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Research Paper

# $(\sigma, \tau)$-DERIVATION ON ORDERED $\Gamma$-SEMIHYPERRINGS

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ABSTRACT. When a suitable partial ordered relation is attached to a $\Gamma$-semihyperring, it results into an ordered $\Gamma$-semihyperring. Concepts of an ordered $\Gamma$-semihyperring, $\Gamma$-band, idempotent $\Gamma$-semihyperring, totally ordered $\Gamma$-semihyperring, positively ordered $\Gamma$-semihyperring, negatively ordered $\Gamma$-semihyperring are introduced which are useful to study derivation on ordered $\Gamma$-semihyperrings. Derivation is nothing but an additive mapping fulfilling the Leibniz rule. In this paper, we introduce the concept of $(\sigma, \tau)$-derivation which is a generalization of $\sigma$-derivation and derivation on $\Gamma$-semihyperring and study some properties of $(\sigma, \tau)$-derivation on an ordered $\Gamma$-semihyperring. Some results reflecting different natures of $(\sigma, \tau)$-derivation depending on natures of the endomorphisms are encountered.

**Key Words:** $\Gamma$-semihyperring, ordered $\Gamma$-semihyperring, $(\sigma, \tau)$-derivation.
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## 1. INTRODUCTION

Algebraic hyperstructures in which both hyperoperations are multivalued have roots in algebraic structures. Thus, hyperoperation is playing a leading role in an algebraic hyperstructure. The first step in the study of hyperstructure was taken by Marty [16] in 1934, by developing the notion of hypergroup with properties at Eight Congress of Scandinavian Mathematics at Stockholm.

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Since then a lot of remarkable work was done in different algebraic hyperstructures such as semihypergroups, hypergroups, $\Gamma$-semihypergroups, hyperrings, $\Gamma$-hyperrings etc. In an algebraic structure, two elements when operated gives back an element, whereas, in an algebraic hyperstructure, one gets a non-empty set when two elements are hyperoperated. Hyperstructure being very flexible, is useful in several fields of science. Different applications of hyperstructure are discussed in [6, 7] by Corsini and Leoreanu. Davvaz and Leoreanu-Fotea [7] generalized ring to hyperring. Later, a generalization of semiring, semihyperring and $\Gamma$-semiring was done in [8] to produce a new hyperstructure which is $\Gamma$-semihyperrings. In recent years, the theory of hyperstructure is flourished intensely. For example, Pawar et al. [19] introduced and studied quasi-ideals on $\Gamma$-semihyperrings. Types of $\Gamma$-semihyperrings like regular and strongly regular $\Gamma$-semihyperring were introduced in [21]. In [20], uniformly strongly prime $\Gamma$-semihyperring was introduced and studied with its properties. There, some results dealing with an ideal were proved with the help of sp-system and super sp-system.

When a partial order relation $\leq$ holding a monotone condition is connected to an algebraic structure, there comes orderedness. Why ordered algebraic structures became a center of attraction? The reason is that many characteristics don't work on rings, but same properties work when one considers an ordered ring. Heidari and Davvaz [10] dealt with a semihypergroup $(H, \circ)$ along with a partial order relation $\leq$ which turned as a binary relation which is compatible with $\circ$. This gave ordered semihypergroups, a generalization of ordered semigroups. Chvalina [4] gave a particular type of hypergroup known as ordering hypergroups. Later, many researchers shew interest in ordered hypergroups and gave their contribution, refer [5, 11, 12]. Characterization of ordered bi-ideals in ordered $\Gamma$-semigroups was put forward by Iampan [13].

The theory of derivation along with algebraic structures and hyperstructures is one of the most attractive research branches. This study on rings was initiated by Posner [22] in 1957. Followed by him, in 1987, Jing [14] introduced the notion of derivation on $\Gamma$-rings. Within a short period of time, this topic of research got attraction of many researchers. Differential Krasner hyperrings introduced by Asokkumar [3] in 2013 became a fascinating concept in the study of hyperstructures. In [1], work on differential $\Gamma$-semihyperrings and $\Gamma$-hyperrings was done. Moreover, in [17], various results on $\Gamma$-semihyperrings equipped with derivations were encountered. Applications of the notion of derivation are given in [18] by using derivation to comprehend the structure of a $\Gamma$-semihyperring. The notion of derivation on ring has also been generalized in various directions such as right derivation, left derivation, reverse derivation, orthogonal derivation, generalised derivation etc.

The notion of derivation on ring plays a vital role in understanding the design of ring. Like, in [23], derivation is used to determine whether the ring is commutative or not. Kaya [15] generalises few results of derivation on ring to a different type of derivation known by $(\sigma,\tau)$-derivation. Recently many authors have worked on $(\sigma,\tau)$-derivations and have given interesting outcomes. Under specific constraints, every Jordan $(\sigma,\tau)$-higher derivation is a $(\sigma,\tau)$-higher derivation on ring $R$ which is prime, this result was given by Khan, Ashraf and Haetinger [2]. Golbas and Koc [9] proved the result showing $(f,d)$ derivation on a ring $R$ is a generalized $(\sigma,\tau)$-derivation. Similarly, Rao [24] studied some properties of $(f,g)$ derivation of ordered $\Gamma$-semirings. We extend here the study of $(\sigma,\tau)$-derivation on ordered $\Gamma$-semihyperrings and investigate some properties of an additive mapping $(\sigma,\tau)$-derivation on an ordered $\Gamma$-semihyperring $R$.

The present paper consists of five sections, section 2 is nothing but the compilation of definitions which are necessary to understand a $\Gamma$-semihyperring. In section 3, we have introduced the notion of an ordered $\Gamma$-semihyperring along with some new definitions on an ordered $\Gamma$-semihyperring that are necessary to prove the properties of $(\sigma,\tau)$-derivation. Definition of derivation in context with an orderedness on $\Gamma$-semihyperring initiates section 4 and is followed by that of $\sigma$-derivation and $(\sigma,\tau)$-derivation. Characterization of $(\sigma,\tau)$-derivation $d$ with the aid of an endomorphism have been taken into account.

## 2. Preliminaries

This section is devoted to some vital terminologies which are necessary for further discussion. For detailed study readers are requested to refer [8].

**Definition 2.1.** [8] A hypergroupoid is a couple $(H,\bullet)$ wherein $H$ is a nonempty set, $\bullet : H\times H \to \mathscr{P}^*(H)$ is a hyperoperation, in which $\mathscr{P}^*(H)$ represents a cluster of all non-empty subsets of $H$.

**Definition 2.2.** [8] A hypergroupoid for which $(x\bullet y)\bullet z = x\bullet(y\bullet z)$, for every $x, y, z\in H$ is known as a semihypergroup, also, if for each $x\in H$, $x\bullet H = H = H\bullet x$, then $(H,\bullet)$ is known as a hypergroup.

**Definition 2.3.** [8] A hypergroup $(H,\bullet)$ is known as canonical if below properties are true

(1) $H$ is commutative;

(2) $H$ must have an identity known as a scalar identity or a scalar unit, in other words, there is $e \in H$ such that $x \bullet e = \{x\}$, for each $x \in H$;
(3) There exists a unique inverse for each element, that is, for each $x \in H$ there exists a unique $x^{-1} \in H$ such that $e \in x^{-1} \bullet x$;
(4) Reversibility must be hold by $H$, means for $x \in y \bullet z$, we have $z \in y^{-1} \bullet x$ and $y \in x \bullet z^{-1}$.

One can turn attention to the detailed insights provided in reference [7] for hypergroups and semihypergroups.

**Definition 2.4.** [8] A nonempty set $S$ is known as a $\Gamma$-semihypergroup where $\Gamma$ is also a nonempty set if for each $x, y, z \in S$, $\alpha, \beta \in \Gamma$, we get, $x\alpha y \subseteq S$ and $x\alpha(y\beta z) = (x\alpha y)\beta z$.

**Definition 2.5.** [8] The algebraic structure satisfying the ring like postulates is called a hyperring, that is, hyperring $(R, +, \cdot)$ in which hyperoperations are '$+$' and '$\cdot$' and $R$ forms a hypergroup with $+$ and distributive property is satisfied by an associated hyperoperation $\cdot$.

Hyperrings are distinguished into various kinds. To understand the notions of Krasner hyperring, multiplicative hyperring and $\Gamma$-hyperring, refer [8].

**Definition 2.6.** [8] $(R, +, \Gamma)$ is known as a $\Gamma$-semihyperring wherein $R$ is a commutative semihypergroup and $\Gamma$ is a commutative group if there is a map $R \times \Gamma \times R \to \mathscr{P}^*(R)$ such that for each $x, y \in R$, $\alpha \in \Gamma$, $x\alpha y$ denotes the images and $\mathscr{P}^*(R)$ stands for a cluster of all nonempty subsets of $R$ fulfilling the constraints:
(1) $x\alpha(y+z) = x\alpha y + x\alpha z$;
(2) $(x+y)\alpha z = x\alpha z + y\alpha z$;
(3) $x\alpha(y\beta z) = (x\alpha y)\beta z$;
(4) $x(\alpha+\beta)y = x\alpha y + x\beta y$.

**Definition 2.7.** [8] If $R$ is a semigroup in above definition, then $R$ is known as a multiplicative $\Gamma$-semihyperring.

**Definition 2.8.** [8] If $x\alpha y = y\alpha x (x\alpha y \bigcap y\alpha x \neq \phi)$, for each $x, y \in R$ and $\alpha \in \Gamma$, then $R$ is termed as commutative(weak commutative) $\Gamma$-semihyperring.

**Definition 2.9.** [8] A $\Gamma$-semihyperring $R$ is said to be with zero if for each $x \in R$, $\alpha \in \Gamma$ there exists $0 \in R$ such that $x \in x + 0$, $0 \in x\alpha 0$ and $0 \in 0\alpha x$.

**Definition 2.10.** [8] A $\Gamma$-semihyperring $R$ with zero is a prime if $0 \in x\alpha r\beta y$, for each $x, y, r \in R$, $\alpha, \beta \in \Gamma$, then either $x = 0$ or $y = 0$.

Now, we are considering $R$ as a multiplicative $\Gamma$-semihyperring onward.

## 3. Ordered Γ-semihyperring

In introducing and studying an ordered Γ-semihyperring, which is a Γ-semihyperring associated with a competent partial order relation, there is no way to avoid new terms and definitions. Hence, this section aims to introduce several new definitions and few examples. Definitions of Γ-band and idempotent Γ-semihyperring in reference to orderedness are introduced, which are playing a key role in analysing the nature of $(\sigma,\tau)$ derivation in the fourth section of the paper.

**Definition 3.1.** [21] If there exists $\alpha \in \Gamma$ such that $e\alpha e = e$, wherein $e \in R$, then $e$ is known as an idempotent element of a Γ-semihyperring $R$. Here $e$ is called $\alpha$-idempotent.

**Definition 3.2.** A right(left) Γ-identity of a Γ-semihyperring $R$ is an element $e \in R$ such that $x = x\alpha e (x = e\alpha x)$, for each $x \in R$, $\alpha \in \Gamma$.

**Definition 3.3.** [21] An element $e$ in $R$ such that $x\alpha e = e\alpha x = x$, for each $x \in R$, $\alpha \in \Gamma$ is known as Γ-identity of a Γ-semihyperring.

**Definition 3.4.** Let $(R,+,\Gamma)$ be a Γ-semihyperring. Then a semigroup $(R,+)$ is said to be a band if $x = x+x$, for all $x \in R$.

**Definition 3.5.** A Γ-semihyperring $(R,+,\Gamma)$ is said to be Γ-band if $x\alpha y = x\alpha y + x\alpha y$, for all $x,y \in R$ and $\alpha \in \Gamma$.

**Definition 3.6.** If every element of a Γ-semihyperring $R$ is an idempotent of $R$ and a semigroup $(R,+)$ is a band, then $R$ is said to be an idempotent Γ-semihyperring.

**Definition 3.7.** A zero divisor of a Γ-semihyperring $R$ containing zero is a non-zero element $x \in R$ such that $0 \in x\alpha y$ and $0 \in y\alpha x$, wherein $\alpha \in \Gamma$, for some nonzero element $y$ in $R$.

**Definition 3.8.** A Γ-semihyperring $R$ with an identity as well as zero element is known as an integral Γ-semihyperring if it has no zero divisor.

**Definition 3.9.** A Γ-semihyperring $(R,+,\Gamma)$ with an appropriate partial order $\leq$ is known as an ordered Γ-semihyperring if

(1) For each $x,y,z \in R$, $x \leq y$, we have $x+z \leq y+z$ i.e., for each $a \in x+z$ there is $b \in y+z$ such that $a \leq b$.
(2) For each $x,y,z \in R$, $\alpha \in \Gamma$, $x \leq y$ and $0 \leq z$, we have $x\alpha z \leq y\alpha z$ that is for any $a \in x\alpha z$ there is $b \in y\alpha z$ such that $a \leq b$.
(3) For each $x,y,z \in R$, $\alpha \in \Gamma$, $x \leq y$ and $0 \leq z$, we have $z\alpha x \leq z\alpha y$.

**Definition 3.10.** If any two elements of an ordered $\Gamma$-semihyperring $R$ are comparable, then $R$ is known as totally ordered $\Gamma$-semihyperring.

**Definition 3.11.** In case of an ordered multiplicative $\Gamma$-semihyperring, $R$ becomes a semigroup with $+$, hence,

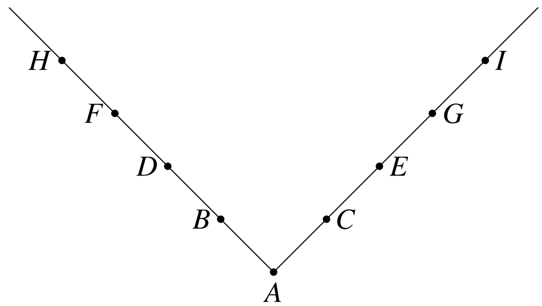
(1) The semigroup $(R,+)$ is known as positively ordered if $x \leq x+y$ and $y \leq x+y$, for every $x,y \in R$.
(2) The semigroup $(R,+)$ is known as negatively ordered if $x+y \leq x$ and $x+y \leq y$, for every $x,y \in R$.
(3) The $\Gamma$-semihypergroup is known as positively ordered if $x \leq x\alpha y$ and $y \leq x\alpha y$, for every $x,y \in R$, $\alpha \in \Gamma$.
(4) The $\Gamma$-semihypergroup is known as negatively ordered if $x\alpha y \leq x$ and $x\alpha y \leq y$, for every $x,y \in R$, $\alpha \in \Gamma$.

**Definition 3.12.** An ordered $\Gamma$-semihyperring $(R,+,\Gamma,\leq)$ is known as positively(negatively) ordered if $x\alpha y \leq x\alpha y + z\alpha w (x\alpha y + z\alpha w \leq x\alpha y)$ and $z\alpha w \leq x\alpha y + z\alpha w (x\alpha y + z\alpha w \leq z\alpha w)$, for every $x,y,z,w \in R$, $\alpha \in \Gamma$.

**Definition 3.13.** Let $A$ be a nonempty subset of an ordered $\Gamma$-semihyperring $R$ satisfying the binary property under both the hyperoperations of $R$. Then $A$ is known as an ordered $\Gamma$-subsemihyperring.

**Definition 3.14.** A right (left) ideal of an ordered $\Gamma$-semihyperring $R$ is a non-empty subset $I$ satisfying closure property with respect to $+$ and $I\Gamma R \subseteq I (R\Gamma I \subseteq I)$ and if for any $x \in R$, $y \in I$, $x \leq y \implies x \in I$.
$I$ becomes an ideal of $R$ if it is left and right ideal as well.

*Example* 3.15. Let $X$ be a non-empty infinite set. Define the addition and multiplication on $\mathscr{P}^*(X)$ as: $A+B = A \cup B$ and $A\alpha B = A \cap B$, for every $A,B \in \mathscr{P}^*(X)$, $\alpha \in \Gamma$. Then $(\mathscr{P}^*(X),+,\Gamma,\leq)$ is an ordered $\Gamma$-semihyperring where the order relation $\leq$ is defined by
$\leq = \{(A,A);(B,B);(C,C);(D,D);...(A,B);(A,C);(B,D);(C,E);(D,F);(E,G);...\}$.
The figure is given below:




Here, $\mathscr{P}^*(X)$ is not totally ordered $\Gamma$-semihyperring. The semigroup $(\mathscr{P}^*(X),+)$ is positively ordered semigroup but not negatively ordered. The $\Gamma$-semigroup $(\mathscr{P}^*(X),\Gamma)$ is negatively ordered but not positively ordered. One can also verify that, $(\mathscr{P}^*(X),+,\Gamma,\leq)$ is positively ordered $\Gamma$-semihyperring but not negatively ordered $\Gamma$-semihyperring.

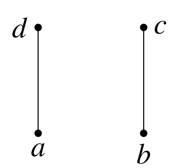
*Example* 3.16. Let $R=\{a,b,c,d\}$. Then $R$ is a commutative semigroup with $+$.

| + | a | b | c | d |
|---|---|---|---|---|
| a | {a} | {b} | {c} | {d} |
| b | {b} | {b} | {c} | {d} |
| c | {c} | {c} | {c} | {d} |
| d | {d} | {d} | {d} | {d} |

Let $\cdot$ be such that $x\alpha y \implies x\cdot y$, for all $x,y\in R$ and $\alpha\in\Gamma$.

| · | a | b | c | d |
|---|---|---|---|---|
| a | {a} | {a, b} | {a, b, c} | R |
| b | {a, b} | {b} | {b, c} | {b, c, d} |
| c | {a, b, c} | {b, c} | {c} | {c, d} |
| d | R | {b, c, d} | {c, d} | {d} |

Then $(R,+,\Gamma)$ is a multiplicative $\Gamma$-semihyperring and $R$ is an ordered $\Gamma$-semihyperring with an order relation $\leq$ defined by
$\leq=\{(a,a);(b,b);(c,c);(d,d);(a,d);(b,c)\}$. The figure of $R$ is given below:



*Example* 3.17. In the Example 3.15, $I=\{A\}$ is both left ideal and right ideal of $\mathscr{P}^*(X)$.

**Definition 3.18.** If $(R,+,\Gamma,\leq)$ is an ordered $\Gamma$-semihyperring, Then a function $\psi : R\to R$ is known as a homomorphism if for each $x,y\in R$, $\alpha\in\Gamma$ the following hold,

(1) $\psi(x+y)=\psi(x)+\psi(y)$;
(2) $\psi(x\alpha y)\subseteq\psi(x)\alpha\psi(y)$;
(3) $x\leq y$ gives $\psi(x)\leq\psi(y)$.

$\psi$ is a good homomorphism if equality holds in (2).

**Definition 3.19.** Let $R$ be an ordered $\Gamma$-semihyperring. Then a mapping $\psi :$ $R\to R$ is an isotone mapping on $R$ if $x\leq y \implies \psi(x)\leq\psi(y)$, for all $x,y\in R$.

**Definition 3.20.** A homomorphism $\psi$ on an ordered $\Gamma$-semihyperring $R$ is called an endomorphism on $R$ if $\psi$ is onto.

## 4. $(\sigma,\tau)$-Derivation

In this section, we have encountered various results dealing with $(\sigma,\tau)$-derivation on an ordered $\Gamma$-semihyperring. Here, one can justify that some results of $(\sigma,\tau)$-derivation that we have proved in the present paper will not be true on a $\Gamma$-semihyperring. But the same results are true on an ordered $\Gamma$-semihyperring. This is possible due to the beauty of an appropriate partial order relation attached to a $\Gamma$-semihyperring.

**Definition 4.1.** Let $R$ be an ordered $\Gamma$-semihyperring. Then a mapping $d:R\to R$ is called a derivation if, for all $x,y\in R$ and $\alpha\in\Gamma$,

(1) $d(x+y)=d(x)+d(y)$;
(2) $d(x\alpha y)\subseteq d(x)\alpha y+x\alpha d(y)$;
(3) If $x\leq y$, then $d(x)\leq d(y)$.

**Definition 4.2.** Let $R$ be an ordered $\Gamma$-semihyperring and $\sigma$ be an endomorphism on $R$. Then a mapping $d:R\to R$ is called a $\sigma$-derivation if, for all $x,y\in R$ and $\alpha\in\Gamma$,

(1) $d(x+y)=d(x)+d(y)$;
(2) $d(x\alpha y)\subseteq d(x)\alpha\sigma(y)+\sigma(x)\alpha d(y)$;
(3) If $x\leq y$, then $d(x)\leq d(y)$.

**Definition 4.3.** Let $R$ be an ordered $\Gamma$-semihyperring and $\sigma,\tau$ be two endomorphisms on $R$. Then a mapping $d:R\to R$ is called a $(\sigma,\tau)$-derivation if, for all $x,y\in R$ and $\alpha\in\Gamma$,

(1) $d(x+y) = d(x) + d(y)$;
(2) $d(x\alpha y) \subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)$;
(3) If $x \leq y$, then $d(x) \leq d(y)$.

**Definition 4.4.** If in above definition equality holds in (2), then $d : R \to R$ is called a strong $(\sigma, \tau)$-derivation on $R$.

*Example* 4.5. Let $R$ be an ordered $\Gamma$-semihyperring. Then $d(x) = 0$, for all $x \in R$ is a $(\sigma, \tau)$-derivation on $R$.

An identity mapping is not a $(\sigma, \tau)$-derivation on $R$. But under certain conditions an identity mapping becomes a $(\sigma, \tau)$-derivation which is given as the current section unfolds.

*Example* 4.6. Consider the Example 3.15. Let $\sigma_l : \mathscr{P}^*(X) \to \mathscr{P}^*(X)$ be such that

$$\sigma_l = \begin{cases} B & \text{for} \quad x = A, \\ D & \text{for} \quad x = B, \\ F & \text{for} \quad x = D, \\ H & \text{for} \quad x = F, \\ \vdots & \qquad\qquad \vdots \end{cases}$$

and $d(A) = A$, for all $A \in \mathscr{P}^*(X)$. Then $\sigma_l$ is an endomorphism and $d$ is a $\sigma_l$ derivation.
Moreover, if $\tau : \mathscr{P}^*(X) \to \mathscr{P}^*(X)$ be such that

$$\tau = \begin{cases} \vdots & \qquad\qquad \vdots \\ G & \text{for} \quad x = I, \\ F & \text{for} \quad x = H, \\ E & \text{for} \quad x = G, \\ D & \text{for} \quad x = F, \\ C & \text{for} \quad x = E, \\ B & \text{for} \quad x = D, \\ A & \text{for} \quad x = C, \\ A & \text{for} \quad x = B. \end{cases}$$

and $\tau(A) = A$, then $\tau$ being an endomorphism, $d$ becomes a $(\sigma_l, \tau)$-derivation on $\mathscr{P}^*(X)$.

*Remark* 4.7. If in above Example, $\sigma_r = \begin{cases} C & \text{for} \quad x = A, \\ E & \text{for} \quad x = C, \\ G & \text{for} \quad x = E, \\ I & \text{for} \quad x = G, \\ \vdots & \qquad \vdots \end{cases}$

and $d(A) = A$, for all $A \in \mathscr{P}^*(X)$, then with the same $\tau$, $d$ is a $(\sigma_r,\tau)$-derivation on $\mathscr{P}^*(X)$.

*Example* 4.8. Consider the Example 3.16. Let $\sigma : R \to R$ be such that $\sigma(x) = x+a$ and $\tau(x) = x+x$, for all $x \in R$. Then both $\sigma$ and $\tau$ are endomorphisms. If $d(x) = a$, then $d$ is individually $\sigma$ derivation and $\tau$ derivation. Here, $d$ is a $(\sigma,\tau)$-derivation on $R$ as well.

**Proposition 4.9.** *Let $R$ be a $\Gamma$-band ordered $\Gamma$-semihyperring. Then $d(x) = x$, for all $x \in R$ is a $(\sigma,\tau)$-derivation on $R$, where $\sigma(x) \leq x$ and $\tau(x) \leq x$.*

*Proof.* Straight forward. □

The following five theorems speak about how $\sigma$, $\tau$ make an effect on derivation $d$.

**Theorem 4.10.** *Let $R$ be an idempotent ordered $\Gamma$-semihyperring in which $\Gamma$-semihypergroup is negatively ordered and $d$ be a $(\sigma,\tau)$ derivation such that $\sigma(x) \leq x$ and $\tau(x) \leq x$, for all $x \in R$. Then $d(x) \leq x$.*

*Proof.* Let $x \in R$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$ and $d(x) = d(x\alpha x) \subseteq d(x)\alpha\sigma(x) + \tau(x)\alpha d(x) \leq \sigma(x) + \tau(x) \leq x + x \leq x$. Hence, $d(x) \leq x$. □

**Theorem 4.11.** *Let $R$ be an ordered $\Gamma$-semihyperring in which $\Gamma$-semihypergroup is negatively ordered and $\sigma(x) \leq x$, $\tau(x) \leq x$. Then $d(x\alpha y) \leq d(x+y)$, where $d$ is a $(\sigma,\tau)$ derivation.*

*Proof.* Suppose $x,y \in R$. Then $d(x\alpha y) \subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \leq d(x)\alpha y + x\alpha d(y) \leq d(x) + d(y) \leq d(x+y)$. Hence, proved. □

**Theorem 4.12.** *Let $d$ be a $(\sigma,\tau)$-derivation of an ordered $\Gamma$-semihyperring $R$. If $\sigma(0) = \tau(0) = 0$, then $d(0) = 0$.*

*Proof.* Proof is elementary. □

**Theorem 4.13.** *Let $R$ be an ordered $\Gamma$-band $\Gamma$-semihyperring and $d,\sigma,\tau$ be an identity function from $R \to R$. Then $d$ is a $(\sigma,\tau)$ derivation on $R$ if and only if $d$ is a homomorphism from $R \to R$.*

*Proof.* Consider, $d(x\alpha y) \subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \subseteq x\alpha y + x\alpha y \subseteq x\alpha y \subseteq d(x)\alpha d(y)$. Similarly, $d(x+y) = d(x)+d(y)$, which means $d$ is a homomorphism. Conversely, suppose $d$ is a homomorphism from $R \to R$. Let $x,y \in R$. Then $d(x\alpha y) \subseteq d(x)\alpha d(y) \subseteq x\alpha y \subseteq x\alpha y + x\alpha y$. Hence, $d(x\alpha y) \subseteq d(x)\alpha\sigma(y)+\tau(x)\alpha d(y)$. Now consider, $x \leq y$. Since, $d$ is a homomorphism, $d(x) \leq d(y)$. Therefore, $d$ is a $(\sigma,\tau)$ derivation on $R$. □

**Theorem 4.14.** *Let $R$ be a commutative ordered $\Gamma$-band $\Gamma$-semihyperring. Then a mapping $d_a(x) = x\alpha a$, for all $x \in R$ is a $(\sigma,\tau)$ derivation on $R$, where $\sigma$ and $\tau$ are identity functions.*

*Proof.* Consider, $d_a(x+y) = (x+y)\alpha a = x\alpha a + y\alpha a = d_a(x)+d_a(y)$.
Similarly,

$$\begin{aligned}
d_a(x\alpha y) &= (x\alpha y)\alpha a \\
&= (x\alpha y)\alpha a + (x\alpha y)\alpha a \\
&= x\alpha(y\alpha a) + (x\alpha a)\alpha y \\
&= x\alpha d_a(y) + d_a(x)\alpha y \\
&= d_a(x)\alpha y + x\alpha d_a(y) \\
&= d_a(x)\alpha\sigma(y) + \tau(x)\alpha d_a(y).
\end{aligned}$$

Hence, $d_a(x\alpha y) \subseteq d_a(x)\alpha\sigma(y)+\tau(x)\alpha d_a(y)$. Similarly, let $x \leq y$. Then for $a \in R$, $x\alpha a \leq y\alpha a$. Therefore, $d(x) \leq d(y)$. Hence, $d$ is a $(\sigma,\tau)$ derivation on $R$. □

In next two theorems, $d$ is influenced by any one of the endomorphisms.

**Theorem 4.15.** *Let $d$ be a $(\sigma,\tau)$-derivation of an idempotent commutative ordered $\Gamma$-semihyperring $R$ in which a $\Gamma$-semihypergroup is negatively ordered and a semigroup $(R,+)$ is positively ordered.*

*(1) If $\sigma(x) \leq \tau(x)$, for all $x \in R$, then $d(x) \leq \tau(x)$, for all $x \in R$.*
*(2) If $\tau(x) \leq \sigma(x)$, for all $x \in R$, then $d(x) \leq \sigma(x)$, for all $x \in R$.*

*Proof.* (1) Suppose $\sigma(x) \leq \tau(x)$, for all $x \in R$, then $\sigma(x)+\tau(x) \leq \tau(x)+\tau(x)$ implies $\sigma(x)+\tau(x) \leq \tau(x) \leq \sigma(x)+\tau(x)$. Therefore, $\sigma(x)+\tau(x) = \tau(x)$. Now, as $R$ is idempotent for $x \in R$ there exists $\alpha \in \Gamma$ such that $x = x\alpha x$ implies $d(x) = d(x\alpha x) \subseteq d(x)\alpha\sigma(x)+\tau(x)\alpha d(x) = d(x)\alpha(\sigma(x)+\tau(x)) = d(x)\alpha\tau(x)$. That is $d(x) \subseteq d(x)\alpha\tau(x) \leq \tau(x)$. Hence, $d(x) \leq \tau(x)$, for all $x \in R$.

(2) We can similarly proof this statement.

□

**Theorem 4.16.** *Let $R$ be negatively ordered $\Gamma$-semihyperring and $d$ be a $(\sigma,\tau)$ derivation such that $d(e)=e$.*

*(1) If $R$ has right $\Gamma$-identity and $\tau(x)=x$, then $d(x)\leq x$, for all $x\in R$.*
*(2) If $R$ has left $\Gamma$-identity and $\sigma(x)=x$, then $d(x)\leq x$, for all $x\in R$.*

*Proof.* (1) Let $x\in R$. Then there exists $\alpha\in\Gamma$, for all $x\in R$ such that $x=x\alpha e$ implies $d(x)=d(x\alpha e)\subseteq d(x)\alpha\sigma(e)+\tau(x)\alpha d(e)\leq\tau(x)\alpha d(e)\leq\tau(x)\alpha e\leq x\alpha e=x$. Therefore, $d(x)\leq x$.
(2) On the similar lines of proof of (1).

□

**Theorem 4.17.** *Let $\sigma,\tau$ be two endomorphisms on an ordered idempotent $\Gamma$-band $\Gamma$-semihyperring $R$ in which a semigroup $(R,+)$ is positively ordered. Then $\sigma$ and $\tau$ are $(\sigma,\tau)$-derivations.*

*Proof.* Let $x,y\in R$ and $\alpha\in\Gamma$. Then

$$\begin{aligned}\sigma(x\alpha y)&\subseteq\sigma(x)\alpha\sigma(y)\\&\subseteq\sigma(x)\alpha\sigma(y)+\sigma(x)\alpha\sigma(y)\\&\subseteq\sigma(x)\alpha\sigma(y)+(\sigma(x)+\tau(x))\alpha\sigma(y)\\&\subseteq\sigma(x)\alpha\sigma(y)+\sigma(x)\alpha\sigma(y)+\tau(x)\alpha\sigma(y)\\&\subseteq\sigma(x)\alpha\sigma(y)+\tau(x)\alpha\sigma(y).\end{aligned}$$

As $\sigma$ is an endomorphism, $\sigma(x+y)=\sigma(x)+\sigma(y)$ and $x\leq y$ implies $\sigma(x)\leq\sigma(y)$. Hence, $\sigma$ is a $(\sigma,\tau)$-derivation.
One can similarly prove that $\tau$ is a $(\sigma,\tau)$-derivation. □

**Theorem 4.18.** *Let $d$ be a $(\sigma,\tau)$ derivation on an ordered prime $\Gamma$-semihyperring $R$. If $a\in R$ and $\alpha\in\Gamma$ such that $a\alpha d(x)=0$, for all $x\in R$, then $a=0$ or $d=0$.*

*Proof.* Proof is simple. □

**Corollary 4.19.** *Let $d$ be a $(\sigma,\tau)$ derivation on an ordered prime $\Gamma$-semihyperring $R$ and $a$ be a non-zero element of $R$. If $a\alpha d(x)=0$, for all $x\in R$, then $d$ is a zero derivation on $R$.*

*Proof.* One can easily prove this corollary. □

In the following four theorems, we will see certain equalities involving composition of derivation and endomorphisms to study the various nature of a $(\sigma,\tau)$ derivation.

**Theorem 4.20.** *Let $R$ be an ordered $\Gamma$-semihyperring which is also an idempotent, $d$ be a strong $(\sigma,\tau)$ derivation over $R$ such that $d\circ d=d$ and $\sigma\circ d=\sigma$. Then for each $x\in R$ there exists $\alpha\in\Gamma$ such that $d(x\alpha d(x))=d(x)$.*

*Proof.* Let $x \in R$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha x$. Now, $d(x\alpha d(x)) = d(x)\alpha\sigma(d(x)) + \tau(x)\alpha d(d(x)) = d(x)\alpha(\sigma \circ d(x)) + \tau(x)\alpha(d \circ d(x)) = d(x)\alpha\sigma(x) + \tau(x)\alpha d(x) = d(x\alpha x) = d(x)$. Hence, $d(x\alpha d(x)) = d(x)$. □

**Theorem 4.21.** *Let R be 2-torsion free prime ordered $\Gamma$-semihyperring, d be a $(\sigma, \tau)$ derivation on R such that $\sigma \circ d = d \circ \sigma$ and $\tau(x)$ be an identity function for all $x \in R$. If $d^2 = 0$, then $d = 0$.*

*Proof.* Consider,

$$\begin{aligned} d(x\alpha y) &\subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \\ d(d(x\alpha y)) &\subseteq d(d(x)\alpha\sigma(y) + \tau(x)\alpha d(y)) \\ d^2(x\alpha y) &\subseteq d(d(x))\alpha\sigma(\sigma(y)) + \tau(d(x))\alpha d(\sigma(y)) + d(\tau(x))\alpha\sigma(d(y)) \end{aligned}$$
$$+\tau(\tau(x))\alpha d(d(y)).$$

Since, $d^2 = 0$, we get,

$$\begin{aligned} 0 &\subseteq d(x)\alpha d(\sigma(y)) + d(x)\alpha\sigma(d(y)) \\ &\subseteq d(x)\alpha(d(\sigma(y)) + d(\sigma(y))) \\ &\subseteq d(x)\alpha(2d(\sigma(y)). \end{aligned}$$

That is, $0 \in d(x)\alpha(2d(\sigma(y))$. As $R$ is prime, $d(x) = 0$ or $2d(\sigma(y)) = 0$. Thus $d = 0$. □

**Theorem 4.22.** *Let R be negatively ordered $\Gamma$-semihyperring and a $\Gamma$-semihypergroup is negatively ordered with $d \circ d = d$, $\sigma \circ d = \sigma$ and $\tau \circ d = \tau$ and d is a $(\sigma, \tau)$ derivation. Then $d(d(x)\alpha d(y)) \leq d(x)$ or $d(y)$.*

*Proof.* Consider, $d(d(x)\alpha d(y)) \subseteq d(d(x))\alpha\sigma(d(y)) + \tau(d(x))\alpha d(d(y)) \subseteq d \circ d(x)\alpha\sigma \circ d(y) + \tau \circ d(x)\alpha d \circ d(y) \leq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \leq d(x)\alpha\sigma(y) \leq d(x)$. Hence, $d(d(x)\alpha d(y)) \leq d(x)$.
Similarly, it can be proved that $d(d(x)\alpha d(y)) \leq d(y)$. □

**Theorem 4.23.** *Let R be a commutative ordered $\Gamma$-semihyperring and $d_1$ and $d_2$ be strong $(\sigma, \tau)$ derivation on R where $\tau \circ d_2 = \tau \circ d_1, d_1 \circ \tau = d_2 \circ \tau, \sigma \circ d_2 = \sigma \circ d_1, d_1 \circ \sigma = d_2 \circ \sigma, \sigma \circ \sigma = \sigma$ and $\tau \circ \tau = \tau$.*

*(1) If $d_1 d_2(x) = 0$, then $d_2 d_1$ is a strong $(\sigma, \tau)$ derivation on R.*
*(2) If $d_2 d_1(x) = 0$, then $d_1 d_2$ is a strong $(\sigma, \tau)$ derivation on R.*

*Proof.* (1) Suppose $d_1d_2 = 0$, for all $x, y \in R$ and $\alpha \in \Gamma$, then

$d_1d_2(x\alpha y) = 0$

$d_1(d_2(x)\alpha\sigma(y) + \tau(x)\alpha d_2(y)) = 0$

$d_1(d_2(x)\alpha\sigma(y)) + d_1(\tau(x)\alpha d_2(y)) = 0$

$d_1d_2(x)\alpha\sigma(\sigma(y)) + \tau(d_2(x))\alpha d_1(\sigma(y)) + d_1(\tau(x))\alpha\sigma(d_2(y)) + \tau(\tau(x))\alpha d_1d_2(y) = 0$

$\tau(d_2(x))\alpha d_1(\sigma(y)) + d_1(\tau(x))\alpha\sigma(d_2(y)) = 0$

$\tau \circ d_2(x)\alpha d_1 \circ \sigma(y) + d_1 \circ \tau(x)\alpha\sigma \circ d_2(y) = 0.$

$$\tau \circ d_1(x)\alpha d_2 \circ \sigma(y) + d_2 \circ \tau(x)\alpha\sigma \circ d_1(y) = 0. \tag{4.1}$$

Consider,

$$d_2d_1(x\alpha y) = d_2(d_1(x)\alpha\sigma(y) + \tau(x)\alpha d_1(y)) = d_2(d_1(x))\alpha\sigma \circ \sigma(y) + \tau \circ d_1(x)\alpha d_2 \circ \sigma(y) + d_2 \circ \tau(x)\alpha\sigma \circ d_1(y) + \tau \circ \tau(x)\alpha d_2(d_1(y)).$$

Using Eq. 4.1, we get,
$d_2d_1(x\alpha y) = d_2(d_1(x))\alpha\sigma \circ \sigma(y) + \tau \circ \tau(x)\alpha d_2(d_1(y)) = d_2d_1(x)\alpha\sigma(y) + \tau(x)\alpha d_2d_1(y)$. Hence, $d_2d_1$ is a strong $(\sigma,\tau)$ derivation on $R$.

(2) Similarly one can prove this statement.

□

**Theorem 4.24.** *Let $d_1$ and $d_2$ be two $(\sigma,\tau)$ derivations. Then $d_1d_2$ is also $(\sigma,\tau)$ derivation where $R$ is 2-torsion free ordered $\Gamma$-semihyperring with characteristic 2 and $\tau \circ d_2 = d_1 \circ \tau, d_1 \circ \sigma = \sigma \circ d_2, \sigma \circ \sigma = \sigma$ and $\tau \circ \tau = \tau$.*

*Proof.* Applying $d_1d_2$ on $x\alpha y$ and using the identities given in the statement of the theorem generate the proof. □

**Definition 4.25.** Let $d$ be a derivation on an ordered $\Gamma$-semihyperring $R$. Then the set of all the elements $x \in R$ such that $d(x) = 0$ is called as kernel $d$ denoted by $Kerd$.

**Proposition 4.26.** *Let $d$ be a $(\sigma,\tau)$ derivation on an ordered $\Gamma$-semihyperring $R$. Then $Kerd$ is a $\Gamma$-subsemihyperring of $R$.*

*Proof.* Proof is elementary. □

The following three theorems explain the interconnection between the derivation on ideal and derivation on an ordered $\Gamma$-semihyperring $R$.

**Theorem 4.27.** *Let $I$ be a non-zero ideal of an ordered integral $\Gamma$-semihyperring $R$ in which a $\Gamma$-semihypergroup is negatively ordered. If $d$ is a non-zero $(\sigma,\tau)$-derivation on $R$ where $\tau$ is a non-zero function on $I$, then $d$ is a non-zero $(\sigma,\tau)$-derivation on $I$.*

*Proof.* Let $d$ be a $(\sigma,\tau)$-derivation on $I$ and $\tau$ be a non-zero function on $I$. Suppose, $x \in I$ such that $\tau(x) \neq 0, d(x) = 0$ and $y \in R, \alpha \in \Gamma$, we have, $x\alpha y \leq x$ implies $d(x\alpha y) \leq d(x) = 0$. Hence, $0 = d(x\alpha y) \subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \subseteq \tau(x)\alpha d(y)$. Hence, $0 \in \tau(x)\alpha d(y)$. As, $\tau(x) \neq 0$ and $R$ is an integral ordered $\Gamma$-semihyperring, we have, $d(y) = 0$ implies $d$ is zero $(\sigma,\tau)$-derivation on $R$, which is a contradiction. Hence, $d$ is non-zero $(\sigma,\tau)$-derivation on $I$. □

A stronger argument of the above theorem is given below, where $\Gamma$-semihypergroup need not be negatively ordered.

**Theorem 4.28.** *Let $I$ be a proper ideal of an ordered integral $\Gamma$-semihyperring $R$. If $d$ is a non-zero $(\sigma,\tau)$ derivation on $R$ where $\tau$ is a non-zero on $I$, then $d$ is a non-zero on $I$.*

*Proof.* Let $I$ be a proper ideal of an ordered integral $\Gamma$-semihyperring $R$ and $d$ is a non-zero $(\sigma,\tau)$ derivation on $R$. Suppose, $d(x) = 0$, for all $x \in I$. Let $y \in R$. Then $d(y) \neq 0$. Now, as $x \in I$ and $y \in R$, hence, $x\alpha y \in I$, thus, $0 = d(x\alpha y) \subseteq d(x)\alpha\sigma(y) + \tau(x)\alpha d(y) \subseteq \tau(x)\alpha d(y)$. As $R$ is an integral $d(y) = 0$, for all $y \in R$, which is a contradiction to $d$ is a non-zero derivation on $R$. Hence, $d$ is a non-zero derivation on $I$. □

**Theorem 4.29.** *Let $R$ be an ordered prime $\Gamma$-semihyperring and $I$ be a proper ideal of $R$. If $d$ is a $(\sigma,\tau)$ derivation on $R$ such that $d(u) = 0$, for all $u \in I$ where $\sigma$, $\tau$ are non-zero automorphisms, then $d(r) = 0$, for all $r \in R$.*

*Proof.* Suppose, $0 \neq u \in I$ and $x \in R$, then $x\alpha u \in I$, therefore, $d(x\alpha u) = 0$. Consider, $0 = d(x\alpha u) \subseteq d(x)\alpha\sigma(u) + \tau(x)\alpha d(u) \subseteq d(x)\alpha\sigma(u)$. Now, replace, $x$ by $r\alpha u$. Hence, $0 \subseteq d(r\alpha u)\alpha\sigma(u) \subseteq d(r)\alpha\sigma(u)\alpha\sigma(u) + \tau(r)\alpha d(u)\alpha\sigma(u)$. That is $0 \subseteq d(r)\alpha\sigma(u)\alpha\sigma(u)$. Thus, $R$ being prime and $\sigma$ is non-zero, $d(r) = 0$, for all $r \in R$. Hence, $d(r) = 0$, for all $r \in R$. □

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