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Research Paper

ON SOME TOPOLOGICAL INDICES OVER RECTANGULAR GRIDS

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Abstract. A topological index is a real number related to a graph, which is considered as a structural invariant. Some examples are Sombor index, Randić index, Zagreb indices, and Harmonic index. In the present paper, we consider the function Ind from the set of all rectangular grids to the set of real numbers, which assigns to each rectangular grid, one of its above indices. Then we show that the only non-degenerate indices over retangular grids, are Sombor index and Randić index, while Zagreb indices and the Harmonic index are degenerate. In the following, we determine rectangular grids with fixed diameter d, where maximum and minimum of the above indices occures on them, in the case $m \geq 3, n \geq 3$. Finally, we find the amounts of these indices.

Key Words: Graph, Graphical indices, Isomorphism.
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1. Introduction

A number that is invariant under graph isomorphisms is referred as a graphical invariant. It is often regarded as a structural invariant relevant to a graph. The term topological index is often reserved for graphical invariant in molecular graph theory. In the mathematical and chemical literature, several vertex-degree-based graph invariants usually referred

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to as "topological indices" have been introduced and extensively studied.

Some of these indices are based on the degrees of vertices. Nevertheless, it is not possible to mention all of the studies, related to these indices, that have been published hereunto. But among them, we intend to present a very brief overview of the studies which are done by the researchers.

The first index of these works is Randić index. It was invented in 1975 by Milan Randić and is defined as $R(G) = \Sigma_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$ [24]. This index is certainly the most widely applied in chemistry and pharmacology, in particular for designing quantitative structure-property and structure-activity relations. Details of these applications can be found in [12, 13, 14, 15, 16, 21] and the surveys written by Randić [25, 26].

One of the oldest graph invariants is the well-known Zagreb index, first introduced in [10], where Gutman and Trinajstić examined the dependence of total ϕ-electron energy on molecular structure. Another index based on the degrees of the vertices, is the second Zagreb index. These are respectively, defined as follows:

$$M_1(G) = \Sigma_{v \in V(G)} d_v^2, \;\; M_2(G) = \Sigma_{uv \in E(G)} d_u d_v.$$

Note that some authors call M_1 the Gutman index (see [9, 27]). The main properties of M_1 and M_2 are summarized in [6, 17, 18].

The harmonic index of a graph G is defined as

$$H(G) = \Sigma_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

This index which is extensively studied in the last decade, was introduced at 1987 [5]. For instance, Zhong [30, 31], and Zhong and Xu [32], determined the minimum and maximum values of the harmonic index for simple connected graphs, trees, unicyclic graphs, and bicyclic graphs. Xu [28] established some relationships between the harmonic index of a graph and its other topological indices, and Deng et al. [4] considered the relation between the Harmonic index of a graph and its matching number. Onagh [20] studied the Harmonic index of t-subdivision graphs, S-sum and S_t-sum of graphs.

Recently, Gutman in [7] introduced a new vertex-degree-based molecular structure descriptor, defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

and named it “Sombor index”. He also established some basic properties of it on some molecular graphs in [8]. It is shown that any vertex-degree-based topological index can be considered as a special case of a Sombor type index. Das et al. [3] found some relations on the Somber index with the Zagreb indices. The authors presented several lower and upper bounds on the Sombor index of graphs stated on some useful graph parameters such as deleting and adding edges to graph, maximum and minimum degree of vertices, and etc. With a similar approach, Alidadi et al. in [1] presented the minimum Sombor index for unicyclic graphs with fixed diameter.

Some of the papers in this area such as [23], deals with the calculation of the Sombor index of some new graphs obtained from a graph, such as the subdivision of a graph, the line graph of a graph, the total graph of a graph, the semi-total point graph of a graph, the semi-total line graph of a graph, the double-graph of a graph, the strong-double graph of a graph and finally the generalized transformation graph of a graph.

In the present paper, we study the mentioned indices over rectangular grids, and determine the non-degenerate ones, over them, i.e., the ones that Ind takes different values over non-isomorphic rectangular grids. Finally, we find the rectangular grids with fixed diameter d, which the minimum and maximum of these values occures on them for $m, n \geq 3$. The amounts of the minimum and maximum of the indices over rectangular grids, are also provided.

2. **Preliminaries**

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $|V(G)|$ be the number of vertices, $|E(G)|$ be the number of edges, and $d_v(G)$ be the degree of a vertex $v \in V(G)$. The distance between any two distinct vertices u and v in G, is the number of edges in the shortest path (also called a graph geodesic) connecting them, and denoted by $d_G(u, v)$. The diameter of G, which is the maximum distance between the pair of vertices of G, is denoted by $d(G)$. From now on, we drop the subscript “G” from the notation $d_v(G)$ and $d(G)$ when there is no confusion. An isomorphism of graphs G_1 and G_2, is a bijection between the vertex sets of G_1 and G_2, say $f : V(G_1) \to V(G_2)$, such that any two vertices u_1 and v_1 of G_1 are adjacent in G_1, iff $f(u_1)$ and $f(v_1)$ are adjacent in G_2. If an isomorphism exists between two graphs, then the graphs are called isomorphic and denoted as $G_1 \simeq G_2$. To define the product $G_1 \times G_2$

of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V_1 \times V_2$, then u and v are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 [11]. A path graph, or, a linear graph, is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n-1$. We denote the $m \times n$ rectangular grid which is a product of two path graphs with m and n vertices, by $RG_{m\times n}$ (Figure 1). The set of all rectangular grids $RG_{m\times n}$ for natural numbers m and n, will be denoted by RG.

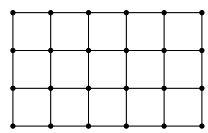
$[RG_{4\times 6}]$  $[RG_{1\times 4} = P_4]$

Figure 1. Examples of Rectangular Grids

We now continue with some lemmas that we need later. The sets of natural numbers, integers, rational numbers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

Lemma 2.1. *Let the funtion $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ defined by $f(m, n) = m + n - 42mn$. Then there are $r, s, p, q \in \mathbb{N}$ such that $r + s \neq p + q$ and $rs \neq pq$, but $f(r, s) = f(p, q)$.*

Proof. As a consequence of the method of solving Diophantine equations having integral coefficients [19], we find that if $r = 3, s = 214, p = 5, q = 128$, then

$$f(r, s) = f(3, 214) = 3 + 214 - 42 \times 3 \times 214 = -26747,$$

and

$$f(p, q) = f(5, 128) = 5 + 128 - 42 \times 5 \times 128 = -26747.$$

□

Lemma 2.2. *Let $a, b, c, d \in \mathbb{Q}$ and $u \in \mathbb{R} - \mathbb{Q}$, such that*

$$au + b = cu + d,$$

then $a = c, b = d$.

Lemma 2.3. *If $x, y, z, t \in \mathbb{Q}$ and $x\sqrt{2} + y\sqrt{5} + z\sqrt{13} = t$ (res. $x\sqrt{2} + y\sqrt{3} + z\sqrt{6} = t$), then $x = y = z = t = 0$.*

Lemma 2.4. *If two natural variables m, n satisfy $m+n=s$, then their product mn satisfies the following statements:*

(a) *If s is even, then $s-1 \leq mn \leq \frac{s^2}{4}$, moreover $mn = s-1$ iff one of the variables equals to 1, and $mn = \frac{s^2}{4}$ iff each of the variables equals to $\frac{s}{2}$,*

(b) *If s is odd, then $s-1 \leq mn \leq \frac{s^2-1}{4}$, moreover $mn = s-1$ iff one of the variables equals to 1, and $mn = \frac{s^2-1}{4}$ iff one of the variables equals to $\frac{s-1}{2}$.*
Moreover, the above inequlities will be changed as follows, if the constriants $m, n \geq 3$ are added:

(c) *If s is even, then $3s-9 \leq mn \leq \frac{s^2}{4}$, moreover $mn = 3s-9$ iff one of the variables equals to 3, and $mn = \frac{s^2}{4}$ iff each of the variables equals to $\frac{s}{2}$,*

(d) *If s is odd, then $3s-9 \leq mn \leq \frac{s^2-1}{4}$, moreover $mn = 3s-9$ iff one of the variables equals to 3, and $mn = \frac{s^2-1}{4}$ iff one of the variables equals to $\frac{s-1}{2}$.*

3. **Main results**

In this section, at first we find the amount of each of the above indices over rectangular grids $RG_{m\times n}$. We show that the only non-degenerate indices over rectangular grids are Sombor index and Randić index. Finally, we present the minimum and maximum amount of the indices for these graphs with fixed diameter d.

Theorem 3.1. *The function $Ind_{SO} : RG \to \mathbb{R}$ given by $Ind_{SO}(RG_{m\times n}) = SO(RG_{m\times n})$ is non-degenerate over rectangular grids.*

Proof. At first we claim that the Sombor index of the $RG_{m\times n}$, in general, is of the form $a + b\sqrt{2} + c\sqrt{5} + d\sqrt{13}$, where a, b, c, d are integers. For this, we calculate the index in different cases as followes:

(a) Let $S_1 = \{RG_{m\times n} | m, n \geq 3\}$. A member of S_1 has 8 edgs of index $\sqrt{13}$, $2(m+n-6)$ edgs of index $\sqrt{18}$, $2(m+n-4)$ edgs of index 5, and $(m-2)(n-3)+(m-3)(n-2)$ edgs of index $\sqrt{32}$ (See Figure 2). Therefore the Sombor index of $RG_{m\times n}$ equals to:

$$\begin{aligned}SO(RG_{m\times n}) &= 2[2\sqrt{13}+(n-3)\sqrt{18}]\\ &\quad+(m-2)[2\times 5+(n-3)\sqrt{32}]\\ &\quad+2[2\sqrt{13}+(m-3)\sqrt{18}]\\ &\quad+(n-2)[2\times 5+(m-3)\sqrt{32}]\\ &= [8mn-14(m+n)+12]\sqrt{2}\\ &\quad+8\sqrt{13}+10(m+n-4),\end{aligned}$$

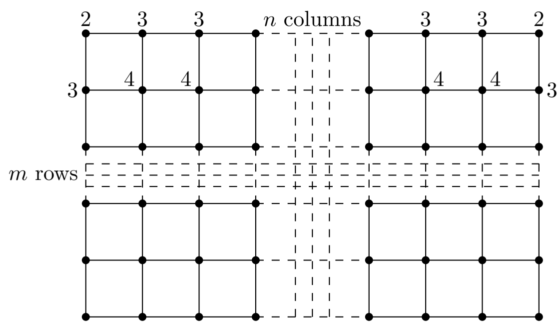


FIGURE 2. $RG_{m\times n},\ (m,n\geq 3)$

(b) Let $S_2=\{RG_{m\times n}|m=2,n\geq 3\}$. A member of S_2 has 2 edges of index $\sqrt{8}$, 4 edges of index $\sqrt{13}$, and $3n-8$ edges of index $\sqrt{18}$ (Figure 3). Therefore the Sombor index of $RG_{m\times n}$ equals to:

$$\begin{aligned}SO(RG_{m\times n}) &= 2\sqrt{8}+4\sqrt{13}+(3n-8)\sqrt{18}\\ &= (9n-20)\sqrt{2}+4\sqrt{13},\end{aligned}$$

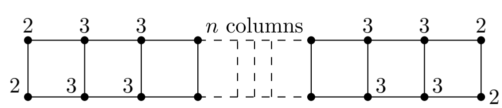


FIGURE 3. $RG_{m\times n},\ (m=2,n\geq 3)$

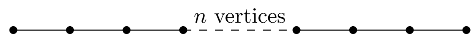


FIGURE 4. $RG_{1\times n},\ (n \geq 3)$

(c) Let $S_3 = \{RG_{m\times n} | m = 1, n \geq 3\}$. A member of S_3 has 2 edges of index $\sqrt{5}$, and $n-3$ edges of index $\sqrt{8}$ (Figure 4). Therefore the Sombor index of $RG_{m\times n}$ equals to:

$$SO(RG_{1\times n}) = 2\sqrt{5} + (n-3)\sqrt{8} = (2n-6)\sqrt{2} + 2\sqrt{5},$$

(d) Let $S_4 = \{RG_{m\times n} | m = 2, n = 2\}$, therefore $SO(RG_{2\times 2}) = 8\sqrt{2}$,

(e) Let $S_5 = \{RG_{m\times n} | m = 1, n = 2\}$, therefore $SO(RG_{1\times 2}) = \sqrt{5}$,

and the claim is proved.

As a consequene of the preceding investigation we have:

1: The Sombor indices of $RG_{m\times n}$ and $RG_{p\times q}$ for $m, n, p, q \geq 3$ are equal if and only if

$$\begin{aligned} 8mn - 14(m+n) + 12]\sqrt{2} + 8\sqrt{13} + 10(m+n-4) \\ = 8pq - 14(p+q) + 12]\sqrt{2} + 8\sqrt{13} + 10(p+q-4). \end{aligned}$$

An application of Lemma 2.2 shows that $m = p, n = q$, or, $m = q, n = p$. Therefore $RG_{m\times n} \simeq RG_{p\times q}$.

2: The Sombor indices of $RG_{2\times n}$ and $RG_{2\times q}$ for $n, q \geq 3$ are equal if and only if $9n - 20 = 9q - 20$, or, $n = q$. Therefore $RG_{2\times n} \simeq RG_{2\times q}$.

3: The Sombor indices of $RG_{1\times n}$ and $RG_{1\times q}$ for $n, q \geq 3$ are equal if and only if $2n-6 = 2q-6$, or, $n = q$. Therefore $RG_{1\times n} \simeq RG_{1\times q}$.

4: Since all of the Sombor indices over rectangular grids is an integer combination of $1, \sqrt{2}, \sqrt{5}, \sqrt{13}$, then Lemma 2.3 asserts that the Sombor indices of the members of S_i does not equal to the ones of the members of S_j for $i \neq j$. □

Theorem 3.2. *The function $Ind_R : RG \to \mathbb{R}$ given by $Ind_R(RG_{m\times n}) = R(RG_{m\times n})$ is non-degenerate over rectangular grids.*

Proof. The Randić index of the $RG_{m\times n}$, in general, is of the form $\alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{6}$, where $\alpha, \beta, \gamma, \delta$ are rational numbers. For an argument, we calculate the index in the following different cases:

(a) Let $R_1 = \{RG_{m\times n} | m, n \geq 3\}$. A member of R_1 has 8 edgs of index $\frac{1}{\sqrt{6}}$, $2(m+n-6)$ edges of index $\frac{1}{3}$, $2(m+n-4)$ edges of index $\frac{1}{\sqrt{12}}$, and $(m-2)(n-3)+(m-3)(n-2)$ edgs of index $\frac{1}{4}$ (Figure 2). Therefore the Randić index of $RG_{m\times n}$ equals to:

$$\begin{aligned}R(RG_{m\times n}) = 2[2\frac{1}{\sqrt{6}}+(n-3)\frac{1}{3}]\\ &+(m-2)[\frac{1}{2\sqrt{12}}+(n-3)\frac{1}{4}]\\ &+2[2\frac{1}{\sqrt{6}}+(m-3)\frac{1}{3}]\\ &+(n-2)[2\frac{1}{\sqrt{12}}+(m-3)\frac{1}{4}]\\ &=[\frac{1}{3}(m+n)-\frac{4}{3}]\sqrt{3}+\frac{4}{3}\sqrt{6}+\frac{1}{2}mn-\frac{7}{12}(m+n)-1,\end{aligned}$$

(b) Let $R_2=\{RG_{m\times n}|m=2,n\geq 3\}$. A member of R_2 has 2 edges of index $\frac{1}{2}$, 4 edges of index $\frac{1}{\sqrt{6}}$, and $3n-8$ edges of index $\frac{1}{3}$ (Figure 3). Therefore the Randić index of $RG_{m\times n}$ equals to:

$$\begin{aligned}R(RG_{m\times n}) &= 2\frac{1}{2}+4\frac{1}{\sqrt{6}}+(3n-8)\frac{1}{3}\\ &=\frac{2}{3}\sqrt{6}+\frac{1}{3}(3n-5),\end{aligned}$$

(c) Let $R_3=\{RG_{m\times n}|m=1,n\geq 3\}$. A member of R_3 has 2 edges of index $\frac{1}{\sqrt{2}}$, and $n-3$ edges of index $\frac{1}{2}$ (Figure 4). Therefore the Randić index of $RG_{m\times n}$ equals to:

$$R(RG_{1\times n})=2\frac{1}{\sqrt{2}}+(n-3)\frac{1}{2}=\sqrt{2}+\frac{1}{2}(n-3),$$

(d) Let $R_4=\{RG_{m\times n}|m=2,n=2\}$, therefore $R(RG_{2\times 2})=2$,

(e) Let $R_5=\{RG_{m\times n}|m=1,n=2\}$, therefore $R(RG_{1\times 2})=\frac{1}{2}\sqrt{2}$,

and the claim is proved.

As a consequene of the preceding investigation we have:

1: The Randić indices of $RG_{m\times n}$ and $RG_{p\times q}$ for $m,n,p,q\geq 3$ are equal if and only if

$$\begin{aligned}&\frac{2}{3}(m+n)\sqrt{3}+\frac{4}{3}\sqrt{6}+\frac{1}{2}mn-\frac{7}{6}(m+n)-1\\ &\qquad=\frac{2}{3}(p+q)\sqrt{3}+\frac{4}{3}\sqrt{6}+\frac{1}{2}pq-\frac{7}{6}(p+q)-1.\end{aligned}$$

Now Lemma 2.2 shows that $m=p,n=q$, or, $m=q,n=p$ and consequently $RG_{m\times n}\simeq RG_{p\times q}$.

2: The Randić indices of $RG_{2\times n}$ and $RG_{2\times q}$ for $n, q \geq 3$ are equal if and only if $3n-5 = 3q-5$, or, $n = q$. Therefore $RG_{2\times n} \simeq RG_{2\times q}$.
3: The Sombor indices of $RG_{1\times n}$ and $RG_{1\times q}$ for $n, q \geq 3$ are equal if and only if $n-3 = q-3$, or, $n = q$, therefore $RG_{1\times n} \simeq RG_{1\times q}$.
4: Since all of the Randić indices over rectangular grids is a rational combination of $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$, then Lemma 2.3 asserts that the Randić indices of the members of R_i does not equal to the ones of the members of R_j for $i \neq j$. $\square$

The following Corollary, is a consequene of Theorems 3.1 and 3.2.

Corollary 3.3. *The Sombor indices of two rectangular grids are the same, if and only if their Randić indices of them are the same.*

A similar argument stated in Theorems 3.1 and 3.2 shows that:

Theorem 3.4. *The first Zagreb index of $RG_{m\times n}$ equals to:*

$$M_1(RG_{m\times n}) = \begin{cases} 16mn - 14(m+n) + 8, & m \geq 3, n \geq 3 \\ 18n - 20, & m = 2, n \geq 3 \\ 4n - 6, & m = 1, n \geq 3 \\ 16, & m = 2, n = 2 \\ 2. & m = 1, n = 2 \end{cases}$$

We now introduce an example to show that $Ind_{M_1} : RG \to \mathbb{R}$ given by $Ind_{M_1}(RG_{m\times n}) = M_1(RG_{m\times n})$, is degenerate over rectangular grids.

Example 3.5. According to Theorem 3.4 we have

$$M_1(RG_{1\times 27}) = M_1(P_{27}) = \Sigma_{v\in V(G)} d_v^2 = 4 \times 27 - 6 = 102,$$

and

$$M_1(RG_{3\times 4}) = \Sigma_{v\in V(G)} d_v^2 = 16 \times 3 \times 4 - 14(3+4) + 8 = 102.$$

The proof of the following Theorem is strightforward.

Theorem 3.6. *The seocnd Zagreb index of $RG_{m\times n}$ equals to:*

$$M_2(RG_{m\times n}) = \begin{cases} 32mn - 38(m+n) + 36, & m \geq 3, n \geq 3 \\ 27n - 40, & m = 2, n \geq 3 \\ 4n - 8, & m = 1, n \geq 3 \\ 16, & m = 2, n = 2 \\ 1. & m = 1, n = 2 \end{cases}$$

In the following example, we show that $Ind_{M_2} : RG \to \mathbb{R}$ given by $Ind_{M_2}(RG_{m\times n}) = M_2(RG_{m\times n})$, is also degenerate over rectangular grids.

Example 3.7. According to Theorm 3.6 we have

$$M_2(RG_{1\times 6}) = M_2(P_6) = \Sigma_{v\in E(G)} d_u d_v = 4 \times 6 - 8 = 16,$$

and

$$M_2(RG_{2\times 2}) = \Sigma_{v\in E(G)} d_u d_v = 16.$$

With a similar argument as in Theorems 3.1 and 3.2 we have:

Theorem 3.8. *The Harmonic index of $RG_{m\times n}$ equals to:*

$$H(RG_{m\times n}) = \begin{cases} -\frac{3}{35} - \frac{1}{84}(m+n-42mn), & m \geq 3, n \geq 3 \\ n - \frac{1}{15}, & m = 2, n \geq 3 \\ \frac{1}{2}n - \frac{1}{6}, & m = 1, n \geq 3 \\ 2, & m = 2, n = 2 \\ 1. & m = 1, n = 2 \end{cases}$$

Now as a consequene of Theorem 3.8 and Lemma 2.1, we have the following Lemma:

Lemma 3.9. *The function $Ind_H : RG \to \mathbb{R}$ is degenerate over rectangular grids.*

3.1. **Maximum and Minimum of Indices.** Finally, we present the minimum and maximum of the mentioned indices over rectangular grids with $m \geq 3, n \geq 3$ and fixed diameter d. In this case the diameter of $RG_{m\times n}$ is equal to $d = m + n - 2$. The next Theorem is a consequence of Lemma 2.4, Theorems 3.1, 3.2, 3.4, 3.6 and 3.8.

Index	Symbol	Min on $RG_{m\times n}$ $(m, n \geq 3)$
Sombor	So	$10(\sqrt{2}+1)d - 40\sqrt{2} + 8\sqrt{13} - 20$
Randić	R	$(\frac{11}{12} + \frac{\sqrt{3}}{3})d - \frac{2}{3}\sqrt{3} + \frac{4}{3}\sqrt{6} - \frac{11}{3}$
First Zagreb	M_1	$34d - 68$
Second Zagreb	M_2	$58d - 136$
Harmonic	H	$\frac{125}{84}d - \frac{169}{105}$

Table 1: $d \geq 6$

Theorem 3.10. *Let $d \geq 6$ be an integer, then*

(a) *The minimum of all mentioned indices over rectangular grids with* $m \geq 3, n \geq 3$ *and fixed diameter* d*, occures iff* $(m = 3, n = d - 1)$*, or* $(m = d - 1, n = 3)$*,*

(b) *If* d *is even, then the maximum of all mentioned indices over rectangular grids with* $m \geq 3, n \geq 3$*, occures iff* $(m = n = \frac{d+2}{2})$*,*

(c) *If* d *is odd, then the maximum of all mentioned indices over rectangular grids with* $m \geq 3, n \geq 3$*, occures iff* $(m = \frac{d+1}{2}, n = \frac{d+3}{2})$ *or* $(m = \frac{d+3}{2}, n = \frac{d+1}{2})$*.*

The minimum and maximum values of the above indices on $RG_{m\times n}$, in the case $m, n \geq 3$, are presented in the Tables 1, 2 and 3. Calculation these values in the cases $m \leq 3$ or $n \leq 3$ is similar.

Index	Symbol	Max on $RG_{m\times n}$ $(m, n \geq 3)$
Sombor	So	$2\sqrt{2}d^2 + (10 - 6\sqrt{2})d - 8\sqrt{2} + 8\sqrt{13} - 20$
Randić	R	$\frac{1}{8}d^2 + (\frac{\sqrt{3}}{3} - \frac{1}{12})d - \frac{2}{3}\sqrt{3} + \frac{4}{3}\sqrt{6} - \frac{5}{3}$
First Zagreb	M_1	$4d^2 + 2d - 4$
Second Zagreb	M_2	$8d^2 - 6d - 8$
Harmonic	H	$\frac{1}{8}d^2 + \frac{41}{84}d + \frac{41}{105}$

Table 2: $d \geq 6$ is even

Index	Symbol	Max on $RG_{m\times n}$ $(m, n \geq 3)$
Sombor	So	$2\sqrt{2}d^2 + (10 - 6\sqrt{2})d - 10\sqrt{2} + 8\sqrt{13} - 20$
Randić	R	$\frac{1}{8}d^2 + (\frac{\sqrt{3}}{3} - \frac{1}{12})d - \frac{2}{3}\sqrt{3} + \frac{4}{3}\sqrt{6} - \frac{43}{24}$
First Zagreb	M_1	$4d^2 + 2d - 8$
Second Zagreb	M_2	$8d^2 - 6d - 16$
Harmonic	H	$\frac{1}{8}d^2 + \frac{41}{84}d + \frac{223}{840}$

Table 2: $d \geq 6$ is odd

4. **Discussions**

Discrimination between the graphs, by using topological indices, is important in studying the structure of molecules in chemistry, or rectangular grid structures in civil and mechanical engeenering [2, 22, 29]. Theorems 3.1 and 3.2 show that Ind_{SO}, Ind_R are non-degenerate on rectangular grids, but examples 3.5, 3.7 and 3.9 show that Ind_{M_1}, Ind_{M_2} and Ind_H are degenerate on them. It can also be seen from the provided Theorems, that non of the above indices are surjetive on rectangular grids. Obviously, non of the the above indices are not surjetive on any classes of graphs! But the following question have yet to be answered:

Problem: Are the indices Ind_{M_1}, Ind_{M_2} and Ind_H non-degenerate on some classes of connected graphs?

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