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Research Paper

UNIQUE COMMON FIXED POINT RESULTS IN C^*-ALGEBRA VALUED METRIC SPACES USING (Φ-C_*)-CONTRACTIONS OF HARDY-ROGERS TYPE

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Abstract. In this paper we have developed C_* -class function and introduced (Φ-C_*)-contractions of Hardy-Rogers type on C^*-algebra valued metric spaces.We have also established some unique common fixed point results for six maps in C^*-algebra valued metric spaces using this type contractions. Some basic definitions, properties and lemmas are also discussed in the introduction and preliminaries parts. Some corollaries and examples are also given on the basis of the results.

Key Words: C^*-algebra valued metric space, C_*-class function, (Φ-C_*)-contractions of Hardy-Rogers type, Norm, weakly compatible mappings.
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1. Introduction

Fixed point theorem is the most important and indispensable part in the theory of metric space for solving different problems in topology, analysis, operator theory, non-linear analysis, differential equations etc. and others fields also. In 1972, the famous Stefan Banach founded The Banach Contraction Principle[3], which is the most valuable and

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powerful tools in the field of Fixed Point Theory. Since then many distinguished authors have been generalizing and developing various type of contractions on different type of metric spaces and establishing fixed point results. C^*-algebra valued metric space is such type of generalization of various metric spaces.

In 2014, Ma et al.[15] introduced and established some fixed point results on C^*- algebra valued metric spaces and then in 2016, Xin et al.[31] established some common fixed point results on C^*- algebra valued metric spaces. In 2016, Piri et al.[21] established fixed point theorems concerning F-contraction in complete metric spaces and in 2016, Shehwar et al.[27] established Caristi's fixed point theorem on C^*-algebra valued metric spaces; in 2017, Radenovic et al.[23] established some coupled fixed point results in the framework of C^*-algebra-valued b-metric spaces and in 2018, Moeini et al.[19] established Zamfirescu type contractions on C^*-algebra valued metric spaces. Later in 2020, Derouche, D. and Ramoul, H. [8] introduced the notions of Extended F-contraction of Hardy-Rogers type, extended F-contraction of Suzuki-Hardy-Rogers type and established fixed point results on complete b-metric spaces with these contractions. Also in 2020, Yang et al.[33] introduced the notion of an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping and prove some fixed point theorems on such contraction mappings in orthogonally metric spaces. In 2021, Zhiqun Xue and Guiwen Lv [32] established fixed point theorem for generalized (ψ, φ)-weak contractions in Branciari type generalized metric spaces, Kumar et al.[13] established some unique common fixed point results on C^*-algebra valued metric spaces using C_*-class function and Hafida Massit and Mohamed Rossafi[16] established fixed point results for (ϕ, F)-contraction on C^*-algebra valued metric spaces. Recently, in 2022, Rossafi et al.[25] introduced (ϕ, MF)-contraction on C^*-algebra valued metric spaces and established some fixed point results and its uniqueness.

One important result of Xin et al.[31] is given below:

Theorem 1.1. *([31], Theorem 2.3). Let (X, A, d) be a complete C^*-algebra valued metric space. Suppose that two mappings $T, S : X \to X$ satisfying the following:*

$$d(Tx, Ty) \preceq ad(Tx, Sx) + ad(Ty, Sy),\ \forall\ x, y \in X;$$

where $a \in A'_+$ with $\|a\| < \frac{1}{2}$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X, then T and S have a unique point of coincidence in X.

Furthermore, if T and S are weakly compatible, T and S have a unique common fixed point in X.

Discussing, analyzing and Motivating from the results of the article given in the ref. ([1], [3], [6], [7], [8], [10], [16], [20], [21], [22], [19], [26], [30], [32], [33]), we have introduced (Φ-C_*)-contractions of Hardy-Rogers type and established some unique common fixed point results for six mappings on C^*-algebra valued metric spaces, which are the generalizations of the results given in [13] and [31]. .

2. Preliminaries

Discussing the articles given in the ref. ([5],[13],[14],[18],[23],[24]) we are introducing some basic definitions, notations and results which are the following:

A complex algebra A with linear involution $* : A \to A$ satisfying $(uv)^* = v^*u^*$ and $u^{**} = u$, for all $u, v \in A$ is said to Banach $*$-algebra if it is complete with respect to sub-multiplicative norm $\|.\|$ such that for all $u \in A$, $\|u^*\| = \|u\|$. A Banach $*$- algebra satisfying $\|u^*u\| = \|u\|^2$ is called C^*-algebra. In this paper, we denote A by an unital(unity element I_A) C^*-algebra with linear involution $*$.
An element $u \in A$ is called a positive element and denote it by $\theta_A \preceq u$, (where θ_A is the zero element in A) if $u = u^*$ and $\rho(u) \subset [0, +\infty)$, where $\rho(u)$ is the spectrum of u. A norm on A is defined by $\|u\| = (u^*u)^{\frac{1}{2}}$, $\forall u \in A$. Set $A_h = \{u \in A : u = u^*\}$ and we define a partial ordering $\preceq$ on A_h by $u \preceq v$ if and only if $\theta_A \preceq v - u$. Here A_+ and A' are defined by $A_+ = \{u \in A : \theta_A \preceq u\}$ and $A' = \{u \in A : uv = vu, \forall v \in A\}$. As A is a unital C^*-algebra, then for all $u \in A_+$ we have $u \preceq I_A \Leftrightarrow \|u\| \leq 1$ and $\|I_A\| = 1$.

Definition 2.1. [13]. Let X be a non-empty set and $d : X \times X \to A$ be a mapping such that for $x, y, z \in X$, where z is different from x and y, satisfying the followings:

(c_1) $d(x, y) = \theta_A$ if and only if $x = y$; and $\theta_A \preceq d(x, y)$, for all $x, y \in X$;
(c_2) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(c_3) $d(x, z) \preceq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Then (X, A, d) is called a C^*-algebra valued metric space.

Note: If $A = \mathbb{R}$, then, C^*-algebra valued metric space becomes equivalent to the definition to the real metric space.

Example 2.1. *Let $X = [0, 2]$, $A = \mathbb{R}^2$. Then A is C^*-algebra with norm on A is defined by*

$$\|x, y\| = |x| + |y|;\ \forall\ (x, y) \in \mathbb{R}^2.$$

Also define C^*-algebra valued metric on X by

$$d(x, y) = (2|x - y|, 3|x - y|);\ \forall\ x, y \in X,$$

with the ordering on A by $(e, f) \preceq (g, h)$ if and only if $e \le g$ and $f \le h$.

Definition 2.2. [13]. Let (X, A, d) be a C^*-algebra valued metric space. A sequence $\{x_n\} \subset X$ is said to be convergent to x in X with respect to A if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N, \|d(x_n, x)\| < \epsilon$.

Definition 2.3. [16]. Let (X, A, d) be a C^*-algebra valued metric space.A sequence $\{x_n\} \subset X$ is said to be Cauchy in X with respect to A if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N, \|d(x_n, x_m)\| < \epsilon$.

The space (X, A, d) is complete C^*-algebra valued metric space if every Cauchy sequence in X with respect to A is convergent to a point in X.

Definition 2.4. ([11],[28]). Let f and g be two maps on a metric space (X, d). If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g; and w is called a point of coincidence of f and g.

Definition 2.5. ([11],[28]). A pair of maps f and g on a metric space (X, d) is called weakly compatible if they commute at coincidence points.

Example 2.2. *Let $X = [0, 3]$ and define two functions f, g on X as*

$$\begin{aligned} f(x) &= 2x^2, & 0 \le x \le 1; \\ &= x + 1, & 1 \le x \le 2; \\ &= 6 - x, & 2 < x \le 3. \end{aligned} \quad \text{and} \quad \begin{aligned} g(x) &= \tfrac{x}{2}, & 0 \le x < 1; \\ &= 3, & 1 \le x \le 2; \\ &= 12 - x^2, & 2 < x \le 3. \end{aligned}$$

Here, $f(0) = g(0) = 0; f(2) = g(2) = 3$ and $f(3) = g(3) = 3$. So, $fg(0) = gf(0) = 0; fg(2) = gf(2) = 3$ and $fg(3) = gf(3) = 3$. Then the pair $\{f, g\}$ is weakly compatible.

Lemma 2.1. [34]*If (E, τ) is a topological vector space ordered by a closed cone K and if C is a compact subset of (E, τ), the supremum(imfimum) of each increasing(decreasing) net in C exists and the net converges to it with respect to τ.*

Lemma 2.2. [31]. *Let (X, A, d) be a C^*-algebra valued metric space. Then following are hold:*

(l_1) *If $\{U_n\}_{n=1}^{\infty} \subset A$ and $\lim_{n\to\infty} U_n = \theta_A$, then for any $V \in A$, $\lim_{n\to\infty} V^* U_n V = \theta_A$.*

(l_2) *If $U, V \in A_+$ and $W \in A'_+$, then $U \preceq V$ deduces $WU \preceq WV$, where $A'_+ = A_+ \cap A'$*

(l_3) *Limit of a convergent sequence in a C^*- algebra valued metric space is unique.*

Definition 2.6. [21]. A function $F : (0, \infty) \to \mathbb{R}$ is said to be F-contraction if it satisfies the following conditions:
(i) F is strictly increasing; (ii) For every sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} F(t_n) = -\infty$; (iii) There exists a constant $l \in (0, 1)$ such that $x^l F(t) \to 0$, when $t \to 0^+$.

Definition 2.7. [16]. Let (X, d) be a metric space such that the mappings $F : (0, \infty) \to \mathbb{R}$ and $\phi : (0, \infty) \to (0, \infty)$ are satisfying the following:
(i) F is strictly increasing, i.e., $x < y$ implies $F(x) < F(y)$, for all $x, y \in (0, \infty)$; (ii) $\lim_{\alpha\to 0^+} \phi(\alpha) > 0$, for all $s > 0$.
A mapping $T : X \to X$ is called an (ϕ, F)-contraction on (X, d) if

$$\phi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y)),\ \forall x, y \in X \text{ for which } Tx \neq Ty.$$

3. Main Results

To established our results first we define the following:

Definition 3.1. A function $\psi : A_+ \to A_+$ is said to be monotonic non-decreasing with respect to $\preceq$ if

$$U \preceq V \Rightarrow \psi(U) \preceq \psi(V),\ \forall\, U, V \in A_+.$$

Definition 3.2. A function $J_* : A_+ \times A_+ \to A_+$ is called a non-decreasing function with respect to $\preceq$ if

$$U \subset W, V \subset Z \Rightarrow J_*(U, V) \preceq J_*(W, Z).$$

Definition 3.3. A sequence $\{W_n\} \subset A_+$ is said to be bounded below with respect to $\preceq$ if there exists $L \in A_+$ such that $L \preceq W_n$ with $\|L\| \leq \|W_n\|, \forall n \in \mathbb{N}$.

Definition 3.4. (C_* -class function). Let A be a unit C^*-algebra. Then a continuous function $J_* : A_+ \times A_+ \to A_+$ is called a C_*-class function if for any $U, V \in A_+$, the following conditions hold:
(i) J_* is non-decreasing; (ii) $J_*(U, V) \preceq U$; (iii) $J_*(U, V) = U$ implies either $U = \theta_A$, or, $V = \theta_A$; (iv) $J_*(\theta_A, \theta_A) = \theta_A$.
Note: Here C_* will denote the class of all C_*-class function.

Definition 3.5. Φ be the set of all functions $\phi : A_+ \to A_+$ such that the following conditions hold: (i) ϕ is monotonic non-decreasing; (ii) $\lim_{Y \to U} \phi(Y)$ exists, for all $U \in A_+$; (iii) ϕ is continuous at θ_A and $\phi(U) = \theta_A \Leftrightarrow U = \theta_A$.

Definition 3.6. ((Φ-C_*)-contraction of Hardy-Rogers type). Let (X, A, d) be a complete C^*-algebra valued metric space. A function T on X is said to be (Φ-C_*)-contraction of Hardy-Rogers type if for all $x, y \in X$, the following conditions hold:

$$\psi\{d(Tx, Ty)\} \preceq J_*\{\psi\{P(x, y)\}, \phi\{P(x, y)\}\};\ \psi, \phi \in \Phi \text{ and } J_* \in C_*;$$

$$\begin{aligned}\text{where, } P(x, y) =&W_1 d(x, y) + W_2 d(x, Ty) + W_3 d(y, Tx) + W_4 d(x, Tx)+\\ &W_5 d(y, Ty);\ W_i \in A',\ \forall\ i = 1, 2, 3, 4, 5.\end{aligned}$$

with $W_1 + W_2 + W_3 + W_4 + W_5 \preceq I_A;\ \theta_A \preceq W_i,\ \forall\ i = 1, 2, 3, 4, 5.$

Note: This contraction is the generalizations of F-contraction, (ϕ, F)-contraction and (ψ, φ)-weak contraction.

Theorem 3.1. *Let (X, A, d) be a complete C^*-algebra valued metric space and F, G, S, T, H and L be self maps on X such that for all $x, y \in X$, the following conditions hold:*

(3.1)
$$\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{P(x, y)\}, \phi\{P(x, y)\}\},\ \psi, \phi \in \Phi \textit{ and } J_* \in C_*;$$

where,

(3.2)
$$\begin{aligned}P(x, y) = &W_1 d(FGx, Hx) + W_2 d(STy, Ly) + W_3 d(STy, Hx)+\\ &W_4 d(FGx, Ly) + W_5 d(FGx, STy);\ W_i \in A',\ \forall\ i = 1, 2, 3, 4, 5.\end{aligned}$$

with $W_1 + W_2 + 2(W_3 + W_4) + W_5 \preceq I_A;\ \theta_A \preceq W_i,\ \forall\ i = 1, 2, 3, 4, 5;$
and $\|W_1\| + \|W_2\| + 2(\|W_3\| + \|W_4\|) + \|W_5\| \leq 1;$
Also
(i)$H(X) \subset ST(X), L(X) \subset FG(X)$*; (ii)* $FG = GF, ST = TS, HG =$

$GH, LT = TL$; (iii) The pairs $\{H, FG\}$ and $\{L, ST\}$ are weekly compatible and one of the ranges $H(X), FG(X), L(X)$ and $ST(X)$ is complete in X.
Then F, G, S, T, H, L have a unique common fixed point in X.

Proof: We will prove the fixed point result by step by step process.
Let $x_0 \in X$. From condition (i), there exist $x_1, x_2 \in X$ such that $Hx_0 = STx_1 = y_0$ and $Lx_1 = FGx_2 = y_1$. Proceeding inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = Hx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Lx_{2n+1} = FGx_{2n+2}$, for $n = 0, 1, 2, ...$

Case-I: Assume for some $n \in \mathbb{Z}^+,\ y_n = y_{n+1}$ implies $y_{n+1} = y_{n+2}$.
(3.3)
Now for is n even i.e., for $n = 2m, m \in \mathbb{Z}^+,$ we have, $y_{2m} = y_{2m+1}$

Now from (3.2) we have,

(3.4)
$$\begin{aligned}
&P(x_{2m+2}, x_{2m+1}) \\
&= W_1 d(FGx_{2m+2}, Hx_{2m+2}) + W_2 d(STx_{2m+1}, Lx_{2m+1}) + \\
&W_3 d(STx_{2m+1}, Hx_{2m+2}) + W_4 d(FGx_{2m+2}, Lx_{2m+1}) + \\
&W_5 d(FGx_{2m+2}, STx_{2m+1}) \\
&= W_1 d(y_{2m+1}, y_{2m+2}) + W_2 d(y_{2m}, y_{2m+1}) + W_3 d(y_{2m}, y_{2m+2}) + \\
&W_4 d(y_{2m+1}, y_{2m+1}) + W_5 d(y_{2m+1}, y_{2m}) \\
&= W_1 d(y_{2m+1}, y_{2m+2}) + W_2 d(y_{2m}, y_{2m+1}) + W_3 d(y_{2m+1}, y_{2m+2}) + \\
&W_5 d(y_{2m+1}, y_{2m+1}) \\
&= (W_1 + W_3) d(y_{2m+1}, y_{2m+2}) \\
&\preceq d(y_{2m+1}, y_{2m+2})
\end{aligned}$$

Now from (3.1) we have,

$$\begin{aligned}
\psi\{d(y_{2m+2}, y_{2m+1})\} &\preceq J_*\{\psi\{P(x_{2m+2}, x_{2m+1})\}, \phi\{P(x_{2m+2}, x_{2m+1})\}\} \\
&\preceq J_*\{\psi\{d(y_{2m+1}, y_{2m+2})\}, \phi\{d(y_{2m+1}, y_{2m+2})\}\} \\
&\preceq \psi\{d(y_{2m+1}, y_{2m+2})\}, \text{ which gives}
\end{aligned}$$

$J_*\{\psi\{d(y_{2m+1}, y_{2m+2})\}, \phi\{d(y_{2m+1}, y_{2m+2})\}\} = \psi\{d(y_{2m+1}, y_{2m+2})\}$
So, by definition either

$$\psi\{d(y_{2m+1}, y_{2m+2})\} = \theta_A,$$

or

$$\phi\{d(y_{2m+1}, y_{2m+2})\} = \theta_A.$$

Hence

$$d(y_{2m+1}, y_{2m+2}) = \theta, \text{ implies } y_{2m+1} = y_{2m+2}.$$

Therefore

$$y_{2m} = y_{2m+1} \text{ implies } y_{2m+1} = y_{2m+2}. \tag{3.5}$$

For n is odd i.e., for $n = 2m+1, m \in \mathbb{N} \cup \{0\}$, then

$$y_{2m+1} = y_{2m+2}, \text{ implies } y_{2m+2} = y_{2m+3}. \tag{3.6}$$

From (3.5) and (3.6) we have,

$$y_n = y_{n+1}, \text{ implies } y_{n+1} = y_{n+2}, \ \forall\ n = 1, 2, 3, ...$$

Proceeding in this manner we have $y_n = y_{n+1}$ implies $y_n = y_{n+k}$, for all $k = 1, 2, 3, ...$

Therefore, $\{y_n\}$ becomes a constant sequence and hence a Cauchy one in X.

Case-II: Assume $y_n \neq y_{n+1}$, for each $n = 1, 2, 3, ...$

Now for $n = 2m$, from (3.2) we have,

(3.7)

$$\begin{aligned}
&P(x_{2m}, x_{2m+1})\\
&= W_1 d(FGx_{2m}, Hx_{2m}) + W_2 d(STx_{2m+1}, Lx_{2m+1}) + W_3 d(STx_{2m+1}, Hx_{2m})\\
&\ + W_4 d(FGx_{2m}, Lx_{2m+1}) + W_5 d(FGx_{2m}, STx_{2m+1})\\
&= W_1 d(y_{2m-1}, y_{2m}) + W_2 d(y_{2m}, y_{2m+1}) + W_3 d(y_{2m}, y_{2m}) + W_4 d(y_{2m-1}, y_{2m+1})\\
&\ + W_5 d(y_{2m-1}, y_{2m})\\
&= W_1 d(y_{2m-1}, y_{2m}) + W_2 d(y_{2m}, y_{2m+1}) + W_4 d(y_{2m-1}, y_{2m}) + W_4 d(y_{2m}, y_{2m+1})\\
&\ + W_5 d(y_{2m-1}, y_{2m})
\end{aligned}$$

If $d(y_{2m}, y_{2m-1}) \preceq d(y_{2m+1}, y_{2m})$, then from (3.2) and using (3.7) we have,

$$\begin{aligned}
P(x_{2m}, x_{2m+1}) &\preceq (W_1 + W_2 + 2W_4 + W_5) d(y_{2m+1}, y_{2m})\\
&\preceq d(y_{2m+1}, y_{2m}) \ [as\ (W_1 + W_2 + 2W_4 + W_5) \preceq I_A.]
\end{aligned} \tag{3.8}$$

Now from (3.1) we have,

$$\begin{aligned}
\psi\{d(Hx_{2m}, Lx_{2m+1})\} &\preceq J_*\{\psi\{P(x_{2m}, x_{2m+1})\}, \phi\{P(x_{2m}, x_{2m+1})\}\}\\
\text{or, } \psi\{d(y_{2m}, y_{2m+1})\} &\preceq J_*\{\psi\{d(y_{2m+1}, y_{2m})\}, \phi\{d(y_{2m+1}, y_{2m})\}\}\\
&\preceq \psi\{d(y_{2m+1}, y_{2m})\}, \text{ which implies}
\end{aligned}$$

$J_*\{\psi\{d(y_{2m+1}, y_{2m})\}, \phi\{d(y_{2m+1}, y_{2m})\}\} = \psi\{d(y_{2m+1}, y_{2m})\}$.

So, by definition either $\psi\{d(y_{2m+1}, y_{2m})\} = \theta_A$, or, $\phi\{d(y_{2m+1}, y_{2m})\} = \theta_A$.

Hence, $d(y_{2m+1}, y_{2m}) = \theta_A$, implies $y_{2m+1} = y_{2m}$, which is a contradiction.

$$\text{Hence, } d(y_{2m+1}, y_{2m}) \prec d(y_{2m}, y_{2m-1}). \tag{3.9}$$

For n is odd i.e., for $n = 2m+1, m \in \mathbb{N} \cup \{0\}$, then from (3.2) we have,

$$\begin{aligned} & P(x_{2m+2}, x_{2m+1}) \\ & = W_1 d(FGx_{2m+2}, Hx_{2m+2}) + W_2 d(STx_{2m+1}, Lx_{2m+1}) + \\ & W_3 d(STx_{2m+1}, Hx_{2m+2}) + W_4 d(FGx_{2m+2}, Lx_{2m+1}) + \\ & W_5 d(FGx_{2m+2}, STx_{2m+1}) \\ & = W_1 d(y_{2m+1}, y_{2m+2}) + W_2 d(y_{2m}, y_{2m+1}) + \\ & W_3 d(y_{2m}, y_{2m+2}) + W_4 d(y_{2m+1}, y_{2m+1}) + \\ & W_5 d(y_{2m+1}, y_{2m}) \\ & \preceq W_1 d(y_{2m+1}, y_{2m+2}) + W_2 d(y_{2m}, y_{2m+1}) + W_3 d(y_{2m}, y_{2m+1}) + \\ & W_4 d(y_{2m+1}, y_{2m+2}) + W_5 d(y_{2m+1}, y_{2m}) \end{aligned} \tag{3.10}$$

If $d(y_{2m}, y_{2m+1}) \preceq d(y_{2m+1}, y_{2m+2})$, then from (3.2) we have,

$$\begin{aligned} P(x_{2m+2}, x_{2m+1}) & \preceq (W_1 + W_2 + 2W_3 + W_5) d(y_{2m+1}, y_{2m+2}) \\ & \preceq d(y_{2m+1}, y_{2m}) \ [\text{as } (W_1 + W_2 + 2W_3 + W_5) \preceq I_A.] \end{aligned} \tag{3.11}$$

Now from (3.1) we have,

$$\begin{aligned} \psi\{d(Hx_{2m+2}, Lx_{2m+1})\} & \preceq J_*\{\psi\{P(x_{2m+2}, x_{2m+1})\}, \phi\{P(x_{2m+2}, x_{2m+1})\}\} \\ \text{or}, \psi\{d(y_{2m+2}, y_{2m+1})\} & \preceq J_*\{\psi\{d(y_{2m+2}, y_{2m+1})\}, \phi\{d(y_{2m+2}, y_{2m+1})\}\} \\ & \preceq \psi\{d(y_{2m+2}, y_{2m+1})\}, \text{ which gives} \end{aligned}$$

$J_*\{\psi\{d(y_{2m+2}, y_{2m+1})\}, \phi\{d(y_{2m+2}, y_{2m+1})\}\} = \psi\{d(y_{2m+2}, y_{2m+1})\}$.
So, by definition either $\psi\{d(y_{2m+2}, y_{2m+1})\} = \theta_A$, or, $\phi\{d(y_{2m+2}, y_{2m+1})\} = \theta_A$.
Hence, $d(y_{2m+2}, y_{2m+1}) = \theta_A \Rightarrow y_{2m+2} = y_{2m+1}$, which is a contradiction.
Hence,

$$d(y_{2m+1}, y_{2m+2}) \prec d(y_{2m}, y_{2m+1}) \tag{3.12}$$

From (3.9) and (3.12) we have,

$$d(y_n, y_{n+1}) \prec d(y_{n-1}, y_n), \ \forall \ n = 1, 2, 3, ... \tag{3.13}$$

Therefore, the sequence $\{d(y_n, y_{n+1})\}$ is monotonic decreasing with respect to $\preceq$ and bounded below by θ_A and hence convergent.

Let $\lim_{n\to\infty} d(y_n, y_{n+1}) = U$, $\lim_{n\to\infty} \psi\{d(y_n, y_{n+1})\} = V$ and $\lim_{n\to\infty} \phi\{d(y_n, y_{n+1})\} = W$, where $U, V, W \in A_+$, i.e., $\theta \preceq U, V, W$.
Claim: $U = \theta_A$ i.e., $\lim_{n\to\infty} d(y_n, y_{n+1}) = \theta_A$.
Now from (3.1) we have,
(3.14)
$$\begin{aligned} \psi\{d(y_{2m}, y_{2m+1})\} &\preceq J_*\{\psi\{P(x_{2m}, x_{2m+1})\}, \phi\{P(x_{2m}, x_{2m+1})\}\} \\ \text{or, } \psi\{d(Hx_{2m}, Lx_{2m+1})\} &\preceq J_*\{\psi\{d(y_{2m-1}, y_{2m})\}, \phi\{d(y_{2m-1}, y_{2m})\}\} \\ &\preceq \psi\{d(y_{2m-1}, y_{2m})\} \end{aligned}$$

Taking limit as $m \to \infty$ on both sides of (3.14) we have,

$$\begin{aligned} \lim_{m\to\infty} \psi\{d(y_{2m}, y_{2m+1})\} &\preceq \lim_{m\to\infty} J_*\{\psi\{P(x_{2m}, x_{2m+1})\}, \phi\{P(x_{2m}, x_{2m+1})\}\} \\ &\preceq \lim_{m\to\infty} J_*\{\psi\{d(y_{2m-1}, y_{2m})\}, \phi\{d(y_{2m-1}, y_{2m})\}\} \\ &= J_*\{\lim_{m\to\infty} \psi\{d(y_{2m-1}, y_{2m})\}, \lim_{m\to\infty} \phi\{d(y_{2m-1}, y_{2m})\}\} \\ &\preceq \lim_{m\to\infty} \psi\{d(y_{2m-1}, y_{2m})\}, \text{ which gives} \end{aligned}$$

(3.15)
$$\begin{aligned} &V \preceq J_*\{\lim_{m\to\infty} \psi\{d(y_{2m-1}, y_{2m})\}, \lim_{m\to\infty} \phi\{d(y_{2m-1}, y_{2m})\}\} \preceq V \\ &\text{or, } V \preceq J_*(V, W) \preceq V, \text{ which gives } J_*(V, W) = V \end{aligned}$$

Therefore, either $V = \theta_A$, or, $W = \theta_A$, which imply that
either $\lim_{m\to\infty} \psi\{d(y_{2m-1}, y_{2m})\} = \theta_A$, or, $\lim_{m\to\infty} \phi\{d(y_{2m-1}, y_{2m})\} = \theta_A$.
Therefore, in both cases $\lim_{m\to\infty} d(y_{2m-1}, y_{2m}) = \theta_A$.
Hence,

(3.16)
$$\lim_{m\to\infty} d(y_{2m-1}, y_{2m}) = \theta_A$$

Similarly for $n = 2m + 1$, we have $U = \theta_A$ i.e.,

(3.17)
$$\lim_{m\to\infty} d(y_{2m+1}, y_{2m}) = \theta_A$$

Hence from (3.16) and (3.17) we have,

(3.18)
$$\lim_{n\to\infty} d(y_{n+1}, y_n) = \theta_A$$

and hence,

(3.19)
$$\lim_{n\to\infty} \|d(y_{n+1}, y_n)\| = 0$$

Now we prove that $\{y_n\}$ is Cauchy in X with respect to A. For this it is sufficient to show that $\{y_{2n}\}$ is Cauchy in X.

If not, then for $\epsilon > 0$, there exist integers $2n_k$ and $2m_k$ with $2m_k > 2n_k > k$ such that

$$\|d(y_{2m_k}, y_{2n_k})\| > \epsilon \text{ and } \|d(y_{2m_k-2}, y_{2n_k})\| \leq \epsilon \tag{3.20}$$

Now,

$$\epsilon < \|d(y_{2m_k}, y_{2n_k})\| \leq \|d(y_{2n_k}, y_{2m_k-2})\| + \|d(y_{2m_k-2}, y_{2m_k-1})\| + \|d(y_{2m_k-1}, y_{2m_k})\|.$$

Taking limit as $k \to \infty$ we get $\epsilon \leq \lim_{k\to\infty} \|d(y_{2m_k}, y_{2n_k})\| \leq \epsilon$, which gives

$$\lim_{k\to\infty} \|d(y_{2m_k}, y_{2n_k})\| = \epsilon \tag{3.21}$$

Again $\epsilon < \|d(y_{2m_k+1}, y_{2n_k})\| \leq \|d(y_{2m_k}, y_{2m_k+1})\| + \|d(y_{2m_k+1}, y_{2n_k})\|$
Taking limit as $k \to \infty$ we get

$$\epsilon \leq \lim_{k\to\infty} \|d(y_{2m_k+1}, y_{2n_k})\| \leq \epsilon, \text{which gives } \lim_{k\to\infty} \|d(y_{2m_k+1}, y_{2n_k})\| = \epsilon \tag{3.22}$$

By similar process we obtained that

$$\lim_{k\to\infty} \|d(y_{2m_k}, y_{2n_k+1})\| = \epsilon \text{ and } \lim_{k\to\infty} \|d(y_{2n_k-1}, y_{2m_k+1})\| = \epsilon \tag{3.23}$$

Therefore, there exists $U \in A_+$ with $\|U\| = \epsilon$ such that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k+1}) = U.$$

Furthermore, there exists $K \in \mathbb{N}$ and with $\epsilon > 0$ such that

$$\|d(y_{2n_k-1}, y_{2m_k}\| > \tfrac{\epsilon}{2},\ \|d(y_{2n_k-1}, y_{2n_k}\| < \tfrac{\epsilon}{2},\ \|d(y_{2m_k+1}, y_{2m_k})\| < \tfrac{\epsilon}{2},$$
$$\|d(y_{2n_k}, y_{2m_k+1})\| > \tfrac{\epsilon}{2}\ \forall\ 2m_k, 2n_k > K.$$

Now for $2m_k, 2n_k > K$, from (3.2) we have,

$$\begin{aligned} &P(x_{2n_k}, x_{2m_k+1}) \\ =&W_1 d(FGx_{2n_k}, Hx_{2m_k}) + W_2 d(STx_{2m_k+1}, Lx_{2m_k+1}) + \\ &W_3 d(STx_{2m_k+1}, Hx_{2n_k}) + W_4 d(FGx_{2n_k}, Lx_{2m_k+1}) + \\ &W_5 d(FGx_{2n_k}, STx_{2m_k+1}) \\ =&W_1 d(y_{2n_k-1}, y_{2n_k}) + W_2 d(y_{2m_k}, y_{2m_k+1}) + W_3 d(y_{2m_k}, y_{2n_k}) + \\ &W_4 d(y_{2n_k-1}, y_{2m_k+1}) + W_5 d(y_{2n_k-1}, y_{2m_k}) \end{aligned} \tag{3.24}$$

Therefore, from (3.24) we have,

(3.25)
$$\begin{aligned}\|P(x_{2n_k}, x_{2m_k+1})\| \leq &\|W_1\|\|d(y_{2n_k-1}, y_{2n_k})\| + \|W_2\|\|d(y_{2m_k}, y_{2m_k+1})\|+\\ &\|W_3\|\|d(y_{2m_k}, y_{2n_k})\| + \|W_4\|\|d(y_{2n_k-1}, y_{2m_k+1})\|+\\ &\|W_5\|\|d(y_{2n_k-1}, y_{2m_k})\|.\end{aligned}$$

Taking limit as $k \to \infty$ on both sides of (3.24) we have,

(3.26)
$$\begin{aligned}\lim_{n\to\infty} P(x_{2n_k}, x_{2m_k+1}) = &\lim_{n\to\infty}\{W_1 d(y_{2n_k-1}, y_{2n_k}) + W_2 d(y_{2m_k}, y_{2m_k+1})+\\ &W_3 d(y_{2m_k}, y_{2n_k}) + W_4 d(y_{2n_k-1}, y_{2m_k+1})+\\ &W_5 d(y_{2n_k-1}, y_{2m_k})\}\\ =&(W_3 + W_4 + W_5)U\\ \preceq &U\end{aligned}$$

Now (3.1) gives,

(3.27)
$$\begin{aligned}\psi\{d(y_{2n_k}, y_{2m_k+1})\} &= \psi\{d(Hx_{2n_k}, Lx_{2m_k+1})\}\\ &\preceq J_*\{\psi\{P(x_{2n_k}, x_{2m_k+1})\}, \phi\{P(x_{2n_k}, x_{2m_k+1})\}\}\\ &\preceq \psi\{P(x_{2n_k}, x_{2m_k})\}\end{aligned}$$

Taking lower limit as $k \to \infty$ on both sides of (3.27) we have,

(3.28)
$$\begin{aligned}&\lim_{k\to\infty} \psi\{d(y_{2n_k}, y_{2m_k+1})\} \preceq \lim_{k\to\infty} J_*\{\psi\{P(x_{2n_k}, x_{2m_k+1})\}, \phi\{P(x_{2n_k}, x_{2m_k+1})\}\}\\ &\preceq \lim_{k\to\infty} \psi\{P(x_{2n_k}, x_{2m_k})\}\\ &\lim_{k\to\infty} \psi\{d(y_{2n_k}, y_{2m_k+1})\} \preceq J_*\{\lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}, \lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\}\}\\ &\preceq \lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}\end{aligned}$$

As $\lim_{n\to\infty} P(x_{2n_k}, x_{2m_k+1}) \preceq U = \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k+1})$, then from (3.28) we have

$J_*\{\lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}, \lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\}\} = \lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\}$

Hence, either $\lim_{k\to\infty} \psi\{d(x_{2n_k}, x_{2m_k+1})\} = \theta_A$, or, $\lim_{k\to\infty} \phi\{d(x_{2n_k}, x_{2m_k+1})\} = \theta_A$,

which implies that $\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) = \theta_A$, which is a contradiction.

Hence, $\{y_n\}$ is a Cauchy sequence in A.

Since (X, A, d) is complete C^* algebra valued metric space so, there exists $z \in X$ such that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Hx_{2n} = \lim_{n\to\infty} STx_{2n+1} = z. \tag{3.29}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Lx_{2n+1} = \lim_{n\to\infty} FGx_{2n+2} = z. \tag{3.30}$$

Assume $ST(X)$ is complete in X, then there exists $u \in X$ such that $z = STu$.

Now,

Step-1.

Claim: $Lu = z$**.**

Putting $x = x_{2n}$ and $y = u$ in (3.2) we have,

(3.31)

$$P(x_{2n}, u) = W_1 d(FGx_{2n}, Hx_{2n}) + W_2 d(STu, Lu) + W_3 d(STu, Hx_{2n}) + W_4 d(FGx_{2n}, Lu) + W_5 d(FGx_{2n}, STu)$$

Taking limit as $n \to \infty$ on both sides of (3.31) we have,

(3.32)

$$\begin{aligned}\lim_{n\to\infty} P(x_{2n}, u) &= W_1 d(z, z) + W_2 d(z, Lu) + W_3 d(z, z) + W_4 d(z, Lu) + \\ &\quad W_5 d(z, z) \\ &= (W_2 + W_4) d(z, Lu) \\ &\preceq d(z, Lu)\end{aligned}$$

Now from (3.1) and using (3.32) we have,

$$\psi\{d(HGz, Lx_{2n+1})\} \preceq J_*\{\psi\{P(Gz, x_{2n+1})\}, \phi\{P(Gz, x_{2n+1})\}\} \tag{3.33}$$

Taking limit as $n \to \infty$ on both sides of (3.33) we have,

(3.34)

$$\begin{aligned}&\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} \preceq \lim_{n\to\infty} J_*\{\psi\{P(x_{2n}, u)\}, \phi\{P(x_{2n}, u)\}\} \\ \text{or, } &\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} \preceq J_*\{\lim_{n\to\infty} \psi\{P(x_{2n}, u)\}, \lim_{n\to\infty} \phi\{P(x_{2n}, u)\}\} \\ \text{or, } &\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Lu)\}\} \\ &\qquad \preceq \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}\end{aligned}$$

Therefore,

$$J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Lu)\}\} = \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\}$$

Hence, either $\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu)\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Hx_{2n}, Lu)\} = \theta_A$
Thus,

$$\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lu) = \theta_A, \text{ which gives}$$
$$\lim_{n\to\infty} d(Hx_{2n}, Lu) = \theta_A, \text{ implies } d(z, Lu) = \theta_A.$$

Hence,

$$Lu = z \text{ and therefore, } STu = Lu = z. \tag{3.35}$$

Since $\{L, ST\}$ is weakly compatible, so

$$Lz = L(STu) = ST(Lu) = STz. \tag{3.36}$$

Step-2.
Claim: $Lz = z$.
Now from (3.2) we have,

$$\begin{aligned} P(x_{2n}, z) =& W_1 d(FGx_{2n}, Hx_{2n}) + W_2 d(STz, Lz) + W_3 d(STz, Hx_{2n}) + \\ & W_4 d(FGx_{2n}, Lz) + W_5 d(FGx_{2n}, STz) \end{aligned} \tag{3.37}$$

Taking limit as $n \to \infty$ on both sides on (3.37) we have,

$$\begin{aligned} \lim_{n\to\infty} P(x_{2n}, z) =& W_1 d(z, z) + W_2 d(STz, Lz) + W_3 d(Lz, z) + W_4 d(z, Lz) + \\ & W_5 d(z, Lz) \\ & = (W_3 + W_4 + W_5) d(Lz, z) \\ & \preceq d(Lz, z) \end{aligned} \tag{3.38}$$

Now from (3.1) and using (3.38) we have,

$$\psi\{d(Hx_{2n}, Lz)\} \preceq J_*\{\psi\{P(x_{2n}, z)\}, \phi\{P(x_{2n}, z)\}\} \tag{3.39}$$

Taking limit as $n \to \infty$ on both sides of (3.39) we have,

$$\begin{aligned} & \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lz)\} \preceq \lim_{n\to\infty} J_*\{\psi\{P(x_{2n}, z)\}, \phi\{P(x_{2n}, z)\}\} \\ \text{or, } & \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lz)\} \preceq J_*\{\lim_{n\to\infty} \psi\{P(x_{2n}, z)\}, \lim_{n\to\infty} \phi\{P(x_{2n}, z)\}\} \\ \text{or, } & \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lz)\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Lz)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Lz)\}\} \\ & \preceq \lim_{n\to\infty} \psi\{d(Hx_{2n}, Lz)\}. \end{aligned} \tag{3.40}$$

Therefore,
$J_*\{\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)\},\lim_{n\to\infty}\phi\{d(Hx_{2n},Lz)\}\}=\lim_{n\to\infty}\psi\{d(Hx_{2n},Lx_{2n+1})\}$.
Hence, either

$$\lim_{n\to\infty}\psi\{d(Hx_{2n},Lx_{2n+1})\}=\theta_A,$$

or,

$$\lim_{n\to\infty}\phi\{d(Hx_{2n},Lx_{2n+1})\}=\theta_A.$$

Thus,

$$\begin{aligned}&\lim_{n\to\infty}\psi\{d(Hx_{2n},Lz)=\theta_A\\ &\text{or, }\lim_{n\to\infty}d(Hx_{2n},Lz)=\theta_A,\ \text{implies } d(Lz,z)=\theta_A.\end{aligned}$$

Hence,

$$Lz=z. \tag{3.41}$$

Therefore,

$$L(STu)=(ST)Lu=Lz=STz=z. \tag{3.42}$$

Now since $L(X)\subset FG(X)$, then there exists $w\in X$ such that $Lz=FGw=z$.
Then using (3.42), we have

$$Lz=FGw=STz=z. \tag{3.43}$$

Step-3.
Claim: $FGw=Hw$.
Now from (3.2) we have,

$$\begin{aligned}P(w,z)=&W_1d(FGw,Hw)+W_2d(STz,Lz)+W_3d(STz,Hw)+\\&W_4d(FGw,Lz)+W_5d(FGw,STz)\\=&W_1d(FGw,Hw)+W_2d(Lz,Lz)+W_3d(FGw,Hw)+\\&W_4d(FGw,FGw)+W_5d(FGw,FGw)\\=&(W_1+W_3)d(FGw,Hw)\\\preceq& d(FGw,Hw)\end{aligned} \tag{3.44}$$

Now from (3.2) and using (3.44) we have,

$$\begin{aligned}\psi\{d(Hw,FGw)\}&=\psi\{d(Hw,Lz)\}\\&\preceq J_*\{\psi\{P(w,z)\},\phi\{P(w,z)\}\}\\&\preceq J_*\{\psi\{d(FGw,Hw)\},\phi\{d(FGw,Hw)\}\}\\&\preceq\psi\{d(FGw,Hw)\},\end{aligned} \tag{3.45}$$

Therefore, $J_*\{\psi\{d(FGw,Hw)\},\phi\{d(FGw,Hw)\}\} = \psi\{d(FGw,Hw)\}$.
Then, either $\psi\{d(FGw,Hw)\} = \theta_A$, or, $\phi\{d(FGw,Hw)\} = \theta_A$,
which implies $d(FGw,Hw) = \theta_A$.
Hence,

$$FGw = Hw, \text{ which gives } FGw = Hw = z. \tag{3.46}$$

Since $\{H, FG\}$ is weakly compatible so,

$$Hz = HLz = H(FGw) = FG(Hw) = FGz. \tag{3.47}$$

Step-4.
Claim: $Hz = z$.
Now from (3.2) we have,

$$\begin{aligned} &P(z, x_{2n+1}) \\ =&W_1 d(FGz, Hz) + W_2 d(STx_{2n+1}, Lx_{2n+1}) + W_3 d(STx_{2n+1}, Hz) + \\ &W_4 d(FGz, Lx_{2n+1}) + W_5 d(FGz, STx_{2n+1}) \end{aligned} \tag{3.48}$$

Taking limit as $n \to \infty$ on both sides on (3.48) we have,

$$\begin{aligned} \lim_{n\to\infty} P(x_{2n}, z) =& W_1(FGz, Hz) + W_2 d(z,z) + W_3 d(z, Hz) + W_4 d(Hz, z) + \\ & W_5 d(Hz, z) \\ =&(W_3 + W_4 + W_5) d(Hz, z) \\ \preceq& d(Hz, z) \end{aligned} \tag{3.49}$$

Now from (3.1) and using (3.49) we have,

$$\psi\{d(Hz, Lx_{2n+1})\} \preceq J_*\{\psi\{P(z, x_{2n+1})\}, \phi\{P(z, x_{2n+1})\}\} \tag{3.50}$$

Taking limit as $n \to \infty$ on both sides of (3.50) we have,

$$\begin{aligned} &\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{P(z, x_{2n+1})\}, \phi\{P(z, x_{2n+1})\}\} \\ \text{or, } &\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{P(z, x_{2n+1})\}, \lim_{n\to\infty} \phi\{P(z, x_{2n+1})\}\} \\ \text{or, } &\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hz, Lx_{2n+1})\}\} \\ &\preceq \lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}, \text{ which implies} \end{aligned} \tag{3.51}$$

$J_*\{\lim_{n\to\infty}\psi\{d(Hz,Lx_{2n+1})\},\lim_{n\to\infty}\phi\{d(Hz,Lx_{2n+1})\}\}=\lim_{n\to\infty}\psi\{d(Hz,Lx_{2n+1})\}$
Then,
either $\lim_{n\to\infty}\psi\{d(Hz,Lx_{2n+1})\}=\theta_A$, or, $\lim_{n\to\infty}\phi\{d(Hz,Lx_{2n+1})\}=\theta_A$.

$$\text{Hence, } \lim_{n\to\infty}\psi\{d(Hz,Lx_{2n+1})\}=\theta_A$$
$$\text{or, } \lim_{n\to\infty} d(Hz,Lx_{2n+1})=\theta_A, \text{ implies } d(Hz,z)=\theta_A.$$

Hence,

$$Hz=z. \tag{3.52}$$

Thus,
(3.53)
$Hz=HLz=FGz=z$ and therefore, $Hz=HLz=FGz=STz=Lz=z$.

Step-5.
Claim: $Tz=z$.
As $LT=TL$ and $ST=TS$, we have $LTz=TLz=Tz$ and $ST(Tz)=T(STz)=Tz$
Now from (3.2) we have,

$$\begin{aligned}&P(x_{2n},Tz)\\&=W_1d(FGx_{2n},Hx_{2n})+W_2d(STTz,LTz)+W_3d(STTz,Hx_{2n})\\&\quad+W_4d(FGx_{2n},LTz)+W_5d(FGx_{2n},STTz)\end{aligned} \tag{3.54}$$

Taking limit as $n\to\infty$ on both sides on (3.54) we have,

$$\begin{aligned}\lim_{n\to\infty}P(x_{2n},Tz)=&W_1(z,z)+W_2d(Tz,Tz)+W_3d(Tz,z)\\&+W_4d(z,Tz)+W_5d(z,Tz)\\=&(W_3+W_4+W_5)d(z,Tz)\\\preceq&d(z,Tz)\end{aligned} \tag{3.55}$$

Now from (3.1) and using (3.55) we have,

$$\psi\{d(Hx_{2n},LTz)\}\preceq J_*\{\psi\{P(x_{2n},Tz)\},\phi\{P(x_{2n},Tz)\}\} \tag{3.56}$$

Taking limit as $n \to \infty$ on both sides of (3.56)we have,

(3.57)

$$\begin{aligned}
&\lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} \preceq \lim_{n\to\infty} J_*\{\psi\{P(x_{2n}, Tz)\}, \phi\{P(x_{2n}, Tz)\}\} \\
\text{or, } &\lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} \preceq J_*\{\lim_{n\to\infty} \psi\{P(z, x_{2n+1})\}, \lim_{n\to\infty} \phi\{P(z, x_{2n+1})\}\} \\
\text{or, } &\lim_{n\to\infty} \psi\{d(Hx_{2n}, LTz)\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\}\} \\
&\qquad \preceq \lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}
\end{aligned}$$

Therefore,
$J_*\{\lim_{n\to\infty} \psi\{d(x_{2n}, Tz)\}, \lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\}\} = \lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\}$
Then either $\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Hx_{2n}, Tz)\} = \theta_A$.

$$\begin{aligned}
\text{Therefore, } &\lim_{n\to\infty} \psi\{d(Hx_{2n}, Tz)\} = \theta_A, \text{ which gives} \\
&\lim_{n\to\infty} d(Hx_{2n}, Tz) = \theta_A, \text{ implies } d(z, Tz) = \theta_A.
\end{aligned}$$

Hence,

$$Tz = z. \tag{3.58}$$

Now $STz = Tz = z$, which implies $Sz = z$.
Therefore,

$$Sz = Tz = Lz = z. \tag{3.59}$$

Now as $FG = GF,\ HG = GH$, so

$$H(Gz) = G(Hz) = Gz \text{ and } FG(Gz) = GF(Gz) = G(FGz) = Gz.$$

Step-6.
Claim: $Gz = z$.
Now from (3.1) we have,

(3.60)

$$\begin{aligned}
&P(Gz, x_{2n+1}) \\
&= W_1 d(FGGz, Hz) + W_2 d(STx_{2n+1}, Lx_{2n+1}) + W_3 d(STx_{2n+1}, HGz) \\
&\ + W_4 d(FGGz, Lx_{2n+1}) + W_5 d(FGGz, STx_{2n+1})
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides on (3.60) we have,

(3.61)
$$\lim_{n\to\infty} P(Gz,z) = W_1(Gz,z) + W_2 d(z,z) + W_3 d(z,Gz) + W_4 d(Gz,z) + W_5 d(Gz,z)$$
$$= (W_1 + W_3 + W_4 + W_5) d(Gz,z)$$
$$\preceq d(Gz,z)$$

Now from (3.1) and using (3.61) we have,

$$\psi\{d(HGz, Lx_{2n+1})\} \preceq J_*\{\psi\{P(Gz, x_{2n+1})\}, \phi\{P(Gz, x_{2n+1})\}\} \tag{3.62}$$

Taking limit as $n \to \infty$ on both sides of (3.62) we have,

(3.63)
$$\lim_{n\to\infty} \psi\{d(HGz, Lx_{2n+1})\} \preceq \lim_{n\to\infty} J_*\{\psi\{P(Gz, x_{2n+1})\}, \phi\{P(Gz, x_{2n+1})\}\}$$
or,
$$\lim_{n\to\infty} \psi\{d(HGz, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{P(Gz, x_{2n+1})\}, \lim_{n\to\infty} \phi\{P(Gz, x_{2n+1})\}\}$$
or,
$$\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} \preceq J_*\{\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Gz, Lx_{2n+1})\}\}$$
or, $\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} \preceq \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\}$

Therefore,
$J_*\{\lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}, \lim_{n\to\infty} \phi\{d(Hz, Lx_{2n+1})\}\} = \lim_{n\to\infty} \psi\{d(Hz, Lx_{2n+1})\}$
Then either $\lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} = \theta_A$, or, $\lim_{n\to\infty} \phi\{d(Gz, Lx_{2n+1})\} = \theta_A$.

$$\text{Hence, } \lim_{n\to\infty} \psi\{d(Gz, Lx_{2n+1})\} = \theta_A, \text{ which implies}$$
$$\lim_{n\to\infty} d(Gz, Lx_{2n+1}) = \theta_A, \text{ which gives } d(Gz,z) = \theta_A.$$

Hence,

$$Gz = z. \tag{3.64}$$

Therefore,

$$Hz = Gz = Fz = z. \tag{3.65}$$

Hence, from (3.59) and (3.65) we have, $Sz = Tz = Lz = Hz = Gz = Fz = z$.
Hence, z is a common fixed point of F, G, H, L, S and T.

Uniqueness: If possible let, u and z be two distinct fixed points of F, G, H, L, S and T i.e., $Fu = Gu = Hu = Lu = Su = Tu = u$ and $Fz = Gz = Hz = Lz = Sz = Tz = z$ with $u \neq z$.
Now from (3.2) we have,

(3.66)

$$\begin{aligned} &P(u,z)\\ &= W_1 d(FGu, Hu) + W_2 d(STz, Lz) + W_3 d(STz, Hu) + W_4 d(FGu, Lz)\\ &\quad + W_5 d(FGu, STz)\\ &= (W_3 + W_4 + W_5) d(u,z)\\ &\preceq d(u,z) \end{aligned}$$

Now from (3.1) and using (3.66) we have,

$$\begin{aligned} \psi\{d(Hu, Lz)\} &\preceq J_*\{\psi\{P(u,z)\}, \phi\{P(u,z)\}\}\\ &\preceq J_*\{\psi\{d(u,z)\}, \phi\{d(u,z)\}\}\\ &\preceq \psi\{d(u,z)\} \end{aligned}$$

Therefore, $J_*\{\psi\{d(u,z)\}, \phi\{d(u,z)\}\} = \psi\{d(u,z)\}$
Hence, either $\psi\{d(u,z)\} = \theta_A$, or, $\phi\{d(u,z)\} = \theta_A$.
In both cases we have, $d(u,z) = \theta$, which is a contradiction.
Hence, $z = u$.
Therefore, F, G, H, L, S and T have a unique common fixed point in X.
This completes the proof.

Example 3.1. *Assume $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$ be the set of all bounded linear operators on a Hilbert space $\mathbb{R}^2$.*
Define $d(x,y) = \begin{pmatrix} k|x-y| & 0 \\ 0 & |x-y| \end{pmatrix}$, where $k > 0$ is a constant. Then (X, A_+, d) is a complete C^*-algebra valued metric space.

Assume $S(x) = 8x, T(x) = \frac{x}{2}, H(x) = x, L(x) = 2x, F(x) = 6x, G(x) = \frac{x}{3}$, then $ST(x) = 4x, FG(x) = 2x$.

Then $H(X) \subset ST(X), L(X) \subset FG(X)$ and $FG = GF, ST = TS, HG = GH, LT = TL$.

Now

$$d(FGx, Hx) = \begin{pmatrix} k|2x-x| & 0 \\ 0 & |2x-x| \end{pmatrix},\ d(STy, Ly) = \begin{pmatrix} k|4y-2y| & 0 \\ 0 & |4y-2y| \end{pmatrix},$$

$$d(STy, Hx) = \begin{pmatrix} k|4y-x| & 0 \\ 0 & |4y-x| \end{pmatrix},\ d(FGx, Ly) = \begin{pmatrix} k|2x-2y| & 0 \\ 0 & |2x-2y| \end{pmatrix},$$

$$d(FGx,STy)=\begin{pmatrix} k|2x-4y| & 0 \\ 0 & |2x-4y| \end{pmatrix},\ d(Hx,Ly)=\begin{pmatrix} k|x-2y| & 0 \\ 0 & |x-2y| \end{pmatrix}$$

Here

$P(x,y)$

$$=W_1\begin{pmatrix} k|2x-x| & 0 \\ 0 & |2x-x| \end{pmatrix}+W_2\begin{pmatrix} k|4y-2y| & 0 \\ 0 & |4y-2y| \end{pmatrix}+W_3\begin{pmatrix} k|4y-x| & 0 \\ 0 & |4y-x| \end{pmatrix}$$

$$+W_4\begin{pmatrix} k|2x-2y| & 0 \\ 0 & |2x-2y| \end{pmatrix}+W_5\begin{pmatrix} k|2x-4y| & 0 \\ 0 & |2x-4y| \end{pmatrix}.$$

Let

$$\psi(Q)=2Q \text{ and } \phi(Q)=Q,\ \forall Q\in A_+,\ \text{then } \psi\{d(Hx,Ly)\}=\begin{pmatrix} k|2x-4y| & 0 \\ 0 & |2x-4y| \end{pmatrix}.$$

If we take

$$W_1=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ W_2=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ W_3=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ W_4=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$W_5=\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3}; \end{pmatrix};$$

then

$$\psi\{P(x,y)\}=\begin{pmatrix} \frac{4k}{3}|2x-4y| & 0 \\ 0 & \frac{4}{3}|2x-4y| \end{pmatrix} \text{ and } \phi\{P(x,y)\}=\begin{pmatrix} \frac{2k}{3}|2x-4y| & 0 \\ 0 & \frac{2}{3}|2x-4y| \end{pmatrix}.$$

Therefore, the condition $\psi\{d(Hx,Ly)\} \preceq J_*\{\psi\{P(x,y)\},\phi\{P(x,y)\}\}$ *holds.*

Here, $x=0$ *is the unique common fixed point of* F,G,S,T,H *and* L.

Example 3.2. *Let* $X=[0,1], A=\mathbb{C}$ *and a norn on* A *is defined by* $\|(x,y)\|=\sqrt{x^2+y^2},\ \forall\ (x,y)\in\mathbb{C}$. *Also we define* C^**-metric on* X *as* $d(x,y)=(0,|x-y|),\ \forall\ x,y\in X$ *with the partial ordering* $(e,f)\preceq(g,h)$ *by* $e\le g$ *and* $f\le h$.
We consider $Hx=Lx=\frac{x}{8}$ *and* $Fx=Gx=Sx=Tx=\frac{x}{2},\ \forall\ x\in X$.
Then $FGx=STx=\frac{x}{4}$.
Also consider $\psi(U)=5U,\ \phi(U)=4U,\ \forall\ U\in A$.
Now,

$$\begin{aligned} P(x,y)=&W_1d(FGx,Hx)+W_2d(STy,Ly)+W_3d(STy,Hx)+\\ &W_4d(FGx,Ly)+W_5d(FGx,STy)\\ =&W_1d(\frac{x}{4},\frac{x}{8})+W_2d(\frac{y}{4},\frac{y}{8})+W_3d(\frac{y}{4},\frac{x}{8})+W_4d(\frac{x}{4},\frac{y}{8})+W_5d(\frac{x}{4},\frac{y}{4})\\ =&(0,\frac{1}{5}|x-y|),\ \textit{Taking } [W_1=0,W_2=0,W_3=0,W_4=0,W_5=\frac{4}{5}.] \end{aligned}$$

Then $d(Hx,Ly)=d(\frac{x}{8},\frac{y}{8})=(0,\frac{1}{8}|x-y|)$ *and therefore,* $\psi\{d(Hx,Ly)\}=(0,\frac{5}{8}|x-y|)$, $\psi\{P(x,y)\}=(0,|x-y|)$ *and* $\phi\{P(x,y)\}=(0,\frac{4}{5}|x-y|)$.

Therefore, the condition $\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{P(x,y)\}, \phi\{P(x,y)\}\}$ *holds.*
Here, $x = 0$ *is the unique common fixed point of* F, G, S, T, H *and* L.

Corollary 3.1. *Let* (X, A, d) *be a complete* C^**-algebra valued metric space and* F, G, S, T, H, L *be self maps on* X *such that for all* $x, y \in X$*, the following hold:*

$$\psi\{d(Hx, Ly)\} \preceq \psi\{P(x,y\}) - \phi\{P(x,y)\},\ \psi, \phi \in \Phi\ ; \tag{3.67}$$

where,

$$\begin{aligned} P(x,y) =& W_1 d(FGx, Hx) + W_2 d(STy, Ly) + W_3 d(STy, Hx) + \\ & W_4 d(FGx, Ly) + W_5 d(FGx, STy);\ W_i \in A',\ \forall\ i = 1,2,3,4,5. \end{aligned} \tag{3.68}$$

with $W_1 + W_2 + 2(W_3 + W_4) + W_5 \preceq I_A;\ \theta_A \preceq W_i,\ \forall\ i = 1,2,3,4,5;$
and $\|W_1\| + \|W_2\| + 2(\|W_3\| + \|W_4\|) + \|W_5\| \leq 1.$
Also
(i) $H(X) \subset ST(X), L(X) \subset FG(X)$*; (ii)* $FG = GF, ST = TS, HG = GH, LT = TL$; *(iii) The pairs* $\{H, FG\}$ *and* $\{L, ST\}$ *are weekly compatible and one of the ranges* $H(X), FG(X), L(X)$ *and* $ST(X)$ *is complete in* X*. Then* F, G, S, T, H, L *have a unique common fixed point in* X.

Corollary 3.2. *Let* (X, A, d) *be a complete* C^**-algebra valued metric space and* F, S, H, L *be self maps on* X *such that for all* $x, y \in X$*, the following hold:*

$$\psi\{d(Hx, Ly)\} \preceq J_*\{\psi\{P(x,y\}), \phi\{P(x,y)\}\},\ \psi, \phi \in \Phi \text{ and } J_* \in C_*; \tag{3.69}$$

where,

$$\begin{aligned} P(x,y) =& W_1 d(Fx, Hx) + W_2 d(Sy, Ly) + W_3 d(Sy, Hx) + \\ & W_4 d(Fx, Ly) + W_5 d(Fx, STy);\ W_i \in A',\ \forall\ i = 1,2,3,4,5. \end{aligned} \tag{3.70}$$

with $W_1 + W_2 + 2(W_3 + W_4) + W_5 \preceq I_A;\ \theta_A \preceq W_i,\ \forall\ i = 1,2,3,4,5.$
and $\|W_1\| + \|W_2\| + 2(\|W_3\| + \|W_4\|) + \|W_5\| \leq 1.$
Also
(i) $H(X) \subset S(X), L(X) \subset F(X)$*; (ii) The pairs* $\{H, F\}$ *and* $\{L, S\}$ *are weekly compatible and one of the ranges* $H(X), F(X), L(X)$ *and* $S(X)$ *is complete in* X*. Then* F, S, H, L *have a unique common fixed point in* X.

Corollary 3.3. *Let (X, A, d) be a complete C^*-algebra valued metric space and F, G, S, T, H, L are self maps on X such that for all $x, y \in X$, the following hold:*

$$\psi\{d(Hx, Ly)\} \preceq \psi\{P(x,y)\} - \phi\{P(x,y)\}, \ \psi, \phi \in \Phi; \tag{3.71}$$

where,

(3.72)
$$P(x,y) = w_1 d(FGx, Hx) + w_2 d(STy, Ly) + w_3 d(STy, Hx) + w_4 d(FGx, Ly) + w_5 d(FGx, STy);$$

with $w_1 + w_2 + 2(w_3 + w_4) + w_5 \leq 1$; $w_i \geq 0$, $\forall\ i = 1, 2, 3, 4, 5$.
Also
(i) $H(X) \subset ST(X), L(X) \subset FG(X)$; (ii) $FG = GF, ST = TS, HG = GH, LT = TL$; (iii) The pairs $\{H, FG\}$ and $\{L, ST\}$ are weekly compatible and one of the ranges $H(X), FG(X), L(X)$ and $ST(X)$ is complete in X. Then F, G, S, T, H, L have a unique common fixed point in X.

Corollary 3.4. *Let (X, A, d) be a complete C^*-algebra valued metric space and F, S, H, L are self maps on X such that for all $x, y \in X$ satisfying the following:*

(3.73)
$$\psi\{(d(Hx, Ly)\} \preceq J_*\{\psi\{P(x,y)\}, \phi\{P(x,y)\}\}, \ \psi, \phi \in \Phi \text{ and } J_* \in C_*;$$

where,

(3.74)
$$P(x,y) = w_1 d(Fx, Hx) + w_2 d(Sy, Ly) + w_3 d(Sy, Hx) + w_4 d(Fx, Ly) + w_5 d(Fx, STy);$$

with $w_1 + w_2 + 2(w_3 + w_4) + w_5 \leq 1$; $w_i \geq 0$, $\forall\ i = 1, 2, 3, 4, 5$.
Also
(i) $H(X) \subset S(X), L(X) \subset F(X)$; (ii) The pairs $\{H, F\}$ and $\{L, S\}$ are weekly compatible and one of the ranges $H(X), F(X), L(X)$ and $S(X)$ is complete in X. Then F, S, H, L have a unique common fixed point in X.

Note: If the mappings F, G, S, T, H and L satisfy F-contraction and (ϕ, F)-contraction, then the results also hold.

4. Conclusion

C^*-algebra valued metric space is the generalization of various metric spaces. In this paper we have developed C_*-class function and introduced (Φ-C_*)-contraction of Hardy-Rogers type which is the generalization of F-contractions, (ϕ, F)-contractions and (ψ, φ)-weak contractions. Using this contraction we have established unique common fixed point results for six mappings in C^*-algebra valued metric spaces. We have given some relevant corollaries and examples on our results. In this paper some new ideas are given and our results generalize many previous results in the field of the fixed point theory on various metric spaces.

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