rho v(\mathcal{Y})v(\mathcal{X}) = 0,$$
for all $\mathcal{Y}, \mathcal{X} \in \Gamma(\mathcal{D} \oplus \mathcal{D}_\theta \oplus \mathcal{D}^\perp)$.

4. Integrability of Distributions and Decomposition Theorems

For invariant distributions $\mathcal{D}$, slant distributions $\mathcal{D}_\theta$ and anti-invariant distributions $\mathcal{D}^\perp$ we provide the integrability criteria.

Proposition 4.1. The invariant distribution $\mathcal{D}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of nearly trans Sasakian manifold $\bar{\mathcal{M}}$ is not integrable.

Proof. If $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$ and using (2.3), (2.5) and (2.7), we infer

(4.1) $< [\mathcal{X}, \mathcal{Y}], \xi >= 2\mu < \phi\mathcal{X}, \mathcal{Y} > -2\rho < \mathcal{X}, \mathcal{Y} >\neq 0.$

Since $< \phi\mathcal{X}, \mathcal{Y} >\neq 0$, therefore $< [\mathcal{X}, \mathcal{Y}], \xi >\neq 0$. $\square$

Theorem 4.2. The distribution $\mathcal{D} \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is integrable if and only if $\forall\ \mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D} \oplus \{\xi\})$ and $\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^\perp)$, we infer

$$(4.2)\quad <\nabla_\mathcal{X} T\mathcal{Y} - \nabla_\mathcal{Y} T\mathcal{X}, TQ\mathcal{Z}> \quad = \quad <\sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y}),$$
$$NQ\mathcal{Z} + NR\mathcal{Z} >.$$

Proof. Using (2.2), (2.5) and (3.2) we obtain

$$<[\mathcal{X}, \mathcal{Y}], \mathcal{Z}> \quad = \quad -<\bar{\nabla}_\mathcal{Y}\mathcal{X}, \mathcal{Z}> + <\bar{\nabla}_\mathcal{X}\mathcal{Y}, \mathcal{Z}>$$
$$= -<\phi\bar{\nabla}_\mathcal{Y}\mathcal{X}, \phi\mathcal{Z}> + <\phi\bar{\nabla}_\mathcal{X}\mathcal{Y}, \phi\mathcal{Z}>.$$

After some computation, we get

$$(4.3)\quad <\nabla_\mathcal{X} T\mathcal{Y} - \nabla_\mathcal{Y} T\mathcal{X}, TQ\mathcal{Z}> \quad = \quad <\sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y}),$$
$$NQ\mathcal{Z} + NR\mathcal{Z} >.$$

$\square$

Proposition 4.3. A slant distribution $\mathcal{D}_\theta$ of proper quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is not integrable.

Proof. : If $\mathcal{W}, \mathcal{X} \in \Gamma(\mathcal{D}_\theta)$ and using (2.3), (2.5) and (2.7), we infer

$$<[\mathcal{W}, \mathcal{X}], \xi> = 2\mu<\phi\mathcal{W}, \mathcal{X}> -2\rho<\mathcal{W}, \mathcal{X}>\neq 0.$$

Since $<\phi\mathcal{W}, \mathcal{X}>\neq 0$, therefore $<[\mathcal{W}, \mathcal{X}], \xi>\neq 0$. $\square$

Theorem 4.4. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifolds $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is integrable if and only if $\forall\ \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$ and $\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$, we infer

$$(4.4)\quad <\Lambda_{NT\mathcal{Z}}\mathcal{Y} - \Lambda_{NT\mathcal{Y}}\mathcal{Z},\ \ \mathcal{W}\ \ > = <\Lambda_{N\mathcal{Z}}\mathcal{Y} - \Lambda_{N\mathcal{Y}}\mathcal{Z}, TP\mathcal{W}>$$
$$+ <\nabla^\perp_\mathcal{Z} N\mathcal{Y} - \nabla^\perp_\mathcal{Y} N\mathcal{Z}, NR\mathcal{W} >.$$

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$ and $\mathcal{W} = P\mathcal{W} + R\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and using (2.2), (2.5) and (3.2), we infer

$$<[\mathcal{X}, \mathcal{Y}], \mathcal{Z}> = <\phi\bar{\nabla}_\mathcal{Y}\mathcal{Z}, \phi\mathcal{W}> - <\phi\bar{\nabla}_\mathcal{Z}\mathcal{Y}, \phi\mathcal{W}>.$$

By using (2.8), (3.2) and lemma 3.4 we infer,

$$(\sin^2\theta)<[\mathcal{Y}, \mathcal{Z}], \mathcal{W}> \quad = \quad <\Lambda_{NT\mathcal{Z}}\mathcal{Y} - \Lambda_{NT\mathcal{Y}}\mathcal{Z}, \mathcal{W}>$$
$$+ <\nabla^\perp_\mathcal{Y} N\mathcal{Z} - \nabla^\perp_\mathcal{Z} N\mathcal{Y}, NR\mathcal{W}>$$
$$- <\Lambda_{N\mathcal{Z}}\mathcal{Y} - \Lambda_{N\mathcal{Y}}\mathcal{Z}, TP\mathcal{W}>.$$

$\square$

This leads to the following conclusion:

Theorem 4.5. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifolds $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is integrable if

$$\nabla^\perp_{\mathcal{U}} N\mathcal{V} - \nabla^\perp_{\mathcal{V}} N\mathcal{U} \in N\mathcal{D}_\theta \oplus \kappa,$$

$$\Lambda_{NT\mathcal{V}}\mathcal{U} - \Lambda_{NT\mathcal{U}}\mathcal{V} \in \mathcal{D}_\theta$$

and

$$\Lambda_{N\mathcal{V}}\mathcal{U} - \Lambda_{N\mathcal{U}}\mathcal{V} \in \mathcal{D}^\perp \oplus \mathcal{D}_\theta,$$

for all $\mathcal{V}, \mathcal{U} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$.

.

Theorem 4.6. The anti-invariant distribution $\mathcal{D}^\perp$ of a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is integrable if and only if $\forall \; \mathcal{Z}, \mathcal{W} \in \Gamma(\mathcal{D}^\perp)$, we infer

$$\nabla^\perp_{\mathcal{Z}} N\mathcal{W} - \nabla^\perp_{\mathcal{W}} N\mathcal{Z} \in N\mathcal{D}^\perp \oplus \kappa.$$

Proof. If $\mathcal{Z}, \mathcal{W} \in \Gamma(\mathcal{D}^\perp), \mathcal{Y} = \mathcal{P}\mathcal{Y} + \mathcal{Q}\mathcal{Y} \in \Gamma(\mathcal{D} \oplus \mathcal{D}_\theta)$ and using (2.2), (2.5), (2.8), (3.2) and lemma 3.7, we infer

$$\begin{aligned}
< [\mathcal{Z}, \mathcal{W}], \mathcal{Y} > \;&=\; < \bar{\nabla}_{\mathcal{Z}} \phi \mathcal{W}, \phi \mathcal{Y} > - < \bar{\nabla}_{\mathcal{W}} \phi \mathcal{Z}, \phi \mathcal{Y} > \\
&= < \Lambda_{\phi \mathcal{Z}} \mathcal{W}, T\mathcal{P}\mathcal{Y} > - < \Lambda_{\phi \mathcal{W}} \mathcal{Z}, T\mathcal{P}\mathcal{Y} > \\
&\quad - < \nabla^\perp_{\mathcal{W}} \phi \mathcal{Z}, N\mathcal{Q}\mathcal{Y} > + < \nabla^\perp_{\mathcal{Z}} \phi \mathcal{W}, N\mathcal{Q}\mathcal{Y} > \\
&= < \nabla^\perp_{\mathcal{Z}} N\mathcal{W}, N\mathcal{Q}\mathcal{Y} > - < \nabla^\perp_{\mathcal{W}} N\mathcal{Z}, N\mathcal{Q}\mathcal{Y} > .
\end{aligned}$$

$\square$

Theorem 4.7. If $\mathcal{M}$ is a proper quasi hemi-slant submanifold of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, then $\mathcal{M}$ is totally geodesic if and only if

$$\begin{aligned}
(4.5) < \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \quad \mathcal{Y} \quad >\;=&< \nabla^\perp_{\mathcal{W}} NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > \\
&+ < \Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > - < \nabla^\perp_{\mathcal{W}} N\mathcal{X}, s\mathcal{Y} > \\
&- \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > .
\end{aligned}$$

Proof. If $\mathcal{W}, \mathcal{X} \in \Gamma(T\mathcal{M}), \mathcal{Y} \in \Gamma(T^\perp\mathcal{M})$ and using (2.2), (2.5), we infer

$$\begin{aligned}
< \bar{\nabla}_{\mathcal{W}} \mathcal{X}, \mathcal{Y} > \;&=\; < \bar{\nabla}_{\mathcal{W}} \mathcal{P}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}} \mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}} \mathcal{R}\mathcal{X}, \mathcal{Y}) \\
&= < \bar{\nabla}_{\mathcal{W}} \phi \mathcal{P}\mathcal{X}, \phi \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}} T\mathcal{Q}\mathcal{X}, \phi \mathcal{Y} > \\
&\quad + < \bar{\nabla}_{\mathcal{W}} N\mathcal{Q}\mathcal{X}, \phi \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}} \phi \mathcal{R}\mathcal{X}, \phi \mathcal{Y} > .
\end{aligned}$$

Using (2.3), (2.7), (2.8), (3.2) and lemma 3.4, we get

$$
\begin{aligned}
< \bar{\nabla}_{\mathcal{W}}\mathcal{X}, \mathcal{Y} > &= \ < \bar{\nabla}_{\mathcal{W}}\mathcal{P}\mathcal{X}, \mathcal{Y} > - < \bar{\nabla}_{\mathcal{W}}T^2\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\
&\quad - < \bar{\nabla}_{\mathcal{W}}NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}}N\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > \\
&\quad + < \bar{\nabla}_{\mathcal{W}}N\mathcal{R}\mathcal{X}, \phi\mathcal{Y} > \\
&= < \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \mathcal{Y} > + \cos^2\theta < \nabla_{\mathcal{W}}\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\
&\quad + \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > - < \nabla^{\perp}_{\mathcal{W}}NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\
&\quad + < -\Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W} + \nabla^{\perp}_{\mathcal{W}}N\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > \\
&\quad + < -\Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W} + \nabla^{\perp}_{\mathcal{W}}N\mathcal{R}\mathcal{X}, \phi\mathcal{Y} > .
\end{aligned}
$$

$$
\begin{aligned}
(4.6)\ < \bar{\nabla}_{\mathcal{W}}\mathcal{X}, \mathcal{Y} > &= \ < \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \mathcal{Y} > \\
&\quad - < \nabla^{\perp}_{\mathcal{W}}NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \nabla^{\perp}_{\mathcal{W}}N\mathcal{X}, f\mathcal{Y} > \\
&\quad - < \Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W} + \Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > \\
&\quad + \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > .
\end{aligned}
$$

□

Examine the geometry of the leaves of the slant, anti-slant, and invariant distributions now.

Proposition 4.8. An invariant distribution $\mathcal{D}$ of proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans -sasakian manifold $\bar{\mathcal{M}}$ is not define a totally geodesic foliation on $\mathcal{M}$.

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D})$ and using (2.3), (2.5), (2.7), we infer

$$
\begin{aligned}
(4.7)\ < \bar{\nabla}_{\mathcal{Y}}\mathcal{Z}, \xi > &= \ < \nabla_{\mathcal{Y}}\mathcal{Z}, \xi > \\
&= \mu < \phi\mathcal{Y}, \mathcal{Z} > - \rho < \mathcal{Y}, \mathcal{Z} > + \rho v(\mathcal{Y})v(\mathcal{Z}) \\
(4.8)\ &\neq 0.
\end{aligned}
$$

Since $< \phi\mathcal{Y}, \mathcal{Z} > \neq 0$, therefore $< \bar{\nabla}_{\mathcal{Y}}\mathcal{Z}, \xi > \neq 0$. □

Theorem 4.9. The distribution $\mathcal{D} \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is define totally geodesic foliation on $\mathcal{M}$ if and only if for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$, $\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^{\perp})$ and $\lambda \in (T^{\perp}\mathcal{M})$, we infer

$$
< \nabla_{\mathcal{X}}T\mathcal{Y}, T\mathcal{Q}\mathcal{Z} > = - < \sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z} >
$$

and

$$
< \nabla_{\mathcal{X}}T\mathcal{Y}, t\lambda > = - < \sigma(\mathcal{X}, T\mathcal{Y}), s\lambda > .
$$

Proof. If $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}), \mathcal{Z} = \mathcal{Q}\mathcal{Z} + \mathcal{R}\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^\perp)$ and using (2.2), (2.5), (3.2) and $N\mathcal{Y} = 0$, we infer

$$
\begin{aligned}
< \bar{\nabla}_\mathcal{X}\mathcal{Y}, \mathcal{Z} > \ &= \ < \bar{\nabla}_\mathcal{X}T\mathcal{Y}, \phi\mathcal{Z} > \\
&= < \nabla_\mathcal{X}T\mathcal{Y}, T\mathcal{Q}\mathcal{Z} > + < \sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z} >,
\end{aligned}
$$

for all $\lambda \in (T^\perp\mathcal{M})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$, we infer

$$
< \bar{\nabla}_\mathcal{X}\mathcal{Y}, \lambda > = < \nabla_\mathcal{X}T\mathcal{Y}, t\lambda > + < \sigma(\mathcal{X}, T\mathcal{Y}), s\lambda > .
$$

$\square$

Proposition 4.10. The slant distribution $\mathcal{D}_\theta$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is not define a totally geodesic foliation on $\mathcal{M}$.

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta)$ and using (2.3), (2.5) and (2.7), we infer

$$
\begin{aligned}
(4.9) \quad < \bar{\nabla}_\mathcal{Y}\mathcal{Z}, \xi > \ &= \ < \nabla_\mathcal{Y}\mathcal{Z}, \xi > \\
&= \mu < \phi\mathcal{Y}, \mathcal{Z} > -\rho < \mathcal{Y}, \mathcal{Z} > +\rho \upsilon(Y)\upsilon(\mathcal{Z}) \\
(4.10) \quad &\neq 0, \qquad for some \quad \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta).
\end{aligned}
$$

Since $< \phi\mathcal{Y}, \mathcal{Z} > \neq 0$, therefore $< \bar{\nabla}_\mathcal{Y}\mathcal{Z}, \xi > \neq 0$. $\square$

Theorem 4.11. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is to define a totally geodesic foliation on $\mathcal{M}$ iff $\forall \quad \mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\}), \mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $\lambda \in (T^\perp\mathcal{M})$, we infer

$$
< \nabla_\mathcal{U}^\perp N\mathcal{V}, N\mathcal{R}\mathcal{W} > = < \Lambda_{N\mathcal{V}}\mathcal{U}, T\mathcal{P}\mathcal{W} > - < \Lambda_{NT\mathcal{V}}\mathcal{U}, \mathcal{W} >
$$

and

$$
< \Lambda_{N\mathcal{V}}\mathcal{U}, t\lambda > = < \nabla_\mathcal{U}^\perp N\mathcal{V}, s\lambda > - < \nabla_\mathcal{U}^\perp NT\mathcal{V}, \lambda > .
$$

Proof. If $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\}), \mathcal{W} = \mathcal{P}\mathcal{W} + \mathcal{R}\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$, and using (2.2), (2.5) and (3.2), we infer

$$
< \bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W} > = < \bar{\nabla}_\mathcal{U}\phi\mathcal{V}, \phi\mathcal{W} > = < \bar{\nabla}_\mathcal{U}T\mathcal{V}, \phi\mathcal{W} > + < \bar{\nabla}_\mathcal{U}N\mathcal{V}, \phi\mathcal{W} > .
$$

Then using (2.8), (3.2) and Lemma 3.4 and the fact that $N\mathcal{P}\mathcal{W} = 0$, we infer

$$
\begin{aligned}
< \bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W} > \ &= \ \cos^2\theta < \bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W} > - < \bar{\nabla}_\mathcal{U}NT\mathcal{V}, \mathcal{W} > \\
&\quad + < \bar{\nabla}_\mathcal{U}N\mathcal{V}, \phi\mathcal{W} >
\end{aligned}
$$

$$(4.11) \qquad \sin^2\theta < \bar{\nabla}_{\mathcal{U}}\mathcal{V}, \mathcal{W} > \; = \; < \Lambda_{NT\mathcal{V}}\mathcal{U}, \mathcal{W} >$$
$$+ < \nabla_{\mathcal{U}}^{\perp}N\mathcal{V}, N\mathcal{R}\mathcal{W} >$$
$$- < \Lambda_{N\mathcal{V}}\mathcal{U}, T\mathcal{P}\mathcal{W} > .$$

Similarly, we get

$$(4.12) \qquad \sin^2\theta < \bar{\nabla}_{\mathcal{U}}\mathcal{V}, V > \; = \; - < \nabla_{\mathcal{U}}^{\perp}NTW, \lambda >$$
$$- < \Lambda_{N\mathcal{V}}\mathcal{U}, t\lambda >$$
$$+ < \nabla_{\mathcal{U}}^{\perp}N\mathcal{V}, s\lambda > .$$

$\square$

References

[1] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Mathematics, **29** (2003), 831–840.

[2] Sahin, A., *Warped product submanifolds of a Kaehler manifold with a slant factor*, Ann. Pol. Math. **95** (2009), 107-126

[3] Lotta, A., *Slant submanifold in contact geometry*, Bull. Math. Soc. Sci. Math. Repub. Soc. Roum. **39** (1996), 183-198

[4] Chen, B. Y., *Geometry of slant submanifolds*, Katholieke Universiteit, Leuven, (1990)

[5] Chen, B. Y., *Slant immersions*, Bull. Austral. Math. Soc. **41** (1990), No. 1, 135-147

[6] Chen, B. Y. and Tazawa, Y., *Slant submanifolds in complex Euclidean spaces*, Tokyo J. Math. **14** (1991), 101-120

[7] Blair, D. E. and Goldberg, S.I., *Topology of almost contact manifolds*, J. Differ. Geom. **1** (1967) 347-354

[8] Mihai, I. and Rosca, R., *On Lorentzian $\mathcal{P}$-Sasakian manifolds*, Classical Analysis, World Scientifc Publ., (1992), 155-169.

[9] Matsumoto, K., *On Lorentzian para contact manifolds*, Bull. of Yamagata Univ., Nat. Sci. **12** (1989), 151-156.

[10] Matsumoto, K. and Mihai, I., *On a certain transformation in Lorentzian para Sasakian manifold*, Tensor, N.S.**47** (1988), 189-197.

[11] Park K. S. and Prasad, R., *Semi-slant submersions*, Bull. Korean Math. Soc. **50** (2013), No. 3, 951-962.

[12] Prasad, R., Verma, S. K. and Kumar, S., *Quasi hemi-slant submanifolds of Sasakian manifolds*, J. Math. Comput. Sci. **10** (2020), No. 2, 418-435.

[13] Rahman, S.,*Contact CR-warped product submanifolds of nearly Lorentzian para-Sasakian manifold*, Turkish Journal of Mathematics and Computer Science, **7** (2017), 40-47.

[14] Rahman, S. and Jun J. B., *CR-submanifolds of a nearly Lorentzian Para-Sasakian manifold*, Far East Journal of Mathematical Sciences (FJMS), **103** (2018), No. 3, 587-602.

[15] Uddin, S. Chen B. Y. and Al-Solamy, F. R. *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math. **14** (2017), No. 2, 14-95.

[16] Shahid M.H. et al., *Slant submanifolds of cosymplectic manifold*, An. S tiint. Univ. Al. I. Cuza Iasi, Mat. **50** (2004), 33-50

[17] Papaghuic, N., *Semi-slant submanifold of Kaehlerian manifold*, An. St. Univ. AI. I. Cuza. Iasi. Math. N.S. **9** (1994), 55-61.

Shamsur Rahman
Faculty of Mathematics , Maulana Azad National Urdu University, Polytechnic Off Campus, Darbhanga Bihar- 846002, India
Email: shamsur@manuu.edu.in, shamsur@rediffmail.com

Amit Kumar Rai
Guru Gobind Singh Indraprastha University, University School of Basic and Applied Science New Delhi-110078, India
Email: rai.amit08au@gmail.com