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The necessary and sufficient condition for Cartan's second curvature tensor which satisfies recurrence and birecurrence property in generalized Finsler spaces

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Abstract. The recurrence and birecurrence property in Finsler space have been studied by the Finslerian geometricians. The aim of this paper is to obtain the necessary and sufficient condition for Cartan's second curvature tensor that is recurrent and birecurrent in generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}P-$birecurrent space, respectively. We discuss certain identities belong to the mentioned spaces. Further, we end up this paper with some illustrative examples.

Keywords: Cartan's second curvature tensor, Berwald's covariant derivative,

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recurrence property, birecurrence property, projection on indicatrix.

1. **Introduction**

Finsler geometry is usually considered as a generalization of Riemannian geometry. The historical studies about development stages for Finsler geometry have been introduced by Matsumoto [16] and Won [35]. The metric tensor in Finsler space is a function of line element i.e. function of positional coordinate and directional coordinate, while the metric tensor in Riemannian space is a function of positional coordinate only [9, 34, 17]. Qasem [22] and Saleem and Abdallah [31] discussed the curvature tensor U^i_{jkh} which satisfies the recurrence property in sense of Berwald and Cartan, respectively.

Mandal [14] generalized the concept of recurrent Finsler connection. Pandey and Shukla [19] generalized and extended some results to a larger class of recurrent spaces. Recently, Pfeifer et al. [21] introduced a necessary and sufficient condition which a Finsler geometry to be Berwald type.

Pandey et al. [20], Qasem and Abdallah [24], Qasem and Baleedi [25] and Awed [8] obtained the necessary and sufficient condition for some tensors that be generalized recurrent in the generalized Hrecurrent Finsler space, generalized $\mathfrak{B}R-$recurrent space, generalized $\mathfrak{B}K-$recurrent space, generalized P^h-recurrent space, respectively. In addition, the necessary and sufficient condition for normal projective curvature tensor N^i_{jkh} that be generalized recurrent in sense of Berwald and Cartan has been obtained by Qasem and Saleem [28] and Saleem [30], respectively.

Zlatanovic and Mincic [37] introduced several identities for some curvature tensors in generalized space. Zafar and Musavvir [36] discussed some identities of W curvature tensor.

Qasem [23] and Qasem and Hadi [26] acquired the necessary and sufficient condition for Berwald curvature tensor H^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} that is generalized birecurrent in sense of Berwald. Also, the necessary and sufficient condition for projective curvature tensor W^i_{jkh} that is generalized birecurrent in sense of Berwald and Cartan has been studied by Qasem and Saleem [27] and Al Qashbari [7], respectively.

The projection on indicatrix for some tensors which behave as recurrent and birecurrent have been discussed by Alaa et al. [1] and Saleem and Abdallah [32]. In this paper, we discuss the necessary and sufficient condition for Cartan's second curvature tensor when behaves as recurrent and birecurrent. Additionally, diverse theorems have been established and proved.

2. **Preliminaries**

In this section, some basic concepts and definitions will be provided for the purpose of this paper. An $n-$dimensional space X_n equipped with a function

$F(x,y)$ which denoted by $F_n = (X_n, F(x,y))$ and is called a Finsler space if the function $F(x,y)$ satisfying the request conditions [5, 11, 18, 10, 33]. The metric tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in its indices which is defined by

$$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y).$$

The metric tensor g_{ij} and its associative g^{ij} are related by

$$g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k, \end{cases} \tag{2.1}$$

where

$$g_{ij} = \delta_i^k \, g_{kj}. \tag{2.2}$$

Matsumoto [15] introduced the $(h)hv-$torsion tensor C_{ijk} that is positively homogeneous of degree -1 in y^i and symmetric in all its indices which is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i \, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k \, F^2.$$

This tensor satisfies the following

$$a)\ C^i_{jk} y_i = 0,\ b)\ C_{ijk} = g_{hj} C^h_{ik},\ c)\ \delta^i_j C^j_{kh} = C^i_{kh} \text{ and } d)\ C^i_{ji} = C_j, \tag{2.3}$$

where C^i_{jk} is called associate tensor of the $(h)hv-$torsion tensor C_{ijk}.

The unit vector l^i and the associative vector l_i with the direction of y^i are given by

$$a)\ l^i = \frac{y^i}{F}, \quad \text{and} \quad b)\ l_i = \frac{y_i}{F}. \tag{2.4}$$

Cartan $h-$covariant differentiation (Cartan's second kind covariant differentiation) with respect to x^k is given by [29]

$$X^i_{|k} = \partial_k X^i - \left(\dot{\partial}_r \, X^i\right) G^{\ r}_{\ k} + X^r \Gamma^{*\ i}_{rk}$$

The $h-$covariant derivative of the vector y^i and the metric tensor g_{ij} are vanish identically i.e.

$$a)\ y^i_{|k} = 0 \text{ and } b)\ g_{ij|k} = 0. \tag{2.5}$$

Berwald's covariant derivative $\mathfrak{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [29]

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$$

Berwald's covariant derivative of the vector y_i is vanish identically i.e.

$$\mathfrak{B}_k \, y_i = 0. \tag{2.6}$$

But the Berwald's covariant derivative of the metric tensor g_{ij} does not vanish in general, i.e. $\mathfrak{B}_k g_{ij} \neq 0$. It is given by

$$\mathfrak{B}_k g_{ij} = -2C_{ijk|h} y^h = -2y^h \mathfrak{B}_h C_{ijk}. \tag{2.7}$$

The processes of Berwald's covariant differentiation with respect to x^h and the partial differentiation with respect to y^k commute according to

$$\left(\dot{\partial}_k \mathfrak{B}_h - \mathfrak{B}_h \dot{\partial}_k\right) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}. \tag{2.8}$$

for an arbitrary tensor field T^i_j.

The tensor P^i_{jkh} is called the *$hv-$curvature tensor* (*Cartan's second curvature tensor*) which is positively homogeneous of degree -1 in y^i is defined by [29]

$$P^i_{jkh} = C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk}, \tag{2.9}$$

and satisfies the relation

$$P^i_{jkh} y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r, \tag{2.10}$$

where P^i_{kh} is called the $(v)hv-$torsion tensor. The associate tensor P_{ijkh}, $P-$Ricci tensor P_{jk} and the tensor $(P_{ij} - P_{ji})$ are given by [29]

$$a)\ P_{ijkh} = g_{ir} P^r_{jkh},\ b)\ P_{jk} = P^i_{jki} \text{ and } c)\ P_{ij} - P_{ji} = P_{ijkh} g^{kh}. \tag{2.11}$$

Also, the $hv-$curvature tensor P^i_{jkh} satisfies the identity

$$P^i_{jkh} - P^i_{kjh} = C^i_{kh|j} + C^i_{sj} P^s_{kh} - j/k. \tag{2.12}$$

Definition 2.1. *Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by $F\left(x_0, x^i\right) = 1$ or by the parametric form defined by $x^i = x^i\left(u^a\right), a = 1, 2, \ \ldots\ , \ n-1$.*

Definition 2.2. *The projection of any tensor T^i_j on indicatrix I_{n-1} is given by* [12]

$$a)\ p.T^i_j = T^a_b h^i_a h^b_j \ \textit{where}\ b)\ h^i_c = \delta^i_c - l^i l_c. \tag{2.13}$$

The projection of the vector y^i, unit vector l^i and metric tensor g_{ij} on the indicatrix are given by $p.y^i = 0$, $p.l^i = 0$ and $p.g_{ij} = h_{ij}$, where $h_{ij} = g_{ij} - l_i l_j$.

Alaa et al. [2, 4, 6] introduced the generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}P-$birecurrent space which are characterized by the conditions

$$\mathfrak{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}),\ P^i_{jkh} \neq 0, \tag{2.14}$$

and

$$\begin{aligned} &\mathfrak{B}_l \mathfrak{B}_m P^i_{jkh} \\ &= a_{lm} P^i_{jkh} + b_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - 2y^t \mu_m \mathfrak{B}_t(\delta^i_j C_{khl} - \delta^i_k C_{jhl}),\ P^i_{jkh} \neq 0, \end{aligned} \tag{2.15}$$

respectively. These spaces are denoted by $G(\mathfrak{B}P) - RF_n$ and $G(\mathfrak{B}P) - BRF_n$.

Let us consider a $G(\mathfrak{B}P)-RF_n$. Transvecting the condition (2.15) by g_{il}, using eqs. (2.11), (2.7) and (2.2), we get

$$\mathfrak{B}_m P_{ljkh} = \lambda_m P_{ljkh} + \mu_m\left(g_{jl}g_{kh} - g_{kl}g_{jh}\right) + 2P^i_{jkh}y^t\mathfrak{B}_t C_{ilm}$$

Transvecting above equation by g^{kh}, using eqs. (2.11), (2.1) and (2.2), we get

$$\mathfrak{B}_m(P_{lj} - P_{jl}) = \lambda_m(P_{lj} - P_{jl}) + 2g^{kh}P^i_{jkh}y^t\mathfrak{B}_t C_{ilm} + P_{ljkh}\mathfrak{B}_m g^{kh}$$

This shows that

$$\mathfrak{B}_m(P_{lj} - P_{jl}) = \lambda_m(P_{lj} - P_{jl}) \tag{2.16}$$

if and only if

$$2g^{kh}P^i_{jkh}y^t\mathfrak{B}_t C_{ilm} + P_{ljkh}\mathfrak{B}_m g^{kh} = 0. \tag{2.17}$$

3. Necessary and Sufficient Condition for P^i_{jkh} to be Recurrent in Generalized $\mathfrak{B}P-$ Recurrent Space

The main aim of this section is studying the necessary and sufficient condition for Cartan's second curvature tensor P^i_{jkh} that satisfies the recurrence property in generalized $\mathfrak{B}P-$recurrent space. Let Cartan's $h-$covariant derivative of first order for the $(h)hv-$torsion tensor C_{ijk} and its associative C^i_{jk} which satisfy

$$\begin{cases} a)\ C^i_{kh|r} = \propto_r C^i_{kh} + \omega_r(\delta^i_k y_h - \delta^i_h y_k) \\ b)\ C_{jkh|r} = \propto_r C_{jkh} + \omega_r(g_{jk}y_h - g_{jh}y_k), \end{cases} \tag{3.1}$$

where $\propto_r$ and ω_r are non - covariant vectors field. Also, let Berwald's covariant derivative of first order for the tensors C^i_{kh} and C_{jkh} which satisfy [3]

$$\begin{cases} a)\ \mathfrak{B}_m C^i_{kh} = \propto_m C^i_{kh} + \omega_m(\delta^i_k y_h - \delta^i_h y_k) \\ b)\ \mathfrak{B}_m C_{jkh} = \propto_m C_{jkh} + \omega_m(g_{jk}y_h - g_{jh}y_k). \end{cases} \tag{3.2}$$

In next theorem we obtain the necessary and sufficient condition for Cartan's second curvature tensor that is recurrent.

Theorem 3.1. *In $G(\mathfrak{B}P)-RF_n$, the behavior of Cartan's second curvature tensor P^i_{jkh} as recurrent if and only if*

$$\begin{aligned} &\Big[\propto_j\omega_m - \propto_m\omega_j + \mathfrak{B}_m\omega_j - \omega_m\omega y_j\Big](\delta^i_k y_h - \delta^i_h y_k) + (\mathfrak{B}_m\propto_j)C^i_{kh} \\ &-(\mathfrak{B}_m\propto^i)C_{jkh} + \Big[\propto_m\omega^i - \propto^i\omega_m - \mathfrak{B}_m\omega^i\Big](g_{jk}y_h - g_{jh}y_k) \\ &+(\mathfrak{B}_m \propto + \propto \propto_m)(C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk}) \\ &+(\mathfrak{B}_m\omega)(y_j C^i_{kh}) + 2\omega^i(y_h C_{jkm|s}y^s - y_k C_{jhm|s}y^s) = 0. \end{aligned} \tag{3.3}$$

Proof. Multiplying eq. (3.1) by y^r, using eqs. (2.5) and (2.10), we get

$$P^i_{kh} = C^i_{kh|r} y^r = \propto C^i_{kh} + \omega(\delta^i_k y_h - \delta^i_h y_k), \tag{3.4}$$

where $\propto = \propto_r y^r$ and $\omega = \omega_r y^r$.

Using eqs. (3.1) and (3.4) in eq. (2.9), using eq. (2.3), we get

$$\begin{aligned} P^i_{jkh} &= \propto_j C^i_{kh} - \propto^i C_{jkh} + \propto (C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk}) + \omega_j(\delta^i_k y_h - \delta^i_h y_k) \\ &\quad -\omega^i (g_{jk} y_h - g_{jh} y_k) + \omega(y_j C^i_{kh}), \end{aligned} \tag{3.5}$$

where $\propto^i = \propto_r g^{ir}$ and $\omega^i = \omega_r g^{ir}$.

Taking $\mathfrak{B}-$covariant derivative for eq. (3.5) with respect to x^m and using eq. (2.6), we get

$$\begin{aligned} &\mathfrak{B}_m P^i_{jkh} \\ =\ & (\mathfrak{B}_m \propto_j) C^i_{kh} + \propto_j \mathfrak{B}_m C^i_{kh} - (\mathfrak{B}_m \propto^i) C_{jkh} - \propto^i \mathfrak{B}_m C_{jkh} + (\mathfrak{B}_m \propto) \\ & (C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk}) \\ & + \propto \left[(\mathfrak{B}_m C^r_{jk}) C^i_{rh} + C^r_{jk} (\mathfrak{B}_m C^i_{rh}) - (\mathfrak{B}_m C^r_{jh}) C^i_{rk} - C^r_{jh} (\mathfrak{B}_m C^i_{rk}) \right] \\ & + (\mathfrak{B}_m \omega_j)(\delta^i_k y_h - \delta^i_h y_k) - (\mathfrak{B}_m \omega^i)(g_{jk} y_h - g_{jh} y_k) + (\mathfrak{B}_m \omega)(y_j C^i_{kh}) \\ & + \omega(y_j \mathfrak{B}_m C^i_{kh}) - \omega^i (y_h \mathfrak{B}_m g_{jk} - y_k \mathfrak{B}_m g_{jh}). \end{aligned}$$

Using eq. (3.2) in above equation, using eq. (2.7), we get

$$\begin{aligned} & \mathfrak{B}_m P^i_{jkh} \\ =\ & \propto_m \left[\propto_j C^i_{kh} - \propto^i C_{jkh} + \propto (C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk}) + \omega(y_j C^i_{kh}) \right] + (\mathfrak{B}_m \propto_j) C^i_{kh} \\ -\ & (\mathfrak{B}_m \propto^i) C_{jkh} + (\mathfrak{B}_m \propto + \propto \propto_m)(C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk}) + (\propto_j \omega_m)(\delta^i_k y_h - \delta^i_h y_k) \\ -\ & \propto^i \omega_m (g_{jk} y_h - g_{jh} y_k) + (\mathfrak{B}_m \omega_j)(\delta^i_k y_h - \delta^i_h y_k) - (\mathfrak{B}_m \omega^i)(g_{jk} y_h - g_{jh} y_k) \\ +\ & (\mathfrak{B}_m \omega)(y_j C^i_{kh}) - \omega_m \omega y_j (\delta^i_k y_h - \delta^i_h y_k) + 2\omega^i (y_h C_{jkm|s} y^s - y_k C_{jhm|s} y^s). \end{aligned}$$

Using eq. (3.5) in above equation, we get

$$\begin{aligned} & \mathfrak{B}_m P^i_{jkh} \\ =\ & \propto_m \left[P^i_{jkh} - \omega_j (\delta^i_k y_h - \delta^i_h y_k) + \omega^i (g_{jk} y_h - g_{jh} y_k) \right] + (\mathfrak{B}_m \propto_j) C^i_{kh} \\ -\ & (\mathfrak{B}_m \propto^i) C_{jkh} + (\mathfrak{B}_m \propto + \propto \propto_m)(C^r_{jk} C^i_{rh} - C^r_{jh} C^i_{rk}) + \propto_j \omega_m (\delta^i_k y_h - \delta^i_h y_k) \\ -\ & \propto^i \omega_m (g_{jk} y_h - g_{jh} y_k) + (\mathfrak{B}_m \omega_j)(\delta^i_k y_h - \delta^i_h y_k) - (\mathfrak{B}_m \omega^i)(g_{jk} y_h - g_{jh} y_k) \\ +\ & (\mathfrak{B}_m \omega)(y_j C^i_{kh}) - \omega_m \omega y_j (\delta^i_k y_h - \delta^i_h y_k) + 2\omega^i (y_h C_{jkm|s} y^s - y_k C_{jhm|s} y^s). \end{aligned}$$

This shows that

$$\mathfrak{B}_m P^i_{jkh} = \propto_m P^i_{jkh} \tag{3.6}$$

The equation (3.6) refers that the Cartan's second curvature tensor P^i_{jkh} behave as recurrent in $G(\mathfrak{B}P) - RF_n$ if and only if eq. (3.3) holds. The proof for this theorem is completed. □

Now, we infer some corollaries related to the previous theorem. Taking $\mathfrak{B}-$covariant derivative for eq. (3.4) with respect to x^m, using eq. (2.6), we get

$$\mathfrak{B}_m P^i_{kh} = (\mathfrak{B}_m \propto) C^i_{kh} + \propto (\mathfrak{B}_m C^i_{kh}) + (\mathfrak{B}_m \omega)(\delta^i_k y_h - \delta^i_h y_k).$$

Using eq. (3.2) in above equation, we get

$$\begin{aligned} &\mathfrak{B}_m P^i_{kh} \\ &= (\mathfrak{B}_m \propto + \propto \propto_m) C^i_{kh} + \propto \omega_m (\delta^i_k y_h - \delta^i_h y_k) + (\mathfrak{B}_m \omega)(\delta^i_k y_h - \delta^i_h y_k). \end{aligned} \tag{3.7}$$

In view of eq. (3.4), we obtain

$$C^i_{kh} = \frac{1}{\propto}\Big[P^i_{kh} - \omega(\delta^i_k y_h - \delta^i_h y_k)\Big]. \tag{3.8}$$

Using eq. (3.8) in eq. (3.7), we get

$$\mathfrak{B}_m P^i_{kh} = \lambda_m P^i_{kh} + \mu_m (\delta^i_k y_h - \delta^i_h y_k), \tag{3.9}$$

where $\lambda_m = \Big(\frac{\mathfrak{B}_m \propto}{\propto} + \propto_m\Big)$ and $\mu_m = \Big[\propto \omega_m + \mathfrak{B}_m \omega - \omega\Big(\frac{\mathfrak{B}_m \propto}{\propto} + \propto_m\Big)\Big]$
Thus, we conclude the following corollary.

Corollary 3.2. *In $G(\mathfrak{B}P) - RF_n$, the $(v)hv-$torsion tensor P^i_{kh} necessarily is given by eq.* (3.9) *[provided eq.* (3.2) *holds].*

Contracting the indices i and h in eq. (2.12), using eqs. (2.11) and (2.3), we get

$$P_{jk} - P_{kj} = C_{k|j} - C^i_{sj} P^s_{ki} - j/k. \tag{3.10}$$

Taking $\mathfrak{B}-$covariant derivative for eq. (3.10) with respect to x^m, we get

$$\mathfrak{B}_m (P_{jk} - P_{kj}) = \mathfrak{B}_m (C_{k|j} - C^i_{sj} P^s_{ki} - j/k).$$

Using eq. (2.16) in above equation, then using eq. (3.10), we get

$$\mathfrak{B}_m (C_{k|j} - C^i_{sj} P^s_{ki} - j/k) = \lambda_m (C_{k|j} - C^i_{sj} P^s_{ki} - j/k). \tag{3.11}$$

Thus, we conclude the following corollary:

Corollary 3.3. *In $G(\mathfrak{B}P) - RF_n$, the behavior of the tensor $(C_{k|j} - C^i_{sj} P^s_{ki} - j/k)$ behaves as recurrent [provided eq.* (2.17) *holds].*

Differentiating eq. (2.16) partially with respect to y^h, we get

$$\dot\partial_h \mathfrak{B}_m (P_{lj} - P_{jl}) = (\dot\partial_h \lambda_m)(P_{lj} - P_{jl}) + \lambda_m \dot\partial_h (P_{lj} - P_{jl}).$$

Using the commutation formula exhibited by eq. (2.8) for $\Big(P_{lj} - P_{jl}\Big)$ in above equation, we get

$$\begin{aligned} &\mathfrak{B}_m \Big[\dot\partial_h (P_{lj} - P_{jl})\Big] - (P_{lr} - P_{rl}) G^r_{mhj} - (P_{rj} - P_{jr}) G^r_{mhl} \\ &= (\dot\partial_h \lambda_m)(P_{lj} - P_{jl}) + \lambda_m \dot\partial_h (P_{lj} - P_{jl}). \end{aligned} \tag{3.12}$$

If the tensor $\dot{\partial}_h\Big(P_{lj} - P_{jl}\Big)$ is recurrent, i.e. satisfies the following

$$\mathfrak{B}_m\Big[\dot{\partial}_h(P_{lj} - P_{jl})\Big] = \lambda_m\dot{\partial}_h(P_{lj} - P_{jl}). \tag{3.13}$$

Using eq. (3.13) in eq. (3.12), we get

$$(\dot{\partial}_h\lambda_m)(P_{lj} - P_{jl}) = -(P_{lr} - P_{rl})G^r_{mhj} - (P_{rj} - P_{jr})G^r_{mhl}.$$

If $\dot{\partial}_h\lambda_l = 0$, then above equation becomes as

$$(P_{lr} - P_{rl})G^r_{mhj} + (P_{rj} - P_{jr})G^r_{mhl} = 0. \tag{3.14}$$

Thus, we conclude the following corollary:

Corollary 3.4. *In $G(\mathfrak{B}P) - RF_n$, we have the identity* (3.14) *[provided eqs.* (3.13), (2.17) *hold and $\dot{\partial}_h\lambda_l = 0$].*

4. Necessary and Sufficient Condition for P^i_{jkh} to be Birecurrent in Generalized $\mathfrak{B}P-$ Birecurrent Space

The main aim of this section is studying the necessary and sufficient condition for Cartan's second curvature tensor P^i_{jkh} that satisfies the birecurrence property in generalized $\mathfrak{B}P-$birecurrent space. Let Berwalds covariant derivative of second order for the $(h)hv-$torsion tensor C_{ijk} and its associative C^i_{jk} which satisfy [13]

$$\begin{cases} a)\ \mathfrak{B}_l\mathfrak{B}_m C^i_{kh} = a_{lm}C^i_{kh} + b_{lm}(\delta^i_k y_h - \delta^i_h y_k) \\ b)\ \mathfrak{B}_l\mathfrak{B}_m C_{jkh} = a_{lm}C_{jkh} + b_{lm}(g_{jk}y_h - g_{jh}y_k), \end{cases} \tag{4.1}$$

where a_{lm} and b_{lm} are non - zero covariant tensors field.

In next theorem we obtain the necessary and sufficient condition for Cartan's second curvature tensor that is birecurrnt.

Theorem 4.1. *In $G(\mathfrak{B}P) - BRF_n$, the behavior of Cartan's second curvature tensor P^i_{jkh} as birecurrent if and only if*

$$\Big[(\mathfrak{B}_l\mathfrak{B}_m\propto_j) + (\mathfrak{B}_m\propto_j)\propto_l + (\mathfrak{B}_l\propto_j)\propto_m + (\mathfrak{B}_l\mathfrak{B}_m\omega)y_j$$
$$+(\mathfrak{B}_m\omega)\propto_l y_j + (\mathfrak{B}_l\omega)\propto_m y_j + a_{lm}y_j\omega\Big]C^i_{kh}$$
$$-\Big[(\mathfrak{B}_l\mathfrak{B}_m\propto^i) + (\mathfrak{B}_m\propto^i)\propto_l + (\mathfrak{B}_l\propto^i)\propto_m\Big]C_{jkh}$$
$$+\Big[\Big(\mathfrak{B}_l\mathfrak{B}_m \propto) + 2\propto_l(\mathfrak{B}_m \propto) + 2\propto_m(\mathfrak{B}_l \propto) + \propto a_{lm}$$
$$+2 \propto \propto_l\propto_m\Big](C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk})$$
$$+\Big[(\mathfrak{B}_m\propto_j)\omega_l + (\mathfrak{B}_l\propto_j)\omega_m + \propto_j b_{lm} + (\mathfrak{B}_l\mathfrak{B}_m\omega_j) + (\mathfrak{B}_m\omega)y_j\omega_l + (\mathfrak{B}_l\omega)y_j\omega_m$$
$$+\omega y_j b_{lm} - 2\omega_l\omega_m y_j - a_{lm}\omega_j\Big](\delta^i_k y_h - \delta^i_h y_k)$$
$$+\Big[(\mathfrak{B}_m\propto^i)\omega_l + (\mathfrak{B}_l\propto^i)\omega_m + (\mathfrak{B}_l\mathfrak{B}_m\omega^i) + \propto^i b_{lm} + a_{lm}\omega^i\Big](g_{jk}y_h - g_{jh}y_k)$$
$$+2(\mathfrak{B}_m\omega^i)(y_hC_{jkl|s}y^s - y_kC_{jhl|s}y^s) + 2(\mathfrak{B}_l\omega^i)(y_hC_{jkm|s}y^s - y_kC_{jhm|s}y^s)$$
$$-\omega^i(y_h\mathfrak{B}_l\mathfrak{B}_m g_{jk} - y_k\mathfrak{B}_l\mathfrak{B}_m g_{jh}) = 0. \tag{4.2}$$

Proof.

Taking $\mathfrak{B}-$ covariant derivative for eq. (3.5) twice with respect to x^m and x^l, successively, using eq. (2.6), we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh}$$
$$= (\mathfrak{B}_l\mathfrak{B}_m\propto_j)C^i_{kh} + (\mathfrak{B}_m\propto_j)(\mathfrak{B}_lC^i_{kh}) + (\mathfrak{B}_l\propto_j)(\mathfrak{B}_mC^i_{kh})$$
$$+\propto_j(\mathfrak{B}_l\mathfrak{B}_mC^i_{kh}) - (\mathfrak{B}_l\mathfrak{B}_m\propto^i)C_{jkh}$$
$$-(\mathfrak{B}_m\propto^i)(\mathfrak{B}_lC_{jkh}) - (\mathfrak{B}_l\propto^i)(\mathfrak{B}_mC_{jkh}) - \propto^i(\mathfrak{B}_l\mathfrak{B}_mC_{jkh})$$
$$+(\mathfrak{B}_l\mathfrak{B}_m \propto)(C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk})$$
$$+(\mathfrak{B}_m \propto)\Big[(\mathfrak{B}_lC^r_{jk})C^i_{rh} + C^r_{jk}(\mathfrak{B}_lC^i_{rh}) - (\mathfrak{B}_lC^r_{jh})C^i_{rk} - C^r_{jh}(\mathfrak{B}_lC^i_{rk})\Big]$$
$$+(\mathfrak{B}_l \propto)\Big[(\mathfrak{B}_mC^r_{jk})C^i_{rh} + C^r_{jk}(\mathfrak{B}_mC^i_{rh}) - (\mathfrak{B}_mC^r_{jh})C^i_{rk} - C^r_{jh}(\mathfrak{B}_mC^i_{rk})\Big]$$
$$+ \propto \Big[(\mathfrak{B}_l\mathfrak{B}_mC^r_{jk})C^i_{rh} + (\mathfrak{B}_mC^r_{jk})(\mathfrak{B}_lC^i_{rh}) + (\mathfrak{B}_lC^r_{jk})(\mathfrak{B}_mC^i_{rh})$$
$$+C^r_{jk}(\mathfrak{B}_l\mathfrak{B}_mC^i_{rh}) - (\mathfrak{B}_l\mathfrak{B}_mC^r_{jh})C^i_{rk} - (\mathfrak{B}_mC^r_{jh})(\mathfrak{B}_lC^i_{rk}) - (\mathfrak{B}_lC^r_{jh})(\mathfrak{B}_mC^i_{rk})$$
$$-C^r_{jh}(\mathfrak{B}_l\mathfrak{B}_mC^i_{rk})\Big] + (\mathfrak{B}_l\mathfrak{B}_m\omega_j)(\delta^i_k y_h - \delta^i_h y_k) - (\mathfrak{B}_l\mathfrak{B}_m\omega^i)(g_{jk}y_h - g_{jh}y_k)$$
$$-(\mathfrak{B}_m\omega^i)(y_h\mathfrak{B}_l g_{jk} - y_k\mathfrak{B}_l g_{jh})$$
$$+(\mathfrak{B}_l\mathfrak{B}_m\omega)(y_jC^i_{kh}) + (\mathfrak{B}_m\omega)(y_j\mathfrak{B}_lC^i_{kh}) + (\mathfrak{B}_l\omega)(y_j\mathfrak{B}_mC^i_{kh}) + \omega(y_j\mathfrak{B}_l\mathfrak{B}_mC^i_{kh})$$
$$-(\mathfrak{B}_l\omega^i)(y_h\mathfrak{B}_m g_{jk} - y_k\mathfrak{B}_m g_{jh}) - \omega^i(y_h\mathfrak{B}_l\mathfrak{B}_m g_{jk} - y_k\mathfrak{B}_l\mathfrak{B}_m g_{jh}).$$

Using eqs. (3.2) and (4.1) in above equation, then using eqs. (2.7) and (2.3), we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} \\
= \;& a_{lm}\Big[\propto_j C^i_{kh} - \propto^i C_{jkh} + \propto (C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk}) + \omega(y_j C^i_{kh})\Big] \\
&+\Big[(\mathfrak{B}_l\mathfrak{B}_m \propto) + 2\propto_l(\mathfrak{B}_m \propto) + 2\propto_m(\mathfrak{B}_l \propto) + \propto a_{lm} \\
&+2\propto\propto_l\propto_m\Big](C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk}) \\
&+\Big[(\mathfrak{B}_m\propto_j)\omega_l + (\mathfrak{B}_l\propto_j)\omega_m + \propto_j b_{lm} + (\mathfrak{B}_l\mathfrak{B}_m\omega_j) + (\mathfrak{B}_m\omega)y_j\omega_l \\
&+(\mathfrak{B}_l\omega)y_j\omega_m + \omega y_j b_{lm} - 2\omega_l\omega_m y_j\Big](\delta^i_k y_h - \delta^i_h y_k) \\
&+\Big[(\mathfrak{B}_m\propto^i)\omega_l + (\mathfrak{B}_l\propto^i)\omega_m + (\mathfrak{B}_l\mathfrak{B}_m\omega^i) + \propto^i b_{lm}\Big](g_{jk}y_h - g_{jh}y_k) \\
&+(\mathfrak{B}_l\mathfrak{B}_m\propto_j)C^i_{kh} + (\mathfrak{B}_m\propto_j)\propto_l C^i_{kh} + (\mathfrak{B}_l\propto_j)\propto_m C^i_{kh} - (\mathfrak{B}_l\mathfrak{B}_m\propto^i)C_{jkh} \\
&-(\mathfrak{B}_m\propto^i)\propto_l C_{jkh} - (\mathfrak{B}_l\propto^i)\propto_m C_{jkh} \\
&+\Big[(\mathfrak{B}_l\mathfrak{B}_m\omega) + (\mathfrak{B}_m\omega)\propto_l + (\mathfrak{B}_l\omega)\propto_m + a_{lm}\omega\Big](y_j C^i_{kh}) \\
&+2(\mathfrak{B}_m\omega^i)(y_h C_{jkl|s}y^s - y_k C_{jhl|s}y^s) + 2(\mathfrak{B}_l\omega^i)(y_h C_{jkm|s}y^s - y_k C_{jhm|s}y^s) \\
&-\omega^i(y_h\mathfrak{B}_l\mathfrak{B}_m g_{jk} - y_k\mathfrak{B}_l\mathfrak{B}_m g_{jh}).
\end{aligned}$$

Using eq. (3.5) in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} \\
= \;& a_{lm}\Big[P^i_{jkh} - \omega_j(\delta^i_k y_h - \delta^i_h y_k) + \omega^i(g_{jk}y_h - g_{jh}y_k)\Big] \\
&+\Big[(\mathfrak{B}_l\mathfrak{B}_m \propto) + 2\propto_l(\mathfrak{B}_m \propto) + 2\propto_m(\mathfrak{B}_l \propto) + \propto a_{lm} + 2\propto\propto_l\propto_m\Big] \\
&\times(C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk}) + \Big[(\mathfrak{B}_m\propto_j)\omega_l \\
&+(\mathfrak{B}_l\propto_j)\omega_m + \propto_j b_{lm} + (\mathfrak{B}_l\mathfrak{B}_m\omega_j) + (\mathfrak{B}_m\omega)y_j\omega_l + (\mathfrak{B}_l\omega)y_j\omega_m + \omega y_j b_{lm} \\
&-2\omega_l\omega_m y_j\Big](\delta^i_k y_h - \delta^i_h y_k) + \Big[(\mathfrak{B}_m\propto^i)\omega_l + (\mathfrak{B}_l\propto^i)\omega_m + (\mathfrak{B}_l\mathfrak{B}_m\omega^i) + \propto^i b_{lm}\Big] \\
&\times(g_{jk}y_h - g_{jh}y_k) + (\mathfrak{B}_l\mathfrak{B}_m\propto_j)C^i_{kh} + (\mathfrak{B}_m\propto_j)\propto_l C^i_{kh} + (\mathfrak{B}_l\propto_j)\propto_m C^i_{kh} \\
&-(\mathfrak{B}_l\mathfrak{B}_m\propto^i)C_{jkh} - (\mathfrak{B}_m\propto^i)\propto_l C_{jkh} - (\mathfrak{B}_l\propto^i)\propto_m C_{jkh} \\
&+\Big[(\mathfrak{B}_l\mathfrak{B}_m\omega) + (\mathfrak{B}_m\omega)\propto_l + (\mathfrak{B}_l\omega)\propto_m + a_{lm}\omega\Big](y_j C^i_{kh}) \\
&+2(\mathfrak{B}_m\omega^i)(y_h C_{jkl|s}y^s - y_k C_{jhl|s}y^s) \\
&+2(\mathfrak{B}_l\omega^i)(y_h C_{jkm|s}y^s - y_k C_{jhm|s}y^s) - \omega^i(y_h\mathfrak{B}_l\mathfrak{B}_m g_{jk} - y_k\mathfrak{B}_l\mathfrak{B}_m g_{jh}).
\end{aligned}$$

This shows that

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{lm}P^i_{jkh} \tag{4.3}$$

The equation (4.3) refers that the Cartan's second curvature tensor P^i_{jkh} behave as birecurrent in $G(\mathfrak{B}P)-BRF_n$ if and only if eq. (4.2) holds. The proof for this theorem is completed.

Now, we infer a corollary related to the previous theorem. Taking $\mathfrak{B}-$covariant derivative for eq. (3.4) twice with respect to x^m and x^l, successively, using eq. (2.6), we get

$$\begin{aligned}&\mathfrak{B}_l\mathfrak{B}_m P^i_{kh}\\ &= (\mathfrak{B}_l\mathfrak{B}_m \propto)C^i_{kh} + (\mathfrak{B}_m \propto)(\mathfrak{B}_l C^i_{kh}) + (\mathfrak{B}_l \propto)(\mathfrak{B}_m C^i_{kh}) + \propto (\mathfrak{B}_l\mathfrak{B}_m C^i_{kh})\\ &+(\mathfrak{B}_l\mathfrak{B}_m\omega)(\delta^i_k y_h - \delta^i_h y_k).\end{aligned}$$

Using eqs. (3.2) and (4.1) in above equation, we get

$$\begin{aligned}&\mathfrak{B}_l\mathfrak{B}_m P^i_{kh}\\ =\ &\Big[(\mathfrak{B}_l\mathfrak{B}_m \propto) + (\mathfrak{B}_m \propto)\propto_l + (\mathfrak{B}_l \propto)\propto_m + \propto a_{lm}\Big]C^i_{kh}\\ &+\Big[(\mathfrak{B}_m \propto)\omega_l + (\mathfrak{B}_l \propto)\omega_m + \propto b_{lm} + (\mathfrak{B}_l\mathfrak{B}_m\omega)\Big](\delta^i_k y_h - \delta^i_h y_k).\end{aligned}$$

Using eq. (3.8) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{kh} = c_{lm}P^i_{kh} + d_{lm}(\delta^i_k y_h - \delta^i_h y_k). \tag{4.4}$$

where

$$c_{lm} = \frac{(\mathfrak{B}_l\mathfrak{B}_m \propto)}{\propto} + \frac{(\mathfrak{B}_m \propto)\propto_l}{\propto} + \frac{(\mathfrak{B}_l \propto)\propto_m}{\propto} + a_{lm}.$$

and

$$\begin{aligned}d_{lm} \quad = \quad &(\mathfrak{B}_m \propto)\omega_l + (\mathfrak{B}_l \propto)\omega_m + \propto b_{lm} + (\mathfrak{B}_l\mathfrak{B}\omega)\\ &-\omega\Big[\frac{(\mathfrak{B}_l\mathfrak{B}_m \propto)}{\propto} + \frac{(\mathfrak{B}_m \propto)\propto_l}{\propto} + \frac{(\mathfrak{B}_l \propto)\propto_m}{\propto} + a_{lm}\Big].\end{aligned}$$

Thus, we conclude the following corollary:

Corollary 4.2. *In $G(\mathfrak{B}P)-BRF_n$, the $(v)hv-$ torsion tensor P^i_{kh}, necessarily is given by eq.* (4.4) *[provided eqs.* (3.2) *and* (4.1) *hold].*

5. **Examples**

In order to illustrate the effectiveness of the proposed findings, some examples related to the previous mentioned theorems will be discussed.

Example 5.1. *The Cartan's second curvature tensor P^i_{jkh} behaves as recurrent if and only if satisfies*

$$\mathfrak{B}_m\left(p.P^i_{jkh}\right) = \propto_m\left(p.P^i_{jkh}\right).$$

Firstly, since Cartan's second curvature tensor P^i_{jkh} behaves as recurrent, then the condition (3.6) is satisfied. In view of eq. (2.13)a, the projection of Cartan's second curvature tensor P^i_{jkh} on indicatrix is given by

$$p.P^i_{jkh} = P^a_{bcd} h^i_a h^b_j h^c_k h^d_h. \tag{5.1}$$

Using $\mathfrak{B}-$covariant derivative for eq. (5.1) with respect to x^m, using eq. (3.6) and the fact that h^a_b is covariant constant in above equation, we get

$$\mathfrak{B}_m \left(p.P^i_{jkh}\right) = \propto_m \left(P^a_{bcd} h^i_a h^b_j h^c_k h^d_h\right).$$

By using eq. (5.1) in above equation, we get

$$\mathfrak{B}_m \left(p.P^i_{jkh}\right) = \propto_m \left(p.P^i_{jkh}\right). \tag{5.2}$$

Above equation means the projection on indicatrix for the Cartan's second curvature tensor P^i_{jkh} behaves as recurrent.

Secondly, let the projection on indicatrix for the Cartan's second curvature tensor P^i_{jkh} is recurrent i.e. satisfy eq. (5.2). Using eq. (2.13)a in eq. (5.2), we get

$$\mathfrak{B}_m \left(P^a_{bcd}\ h^i_a h^b_j h^c_k h^d_h\right) = \propto_m \left(P^a_{bcd}\ h^i_a h^b_j h^c_k h^d_h\right).$$

Using eq. (2.13)b in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_m\big[P^i_{jkh} - P^a_{jkd} l^d l_{\ h} - P^i_{jch} l^c l_k + P^i_{jcd} l^c l_k l^d l_h - P^i_{bkh} l^b l_j \\
&+P^i_{bkd} l^b l_j l^d l_h + P^i_{bch} l^b l_j l^c l_k - P^i_{bcd} l^b l_j l^c l_k l^d l_h - P^a_{jkh} l^i l_a + P^a_{jkd} l^i l_a l^d l_h \\
&+P^a_{jch} l^i l_a l^c l_k - P^a_{jcd} l^i l_a l^c l_k l^d l_h + P^a_{bkh} l^i l_a l^b l_j - P^a_{bkd} l^i l_a l^b l_j l^d l_h \\
&-P^a_{bch} l^i l_a l^b l_j l^c l_k + P^a_{bcd} l^i l_a l^b l_j l^c l_k l^d l_h\big] \\
&= \propto_m\big[P^i_{jkh} - P^i_{jkd} l^d l_h - P^i_{jch} l^c l_k + P^i_{jcd} l^c l_k l^d l_h - P^i_{bkh} l^b l_j \\
&+P^i_{bkd} l^b l_j l^d l_h + P^i_{bch} l^b l_j l^c l_k - P^i_{bcd} l^b l_j l^c l_k l^d l_h - P^a_{jkh} l^i l_a + P^a_{jkd} l^i l_a l^d l_h \\
&+P^a_{jch} l^i l_a l^c l_k - P^a_{jcd} l^i l_a l^c l_k l^d l_h + P^a_{bkh} l^i l_a l^b l_j - P^a_{bkd} l^i l_a l^b l_j l^d l_h \\
&-P^a_{bch} l^i l_a l^b l_j l^c l_k + P^a_{bcd} l^i l_a l^b l_j l^c l_k l^d l_h\big].
\end{aligned}$$

In view of eq. (2.4) and if $P^a_{bcd}\ y_a = P^a_{bcd}\ y^b = P^a_{bcd}\ y^c = P^a_{bcd}\ y^d = 0$, then above equation can be written as

$$\mathfrak{B}_m P^i_{jkh} = \propto_m P^i_{jkh}.$$

Above equation means the Cartan's second curvature tensor P^i_{jkh} behaves as recurrent.

Example 5.2. *The Cartan's second curvature tensor P^i_{jkh} behaves as recurrent if and only if satisfies*

$$\mathfrak{B}_l \mathfrak{B}_m \left(p.P^i_{jkh}\right) = a_{lm} \left(p.P^i_{jkh}\right).$$

Firstly, since Cartan's second curvature tensor P^i_{jkh} behaves as birecurrent, then the condition (4.3) is satisfied.

By using $\mathfrak{B}-$covariant derivative for eq. (5.1) twice with respect to x^m and x^l, respectively, using eq.(4.3) and the fact that h^a_b is covariant constant, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.P^i_{jkh}\right) = a_{lm}\left(P^a_{bcd}h^i_a h^b_j h^c_k h^d_h\right).$$

Using eq. (5.1) in above equation, we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(p.P^i_{jkh}\right) = a_{lm}\left(p.P^i_{jkh}\right). \tag{5.3}$$

Equation (5.3) means the projection on indicatrix for the Cartan's second curvature tensor P^i_{jkh} behaves as birecurrent.

Secondly, let the projection on indicatrix for the Cartan's second curvature tensor P^i_{jkh} is birecurrent, i.e satisfy eq. (5.3). Using eq. (2.13)a in eq. (5.3), we get

$$\mathfrak{B}_l\mathfrak{B}_m\left(P^a_{bcd}\ h^i_a h^b_j h^c_k h^d_h\right) = a_{lm}\left(P^a_{bcd}\ h^i_a h^b_j h^c_k h^d_h\right).$$

By using eq. (2.13)b in above equation, we get

$$\begin{aligned}
&\mathfrak{B}_l\mathfrak{B}_m\big[P^i_{jkh} - P^a_{jkd}l^d l_h - P^i_{jch}l^c l_k + P^i_{jcd}l^c l_k l^d l_h - P^i_{bkh}l^b l_j + P^i_{bkd}l^b l_j l^d l_h\\
&+P^i_{bch}l^b l_j l^c l_k - P^i_{bcd}l^b l_j l^c l_k l^d l_h - P^a_{jkh}l^i l_a + P^a_{jkd}l^i l_a l^d l_h + P^a_{jch}l^i l_a l^c l_k\\
&-P^a_{jcd}l^i l_a l^c l_k l^d l_h + P^a_{bkh}l^i l_a l^b l_j - P^a_{bkd}l^i l_a l^b l_j l^d l_h - P^a_{bch}l^i l_a l^b l_j l^c l_k\\
&+P^a_{bcd}l^i l_a l^b l_j l^c l_k l^d l_h\big]\\
&= a_{lm}\big[P^i_{jkh} - P^a_{jkd}l^d l_h - P^i_{jch}l^c l_k + P^i_{jcd}l^c l_k l^d l_h - P^i_{bkh}l^b l_j\\
&+P^i_{bkd}l^b l_j l^d l_h + P^i_{bch}l^b l_j l^c l_k - P^i_{bcd}l^b l_j l^c l_k l^d l_h - P^a_{jkh}l^i l_a\\
&+P^a_{jkd}l^i l_a l^d l_h + P^a_{jch}l^i l_a l^c l_k - P^a_{jcd}l^i l_a l^c l_k l^d l_h + P^a_{bkh}l^i l_a l^b l_j\\
&-P^a_{bkd}l^i l_a l^b l_j l^d l_h - P^a_{bch}l^i l_a l^b l_j l^c l_k + P^a_{bcd}l^i l_a l^b l_j l^c l_k l^d l_h\big].
\end{aligned}$$

In view of eq. (2.4) and if $P^a_{bcd}y_a = P^a_{bcd}y^b = P^a_{bcd}y^c = P^a_{bcd}y^d = 0$, then above equation can be written as

$$\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} = a_{lm}P^i_{jkh}.$$

Last equation means the Cartans second curvature tensor P^i_{jkh} behaves as birecurrent.

6. **Conclusion**

The necessary and sufficient condition for Cartan's second curvature tensor which satisfies the recurrence and birecurrence property has been obtained in generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}P-$birecurrent space, respectively. Also, certain identities belong to these spaces have been studied. In addition, we find the condition for the projection of Cartan's second curvature tensor on indicatrix to be recurrent and birecurrent tensor.

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