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Two classes of weakly Landsberg Finsler metrics

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Abstract. In this paper, we investigate the mean Landsberg curvature of two subclasses of (α, β)-metrics. We prove that these subclasses of (α, β)-metrics with vanishing mean Landsberg curvature have vanishing S-curvature. Using it, we prove that these Finsler metrics are weakly Landsbergian if and only if they are Berwaldian.

Keywords: Weakly Landsberg metric, (α, β)-metric, S-curvature.

1. Introduction

Consider a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M. Let $G^i = G^i(x, y)$ denote the spray coefficients of F in a local coordinate system.The Landsbergc curvature $\mathbf{L} = L_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is a horizontal on $TM/0$, defined by

$$L_{ijk} := -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j y^k}.$$

Finsler metrics F are called Landsberg metrics if $L_{ijk} = 0$ .The mean Landsbergc curvature $\mathbf{J} = J_i dx^i$, defined by

$$J_k := g^{ij} L_{ijk}$$

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Finsler metrics F with $\mathbf{J} = 0$ are called weakly Landsberg metrics. Clearly, in dimension two, any weakly Landsberg metric must be a Landsberg metric.

In this paper, first we consider the (α, β)-metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ on a manifold M, where c_i are real numbers. This metric is called the Randers-type metric [1]. Indeed, by putting $c_1 = c_2 = c_3 = 1$, we get the Randers metric. We prove the following.

Theorem 1.1. *Let* $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ *be the generalized Randers metric. Then* F *is weakly Landsberg metric if and only if it is a Berwald metric.*

Then, we study the mean Landsberg curvature of the Finsler metric $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ and prove the following.

Theorem 1.2. *Let* $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ *be a* (α, β)*-metric. Then* F *is weakly Landsberg metric if and only if it is a Berwald metric.*

2. Preliminaries

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \quad y \in T_xM. \tag{2.2}$$

$\mathbf{G}$ is called the associated spray to (M, F).

For a non-zero vector $y \in T_eM$, define $\mathbf{B}_y : T_eM \times T_eM \times T_eM \to T_eM$ by $\mathbf{B}_y(v, u, w) = B^m_{ijl} v^i u^j w^l \frac{\partial}{\partial x^m}|_l$, where

$$B^m_{ijl} := \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature, and F is represents a Berwald metric if $\mathbf{B} = 0$.

For a Finsler manifold (M, F), the Busemann-Hausdorf volume form $dV_F = \sigma_F(x) dx^1 ... dx^n$ is defined as follows:

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\big\{(y^t) \in \mathbb{R}^n | F(y^t \frac{\partial}{\partial x^t}|_x) < 1\big\}}.$$

Then, for $y = y^m \partial/\partial x^m|_e \in T_eM$, the S-curvature is defined by

$$\mathbf{S}(y) := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} \Big[ln \sigma_F(x) \Big]. \tag{2.3}$$

The S-curvature has been introduced by Shen for the formulation of a comparison theorem on Finsler manifolds.

The function $F = \alpha\phi(s)$ is a Finsler metric for any $\alpha = \sqrt{a_{ij}y^i y^j}$ and any $\beta = b_i y^i$ with $||\beta_x||_\alpha < b_0$ if and only if ϕ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \leq b < b_0. \tag{2.4}$$

From (2.4), one can see that $\phi = \phi(s)$ must satisfy

$$\phi(s) - s\phi'(s) > 0, \qquad |s| < b_0.$$

For more details, see [4]. A Finsler metric F on a manifold M is called an (α, β)-metric if it is expressed as $F = \alpha\phi(s)$ with $||\beta_x||_\alpha < b_0$, where $\phi(s)$ is a positive C^∞ on $(-b_0, b_0)$ satisfying (2.4). In order to study the geometric properties of (α, β)-metrics, one needs a formula for the spray coefficients of an (α, β)-metric. Let

$$\begin{array}{llll} r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), & & \\ r^i{}_j := a^{is} r_{sj}, & s^i{}_j := a^{is} s_{sj}, & q_{ij} := r_{is} s^s{}_j, & t_{ij} := t_{ik} s^k{}_j, \\ r_j := b^i r_{ij}, & s_j := b^i s_{ij}, & q_j := b^i q_{ij}, & t_j := b^i t_{ij}, \end{array}$$

where "$|$" denotes the covariant derivative with respect to the Levi-Civita connection of α and $b^i := a^{ij} b_j$, a^{ij} is the inverse of a_{ij}. We define $r_{i0} = r_{ij} y^j$ and $r_{00} = r_{ij} y^i y^j$, etc [2]. For a function $\phi = \phi(s)$ satisfying (2.4), we let

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta := 1 + sQ + (b^2 - s^2)Q', \qquad h_j := \alpha b_j - s y_j.$$

3. **Proof of Theorem 1.1**

In this section, we are going to prove Theorem 1.1. In order to prove it, we need to remark some necessary facts. In [5], Benling Li and Zhongmin Shen studied the mean Landsberg curvature of (α, β)-metrics and proved the following.

Lemma 3.1. ([5])*Let*

$$\begin{aligned} \Phi &:= -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'', \\ \psi 1 &:= \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}} \Big(\frac{\sqrt{b^2 - s^2}\,\Phi}{\Delta^{\frac{3}{2}}} \Big)'. \end{aligned}$$

Then the mean Landsberg curvature of F is given by following

$$
\begin{aligned}
J_k :=& -\frac{\Delta}{2\alpha^4}\Big\{\frac{2\alpha^2}{b^2-s^2}\Big(\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big)(s_0-r_0)hj\\
&+\frac{\alpha}{b^2-s^2}\Big(\psi 1+s\frac{\Phi}{\Delta}\Big)(r_{00}-2\alpha Q s_0)hj\\
&+\alpha\Big(-\alpha Q' s_0 hj+\alpha Q(\alpha^2 s_j-y_j s_0)+\alpha^2\Delta s_{j0}\\
&+\alpha^2(r_{j0}-2\alpha Q s_j)-(r_{00}-2\alpha Q s_0)y_j\Big)\frac{\Phi}{\Delta}\Big\},
\end{aligned}
\tag{3.1}
$$

where $s_0 := s_i y^i$, $r_0 := r_i y^i$, $r_{00} := r_{ij} y^i y^j$, $r_{j0} := r_{jk} y^k$ *and* $s_{j0} := s_{jk} y^k$.

Remark 3.2. *For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, if β is parallel with respect to α, then the mean Landsberg curvature vanish. This means that F is weakly Landsberg metric.*

Theorem 3.3. *Let $F=\sqrt{c_1\alpha^2+2c_2\alpha\beta+c_3\beta^2}$ be a weakly Landsberg (α,β)-metric. Then F has vanishing S-curvature.*

Proof. Suppose F is weakly Landsberg metric, i.e., we have

$$
\mathbf{J}=0. \tag{3.2}
$$

By Lemmas 3.1, we calculate $J=J_i b^i=0$ which is equal to following

$$
\mathbf{J}=f_5\alpha^5+f_4\alpha^4+f_3\alpha^3+f_2\alpha^2+f_1\alpha+f_0=0, \tag{3.3}
$$

where

$$A := \sqrt{c_1\alpha^2 + c_2\beta\alpha + c_3\beta^2}\,\alpha^2$$

$$f_0 := -8\beta^8 ns_0 c_2^5 c_3^4, \tag{3.4}$$

$$\begin{aligned} f_1 := {}& 20\beta^6 c_2^4 c_3^4 ns_0 A - 40\beta^7 c_2^4 c_3^4 ns_0 c_1 + 8\beta^5 c_2^6 c_3^2 Anr_{00} - 40\beta^7 c_2^6 c_3^3 ns_0 \\ & - 8\beta^5 c_2^4 c_3^3 Anr_{00} c_1 - 2\beta^6 c_2^4 c_3^4 r_{00} + 4\beta^6 c_2^4 c_3^4 s_0 A + 2\beta^6 c_2^4 c_3^4 nr_{00}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} f_2 := {}& 24\beta^5 c_2^3 c_3^4 s_0 c_1 A - 14\beta^5 c_2^5 c_3^3 r_{00} - 20\beta^6 c_2^3 c_3^5 ns_0 + 8\beta^5 c_2^5 c_3^3 r_0 A \\ & - 2\beta^6 c_2^3 c_3^5 s_0 - 10\beta^5 c_2^3 c_3^4 r_{00} c_1 + 8\beta^5 c_2^5 c_3^3 s_0 A + 14\beta^4 c_2^7 c_3 Anr_{00} \\ & + 104\beta^5 c_2^5 c_3^3 Ans_0 + 32\beta^6 c_2^5 c_3^4 b^2 ns_0 + 18\beta^4 c_2^5 c_3^2 Anr_{00} c_1 - 72\beta^6 c_2^7 c_3^2 ns_0 \\ & - 80\beta^6 c_2^3 c_3^4 ns_0 c_1^2 + 2\beta^5 c_2^5 c_3^3 nr_{00} + 100\beta^5 c_2^3 c_3^4 Ans_0 c_1 + 4\beta^5 c_2^5 c_3^3 nr_0 A \\ & + 22\beta^5 c_2^3 c_3^4 nr_{00} c_1 - 208\beta^6 c_2^5 c_3^3 ns_0 c_1 - \beta^4 c_2^3 c_3^4 Ar_{00} - 2\beta^4 c_2^3 c_3^4 Anr_{00} \\ & - 32\beta^4 c_2^3 c_3^3 Anr_{00} c_1^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} f_3 := {}& 32\beta^3 c_2^4 Ab^2 nr_{00} c_1 c_3^3 + 32\beta^4 c_2^6 Ar_0 c_3^2 - 12\beta^4 c_2^6 As_0 c_3^2 + 6\beta^3 c_2^8 Anr_{00} \\ & - 8\beta^3 c_2^4 Ar_{00} c_3^3 + 2\beta^4 c_2^4 c_3^4 b^2 r_{00} - 152\beta^5 c_2^4 c_3^4 ns_0 - 16\beta^5 c_2^2 c_3^5 s_0 c_1 \\ & - 2\beta^4 c_2^2 c_3^4 r_{00} c_1^2 - 103\beta^4 c_2^4 c_3^3 r_{00} c_1 - 8\beta^5 c_2^4 c_3^4 nr_0 - 25\beta^4 c_2^6 c_3^2 r_{00} - 14\beta^5 c_2^4 c_3^4 r_0 \\ & - 80\beta^5 c_2^2 c_3^4 ns_0 c_1^3 - 440\beta^5 c_2^4 c_3^3 ns_0 c_1^2 + 216\beta^4 c_2^6 Ans_0 c_3^2 - 12\beta^4 c_2^4 Ab^2 s_0 c_3^4 \\ & + 40\beta^4 c_2^4 Ar_0 c_1 c_3^3 + 48\beta^4 c_2^2 As_0 c_1^2 c_3^4 + 72\beta^4 c_2^4 As_0 c_1 c_3^3 - 12\beta^3 c_2^4 Anr_{00} c_3^3 \\ & + 8\beta^4 c_2^2 Ans_0 c_3^5 - 2\beta^4 c_2^4 c_3^4 b^2 nr_{00} - 96\beta^5 c_2^2 c_3^5 ns_0 c_1 + 54\beta^4 c_2^2 c_3^4 nr_{00} c_1^2 \\ & + 16\beta^5 c_2^6 c_3^3 b^2 ns_0 - 384\beta^5 c_2^6 c_3^2 ns_0 c_1 + 54\beta^4 c_2^4 Ans_0 c_1 c_3^3 + 48\beta^3 c_2^6 Anr_{00} c_1 c_3 \\ & - 12\beta^3 c_2^2 Anr_{00} c_1 c_3^4 + 20\beta^4 c_2^4 Anr_0 c_1 c_3^3 - 32\beta^3 c_2^6 Ab^2 nr_{00} c_3^2 - 48\beta^3 c_2^2 Anr_{00} c_3^3 \\ & + 4\beta^4 c_2^2 As_0 c_3^5 - 22\beta^4 c_2^6 c_3^2 nr_{00} + 2\beta^5 c_2^4 c_3^4 s_0 + 16\beta^4 c_2^6 Anr_0 c_3^2 - 4\beta^3 c_2^2 Ar_{00} c_1 c_3^4 \\ & + 98\beta^4 c_2^4 c_3^3 nr_{00} c_1 - 6\beta^3 c_2^4 Anr_{00} c_1^2 c_3^2 + 160\beta^5 c_2^4 c_3^4 b^2 ns_0 c_1 - 56\beta^5 c_2^8 c_3 ns_0 \\ & + 180\beta^4 c_2^2 Ans_0 c_1^2 c_3^4, \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
f_4 :=\ & 140\beta^3 Ans_0 c_1^3 c_2 c_3^4 + 1040\beta^3 Ans_0 c_1^2 c_2^3 c_3^3 + 1164\beta^3 Ans_0 c_1 c_2^5 c_3^2 \\
& - 12\beta^3 b^2 nr_0 c_2^5 c_3^3 A + 128\beta^2 Ab^2 nr_{00} c_1^2 c_2^3 c_3^3 - 320\beta^3 Ab^2 ns_0 c_1 c_2^3 c_3^4 \\
& - 20\beta^2 Ar_{00} c_2^5 c_3^2 + 40\beta^3 r_0 c_2^7 c_3 A - 32\beta^3 s_0 c_2^7 c_3 A + 6\beta^3 Ar_0 c_2^3 c_3^4 + 6\beta^3 c_2^5 b^2 r_{00} \\
& - 472\beta^4 c_2^5 ns_0 c_3^3 - 62\beta^4 c_2 s_0 c_1^2 c_3^5 - 48\beta^3 c_2^7 nr_{00} c_3 + 18\beta^3 c_2 r_{00} c_1^3 c_3^4 \\
& - 270\beta^3 c_2^5 r_{00} c_1 c_3^2 - 56\beta^4 c_2^5 nr_0 c_3^3 - 52\beta^4 c_2^3 r_0 c_1 c_3^4 - 2\beta^4 c_2^3 s_0 c_3^4 + 8\beta^4 c_2^3 b^2 s_0 c_3^5 \\
& + 62\beta^4 c_2^5 s_0 c_3^3 - 10\beta^3 c_2^7 r_{00} c_3 - 12\beta^4 c_2^5 r_0 c_3^3 - 30\beta^2 Ar_{00} c_1 c_2^3 c_3^3 - \beta^2 Anr_{00} c_2^5 c_3^2 \\
& + 8\beta^3 s_0 c_1 c_2^5 c_3^2 A + 176\beta^3 s_0 c_1^2 c_2^3 c_3^3 A + 168\beta^3 r_0 c_1 c_2^5 c_3^2 A + 40\beta^3 s_0 c_1^3 c_2 c_3^4 A \\
& + 80\beta^3 r_0 c_1^2 c_2^3 c_3^3 A - 24\beta^3 b^2 r_0 c_2^5 c_3^3 A - 16\beta^3 b^2 s_0 c_2^5 c_3^3 A + 4\beta^3 Anr_0 c_2^3 c_3^4 \\
& + 22\beta^2 Anr_{00} c_1 c_2^7 + 200\beta^3 Ans_0 c_2^7 c_3 - 15\beta^2 Ar_{00} c_1^2 c_2 c_3^4 + 288\beta^4 c_2^7 b^2 ns_0 c_3^2 \\
& - 840\beta^4 c_2^5 ns_0 c_1^2 c_3^2 - 304\beta^4 c_2^7 ns_0 c_1 c_3 - 48\beta^4 c_2^5 b^4 ns_0 c_3^4 - 40\beta^4 c_2 ns_0 c_1^4 c_3^4 \\
& + 22\beta^3 c_2^5 b^2 nr_{00} c_3^3 + 22\beta^3 c_2^3 b^2 r_{00} c_1 c_3^4 - 132\beta^4 c_2 ns_0 c_1^2 c_3^5 - 800\beta^4 c_2^3 ns_0 c_1 c_3^4 \\
& + 316\beta^3 c_2^3 nr_{00} c_1^2 c_3^3 + 102\beta^3 c_2^5 nr_{00} c_1 c_3^2 - 32\beta^4 c_2^3 nr_0 c_1 c_3^4 + 40\beta^3 nr_0 c_1^2 c_2^3 c_3^3 A \\
& - 32\beta^2 Anr_{00} c_1^4 c_2 c_3^3 - 44\beta^2 Anr_{00} c_1^3 c_2^3 c_3^2 + 54\beta^2 Anr_{00} c_1^2 c_2^5 c_3 - 18\beta^2 Ar_{00} c_1^2 c_3^4 \\
& - 8\beta^2 Anr_{00} c_1 c_2^3 c_3^3 - 22\beta^3 Ab^2 ns_0 c_2^5 c_3^3 - 56\beta^2 Ab^2 nr_{00} c_2^7 c_3 - 5\beta^3 c_2^3 b^2 nr_{00} c_1 c_3^4 \\
& + 32\beta^4 c_2^3 b^2 ns_0 c_1^2 c_3^4 + 832\beta^4 c_2^5 b^2 ns_0 c_1 c_3^3 + 30\beta^3 As_0 c_2^3 c_3^4 + 16\beta^3 As_0 c_1 c_2 c_3^5 \\
& - 80\beta^3 b^2 s_0 c_1 c_2^3 c_3^4 A - 72\beta^2 Ab^2 nr_{00} c_1 c_2^5 c_3^2 - 158\beta^3 c_2^3 r_{00} c_1^2 c_3^3 - 16\beta^4 c_2^9 ns_0 \\
& + 20\beta^3 nr_0 c_2^7 c_3 A + 68\beta^3 Ans_0 c_2^3 c_3^4 - 480\beta^4 c_2^3 ns_0 c_1^3 c_3^3 + 36\beta^4 c_2^3 b^2 ns_0 c_3^5 \\
& + 50\beta^3 c_2 nr_{00} c_1^3 c_3^4 + 84\beta^3 nr_0 c_1 c_2^5 A, \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
f_5 :=\ & 18\beta^2 nr_0 c_1^2 c_2^4 c_3^2 A + 40\beta^2 nr_0 c_1^3 c_2^2 c_3^3 A + 20\beta^2 Ans_0 c_1^3 c_2^2 c_3^3 + 24\beta^2 Ans_0 c_1^2 c_2^4 \\
& + 1108\beta^2 Ans_0 c_1 c_2^6 c_3 - 228\beta Anr_{00} c_1 c_2^4 c_3^2 + 16\beta Anr_{00} c_1^3 c_2^4 c_3 \\
& + 60\beta^2 b^4 ns_0 c_2^4 c_3^4 A + 312\beta^2 Ans_0 c_1 c_2^2 c_3^4 - 12\beta Ab^2 r_{00} c_1 c_2^2 c_3^4 - 16\beta Anr_{00} c_1^2 c_2^2 \\
& + 48\beta Ab^4 nr_{00} c_2^6 c_3^2 - 588\beta^2 Ab^2 ns_0 c_2^6 c_3^2 + 12\beta^2 Anr_0 c_1 c_2^2 c_3^4 - 48\beta^2 b^2 nr_0 c_2^6 c_3^2 A \\
& + 108\beta^2 nr_0 c_1 c_2^6 c_3 A - 4\beta^2 Ab^2 ns_0 c_2^2 c_3^5 - 224\beta^2 b^2 s_0 c_1 c_2^4 c_3^3 A \\
& - 120\beta^2 b^2 r_0 c_1 c_2^4 c_3^3 A - 8\beta Ab^2 nr_{00} c_2^4 c_3^3 - 138\beta^2 b^2 nr_{00} c_1^2 c_2^2 c_3^4 \\
& + 216\beta^3 b^2 ns_0 c_1 c_2^2 c_3^5 - 240\beta^3 b^4 ns_0 c_1 c_2^4 c_3^4 + 320\beta^3 b^2 ns_0 c_1^3 c_2^2 c_3^4 \\
& + 1536\beta^3 b^2 ns_0 c_1 c_2^6 c_3^2 - 600\beta^2 Ab^2 ns_0 c_1^2 c_2^2 c_3^4 - 1620\beta^2 Ab^2 ns_0 c_1 c_2^4 c_3^3 \\
& - 48\beta Ab^4 nr_{00} c_1 c_2^4 c_3^3 - 60\beta^2 b^2 nr_0 c_1 c_2^4 c_3^3 A + 192\beta Ab^2 nr_{00} c_1^3 c_2^2 c_3^3 \\
& + 12\beta c_2^2 b^2 Anr_{00} c_1 c_3^4 + 64\beta^2 As_0 c_2^4 c_3^3 + 8\beta^2 nr_0 c_2^8 A + 52\beta^2 Ar_0 c_2^4 c_3^3 \\
& - 16\beta Ar_{00} c_2^6 c_3 + 36\beta^2 As_0 c_1^2 c_3^5 - 12\beta Ar_{00} c_1^3 c_3^4 + 68\beta^2 Ans_0 c_2^8 + 2\beta^2 b^4 r_{00} c_2^4 c_3^4 \\
& - 88\beta^3 ns_0 c_1 c_2^8 - 45\beta^2 b^2 r_{00} c_2^6 c_3^2 - 56\beta^3 ns_0 c_1^3 c_3^5 - 756\beta^3 ns_0 c_2^6 c_3^2 + 16\beta^2 r_{00} c_3^4 \\
& - 426\beta^3 r_0 c_1 c_2^4 c_3^3 - 320\beta^3 s_0 c_1^2 c_2^2 c_3^4 + 232\beta^3 s_0 c_1 c_2^4 c_3^3 + 16\beta^2 r_0 c_2^8 A \\
& - 24\beta^2 nr_{00} c_2^8 + 12\beta^2 r_{00} c_1^4 c_3^4 - 22\beta^3 r_0 c_2^6 c_3^2 + 12\beta^2 b^4 s_0 c_2^4 c_3^4 A \\
& - 680\beta^3 ns_0 c_1^2 c_2^6 c_3 - 2\beta^2 b^4 nr_{00} c_2^4 c_3^4 + 16\beta^3 b^2 nr_0 c_2^4 c_3^4 + 14\beta Ab^2 r_{00} c_2^4 c_3^3 \\
& - 168\beta^2 b^2 s_0 c_1^2 c_2^2 c_3^4 A - 170\beta^2 b^2 nr_{00} c_1 c_2^4 c_3^3 + 1760\beta^3 b^2 ns_0 c_1^2 c_2^4 c_3^3 \\
& + 24\beta Ab^2 nr_{00} c_1^2 c_2^4 c_3^2 - 192\beta Ab^2 nr_{00} c_1 c_2^6 c_3 + 12\beta^2 s_0 c_1^4 c_3^4 A - 8\beta^3 ns_0 c_1^5 c_3^4 \\
& - 60\beta^2 r_{00} c_1^3 c_3^3 - 48\beta^3 s_0 c_1^3 c_3^5 + 24\beta^3 b^2 ns_0 c_2^8 c_3. \qquad (3.9)
\end{aligned}
$$

By (3.3), we get

$$f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \tag{3.10}$$
$$f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \tag{3.11}$$

By (3.4) and (3.11), observe that $-8\beta^8 ns_0c_2^5c_3^4$ is not divisible by α^2, which give us

$$s_i = 0. \tag{3.12}$$

By putting $s_0 = 0$ into (3.3), we obtain

$$g_5\alpha^5 + g_4\alpha^4 + g_3\alpha^3 + g_2\alpha^2 + g_1\alpha = 0, \tag{3.13}$$

where

$$g_1 := 8\beta^5c_2^6c_3^2Anr_{00} - 8\beta^5c_2^4c_3^3Anr_{00}c_1 - 2\beta^6c_2^4c_3^4r_{00} + 2\beta^6c_2^4c_3^4nr_{00}, \tag{3.14}$$

$$\begin{aligned} g_2 :=& -14\beta^5c_2^5c_3^3r_{00} + 8\beta^5c_2^5c_3^3r_0A - 10\beta^5c_2^3c_3^4r_{00}c_1 + 14\beta^4c_2^7c_3Anr_{00} \\ &+ 18\beta^4c_2^5c_3^2Anr_{00}c_1 - 2\beta^4c_2^3c_3^4Anr_{00} + 2\beta^5c_2^5c_3^3nr_{00} + 4\beta^5c_2^5c_3^3nr_0A \\ &+ 22\beta^5c_2^3c_3^4nr_{00}c_1 - \beta^4c_2^3c_3^4Ar_{00} - 32\beta^4c_2^3c_3^3Anr_{00}c_1^2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} g_3 :=& 32\beta^3c_2^4Ab^2nr_{00}c_1c_3^3 + 32\beta^4c_2^6Ar_0c_3^2 + 6\beta^3c_2^8Anr_{00} - 8\beta^3c_2^4Ar_{00}c_3^3 \\ &- 22\beta^4c_2^6c_3^2nr_{00} - 2\beta^4c_2^2c_3^4r_{00}c_1^2 - 10\beta^4c_2^4c_3^3r_{00}c_1 - 8\beta^5c_2^4c_3^4nr_0 - 25\beta^4c_2^6c_3^2r_{00} \\ &+ 16\beta^4c_2^6Anr_0c_3^2 + 40\beta^4c_2^4Ar_0c_1c_3^3 - 12\beta^3c_2^4Anr_{00}c_3^3 - 4\beta^3c_2^2Ar_{00}c_1c_3^4 \\ &+ 54\beta^4c_2^2c_3^4nr_{00}c_1^2 + 98\beta^4c_2^4c_3^3nr_{00}c_1 + 48\beta^3c_2^6Anr_{00}c_1c_3 - 12\beta^3c_2^2Anr_{00}c_1c_3^4 \\ &+ 20\beta^4c_2^4Anr_0c_1c_3^3 - 32\beta^3c_2^6Ab^2nr_{00}c_3^2 - 48\beta^3c_2^2Anr_{00}c_1^3c_3^3 - 6\beta^3c_2^4Anr_{00}c_1^2 \\ &+ 2\beta^4c_2^4c_3^4b^2r_{00} - 14\beta^5c_2^4c_3^4r_0 - 2\beta^4c_2^4c_3^4b^2nr_{00}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} g_4 :=& -12\beta^3b^2nr_0c_2^5c_3^3A + 128\beta^2Ab^2nr_{00}c_1^2c_2^3c_3^3 - 72\beta^2Ab^2nr_{00}c_1c_2^5c_3^2 \\ &+ 40\beta^3r_0c_2^7c_3A + 6\beta^3Ar_0c_2^3c_3^4 + 6\beta^3c_2^5b^2r_{00}c_3^3 - 48\beta^3c_2^7nr_{00}c_3 \\ &- 158\beta^3c_2^3r_{00}c_1^2c_3^3 - 270\beta^3c_2^5r_{00}c_1c_3^2 - 56\beta^4c_2^5nr_0c_3^3 - 52\beta^4c_2^3r_0c_1c_3^4 \\ &- 102\beta^4c_2^5r_0c_3^3 - 30\beta^2Ar_{00}c_1c_2^3c_3^3 - 24\beta^2Anr_{00}c_2^5c_3^2 + 168\beta^3r_0c_1c_2^5c_3^2A \\ &+ 80\beta^3r_0c_1^2c_2^3c_3^3A - 24\beta^3b^2r_0c_2^5c_3^3A + 4\beta^3Anr_0c_2^3c_3^4 + 22\beta^2Anr_{00}c_1c_2^7 \\ &+ 22\beta^3c_2^5b^2nr_{00}c_3^3 + 22\beta^3c_2^3b^2r_{00}c_1c_3^4 + 50\beta^3c_2nr_{00}c_1^3c_3^4 + 316\beta^3c_2^3nr_{00}c_1^2c_3^3 \\ &- 32\beta^4c_2^3nr_0c_1c_3^4 + 40\beta^3nr_0c_1^2c_2^3c_3^3A + 84\beta^3nr_0c_1c_2^5c_3^2A - 32\beta^2Anr_{00}c_1^4c_2c_3^3 \\ &+ 54\beta^2Anr_{00}c_1^2c_2^5c_3 - 88\beta^2Anr_{00}c_1c_2^3c_3^3 - 56\beta^2Ab^2nr_{00}c_2^7c_3 \\ &- 50\beta^3c_2^3b^2nr_{00}c_1c_3^4 - 20\beta^2Ar_{00}c_2^5c_3^2 + 18\beta^3c_2r_{00}c_1^3c_3^4 - 10\beta^3c_2^7r_{00}c_3 \\ &+ 20\beta^3nr_0c_2^7c_3A - 15\beta^2Ar_{00}c_1^2c_2c_3^4 + 102\beta^3c_2^5nr_{00}c_1c_3^2, \end{aligned} \tag{3.17}$$

$$\begin{aligned}
g_5 := {} & 18\beta^2 n r_0 c_1^2 c_2^4 c_3^2 A + 40\beta^2 n r_0 c_1^3 c_2^2 c_3^3 A - 22\beta A n r_{00} c_1 c_2^4 c_3^2 + 16\beta A n r_{00} c_1^3 c_2^4 c_3 \\
& - 36\beta A n r_{00} c_1^4 c_2^2 c_3^2 - 12\beta A b^2 r_{00} c_1 c_2^2 c_3^4 - 18\beta A n r_{00} c_1^2 c_2^2 c_3^3 + 8\beta A b^4 n r_{00} c_2^6 c_3^2 \\
& - 48\beta^2 b^2 n r_0 c_2^6 c_3^2 A + 18\beta^2 n r_0 c_1 c_2^6 c_3 A - 10\beta^2 b^2 r_0 c_1 c_2^4 c_3^3 A - 8\beta A b^2 n r_{00} c_2^4 c_3^3 \\
& - 18\beta^2 b^2 n r_{00} c_1^2 c_2^2 c_3^4 - 10\beta^2 b^2 n r_{00} c_1 c_2^4 c_3^3 + 24\beta A b^2 n r_{00} c_1^2 c_2^4 c_3^2 - 4\beta A b^4 n r_{00} c_2^4 \\
& - 60\beta^2 b^2 n r_0 c_1 c_2^4 c_3^3 A + 192\beta A b^2 n r_{00} c_1^3 c_2^2 c_3^3 - 192\beta A b^2 n r_{00} c_1 c_2^6 c_3 \\
& + 8\beta^2 n r_0 c_2^8 A + 52\beta^2 A r_0 c_2^4 c_3^3 - 16\beta A r_{00} c_2^6 c_3 - 12\beta A r_{00} c_1^3 c_3^4 + 2\beta^2 b^4 r_{00} c_2^4 c_3^4 \\
& + 16\beta^2 n r_{00} c_1^4 c_3^4 - 60\beta^2 r_{00} c_1^3 c_2^2 c_3^3 - 612\beta^2 r_{00} c_1^2 c_2^4 c_3^2 - 240\beta^2 r_{00} c_1 c_2^6 c_3 \\
& + 28\beta^3 b^2 r_0 c_2^4 c_3^4 - 72\beta^3 r_0 c_1^2 c_2^2 c_3^4 - 426\beta^3 r_0 c_1 c_2^4 c_3^3 + 16\beta^2 r_0 c_2^8 A - 24\beta^2 n r_{00} c_2^8 \\
& - 272\beta^3 r_0 c_2^6 c_3^2 - 96\beta^2 b^2 r_0 c_2^6 c_3^2 A + 80\beta^2 r_0 c_1^3 c_2^2 c_3^3 A + 360\beta^2 r_0 c_1^2 c_2^4 c_3^2 A \\
& - 16\beta A n r_{00} c_2^6 c_3 - 8\beta A n r_{00} c_1^3 c_3^4 + 564\beta^2 n r_{00} c_1^2 c_2^4 c_3^2 - 36\beta^2 n r_{00} c_1 c_2^6 c_3 \\
& - 248\beta^3 n r_0 c_1 c_2^4 c_3^3 - 2\beta^2 b^4 n r_{00} c_2^4 c_3^4 + 16\beta^3 b^2 n r_0 c_2^4 c_3^4 + 14\beta A b^2 r_{00} c_2^4 c_3^3 \\
& + 12\beta c_2^2 b^2 A n r_{00} c_1 c_3^4 - 45\beta^2 b^2 r_{00} c_2^6 c_3^2 - 144\beta^3 n r_0 c_2^6 c_3^2 + 12\beta^2 r_{00} c_1^4 c_3^4 \\
& + 216\beta^2 r_0 c_1 c_2^6 c_3 A - 48\beta^3 n r_0 c_1^2 c_2^2 c_3^4.
\end{aligned} \tag{3.18}$$

By (3.13), we get

$$g_4\alpha^4 + g_2\alpha^2 = 0, \tag{3.19}$$

$$g_5\alpha^4 + g_3\alpha^2 + g_1 = 0. \tag{3.20}$$

By (3.13) and (3.20), one can see that $-2\beta^5 c_2^4 c_3^2 r_{00}(\beta c_3^2 - \beta c_3^2 n - 4Anc_2^2 + 4Anc_1c_3)$ is not divisible by α^2, which implies that

$$r_{ij} = 0. \tag{3.21}$$

In [3], for each (α, β)-metric, the **S**-curvature is calculated. For the given Finsler metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$, we have

$$\begin{aligned}
\mathbf{S} := {} & \Big\{ \frac{(c_3c_1 - c_2^2)A}{2(c_1 + c_2 s)B} - \frac{f'(b)}{bf(b)} \Big\}(r_0 + s_0) + \frac{1}{8(c_1 + c_2 s)^2 \alpha B^2} \Big\{ (6nAc_1c_2s \\
& - 2ns^2Ac_3c_1 - 2ns^3Ac_3c_2 - 4nb^2c_2^2c_1s - 2nb^2c_3sc_1^2 - 2nb^2c_3s^3c_2^2 \\
& + 4ns^4c_3c_1c_2 - 4nAc_2^2s^2 - 2nb^2c_2c_1^2 - 2nb^2c_2^3s^2 + 2ns^2c_2c_1^2 + 4ns^3c_2^2c_1 \\
& + 2ns^3c_3c_1^2 + 2ns^5c_3c_2^2 - 3Ac_1c_2s - 3s^2Ac_3c_1 - s^3Ac_3c_2 + 2Ab^2c_3c_1 \\
& - 2Ab^2c_2^2 + 2ns^4c_2^3 - 4nb^2c_3s^2c_1c_2 - Ac_1^2)c_2(c_1 + 2c_2s + c_3s^2)^{\frac{3}{2}} \Big\}(r_{00} \\
& - \frac{2(c_2 + c_3 s)}{c_1 + c_2 s} s_0),
\end{aligned} \tag{3.22}$$

where

$$A := \sqrt{c_1 + 2c_2 s + c_3 s^2},$$
$$B := (Ac_1 + 2Ac_2 s + s^2 Ac_3 + b^2 c_2 c_1 + b^2 c_2^2 s + b^2 c_3 s c_1 + b^2 c_3 s^2 c_2 - s^2 c_2 c_1 - s^3 c_2^2 - s^3 c_3 c_1 - s^4 c_3 c_2),$$
$$f(b) := \frac{\int_0^\pi sin^{n-2} t T(bcost)dt}{\int_0^\pi sin^{n-2} tdt},$$
$$T := (c_1 + 2c_2 s + c_3 s^2)(\sqrt{c_1 + 2c_2 s + c_3 s^2})^{n-2}.$$

By putting (3.12) and (3.21) into (3.22) , we obtain

$$\mathbf{S} = 0. \tag{3.23}$$

This completes the proof. □

Proof of Theorem 1.1: in [6] Najafi-Tayebi showed that every weakly Landsberg (α, β)-metric with vanishing S- curvature on a manifold M of dimension $n \geq 3$ is a Berwald metric. By theorem (3.3), every weakly Landsberg (α, β)-metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ on M of dimension $n \geq 3$ is a Berwald metric.

Now, we consider the class $(\alpha, \beta)-$ metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ of dimension $n = 2$. We know that Every 2-dimensional Finsler manifold is C-reducible

$$C_{ijk} = \frac{1}{3}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\}. \tag{3.24}$$

By using

$$\mathbf{J}_k = I_{k|m}y^m, \qquad \mathbf{S}_{ij} = E_{ij}, \tag{3.25}$$

and by deriving of (3.24) yields

$$L_{ijk} = \frac{1}{3}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\}. \tag{3.26}$$

By putting $\mathbf{J} = 0$ in (3.26) implies that $\mathbf{L} = 0$. On the other hand, the Berwald curvature Finsler manifold of dimensional $n = 2$ can be written as follows

$$B^i{}_{jkl} = -\frac{2}{F}L_{jkl}l^i + \frac{2}{3}\Big\{E_{jk}h^i_l + E_{kl}h^i_j + E_{lj}h^i_k\Big\}. \tag{3.27}$$

By Putting $\mathbf{L} = 0$ and $\mathbf{E} = 0$ in (3.27), we conclude that F is a Berwald metric. The proof is complete. □

4. **Proof of Theorem 1.2**

Theorem 4.1. *Let $F = c_1\alpha+c_2\beta+c_3\beta^2/\alpha$ be a weakly Landsberg (α,β)-metric. Then F has vanishing S-curvature.*

Proof. Suppose F is weakly Landsberg, that is

$$\mathbf{J} = 0. \tag{4.1}$$

By Lemmas 3.1, we calculate $J = J_i b^i = 0$ which give us

$$f_5\alpha^5 + f_4\alpha^4 + f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0 = 0, \tag{4.2}$$

where

$$\begin{aligned}
f_5 = &-448\beta s_0 c_1 c_3^2 + 33c_2c_3r_{00} + 192\beta b^2 r_0 c_3^3 + 96c_3^3\beta b^2 nr_0 - 448c_3^2 nc_1 r_{00} b^2 \\
&+ 32c_3 nc_1^2 r_{00} - 66c_3 nc_2 r_{00} - 464\beta ns_0 c_1 c_2^3 + 640c_3^3\beta ns_0 b^2 + 16\beta s_0 c_2^2 c_3 \\
&- 32c_3\beta nr_0 c_2^2 + 18c_3^3\beta b^4 ns_0 c_2 + 320\beta r_0 c_1 c_3^2 - 160\beta b^2 s_0 c_3^3 + 28c_3 nc_2^2 r_{00} b^2 \\
&+ 64c_3^2\beta ns_0 c_1 - 352c_3\beta b^2 ns_0 c_2^3 + 3040c_3\beta ns_0 c_1^2 c_2 - 288c_3^3 b^4 nr_{00} \\
&+ 5888c_3^2\beta b^2 ns_0 c_1 c_2 - 64\beta r_0 c_2^2 c_3 + 160c_3^2\beta nr_0 c_1 + 26nc_1 c_2^2 r_{00},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
f_4 = &\,2560c_3^3\beta^2 b^2 ns_0 c_1 + 312c_3\beta nc_1 c_2 r_{00} + 12\beta^2 ns_0 c_2^4 + 1280c_3^2\beta^2 ns_0 c_1^2 \\
&+ 400c_3^2\beta nc_2 r_{00} b^2 - 24c_3^2\beta nr_{00} - 56c_3^2\beta^2 nr_0 c_2 - 288c_3^2\beta^2 ns_0 c_2 \\
&+ 768c_3^4\beta^2 b^4 ns_0 - 12\beta nc_2^3 r_{00} - 960c_3^2\beta^2 b^2 ns_0 c_2^2 - 1248c_3\beta^2 ns_0 c_1 c_2^2 \\
&+ 88\beta^2 s_0 c_2 c_3^2 - 112\beta^2 r_0 c_2 c_3^2,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
f_3 = &-62c_3\beta^2 nc_2^2 r_{00} - 1152c_3^3\beta^3 b^2 ns_0 c_2 - 1472c_3^2\beta^3 ns_0 c_1 c_2 + 88c_3\beta^3 ns_0 c_2^3 \\
&- 128c_3^3\beta^3 ns_0 + 112c_3^2\beta^2 nc_1 r_{00} + 64\beta^3 s_0 c_3^3 - 64\beta^3 r_0 c_3^3 - 32c_3^3\beta^3 nr_0 \\
&+ 192c_3^3\beta^2 nr_{00} b^2,
\end{aligned} \tag{4.5}$$

$$f_2 = -640c_3^3\beta^4 ns_0 c_1 + 240c_3^2\beta^4 ns_0 c_2^2 - 100c_3^2\beta^3 nc_2 r_{00} - 512c_3^4\beta^4 b^2 ns_0 \tag{4.6}$$

$$f_1 = -48c_3^3\beta^4 nr_{00} + 288c_3^3\beta^5 ns_0 c_2, \tag{4.7}$$

$$f_0 = +128c_3^4\beta^6 ns_0. \tag{4.8}$$

By (4.2), we get

$$f_5\alpha^4 + f_3\alpha^2 + f_1 = 0, \tag{4.9}$$

$$f_4\alpha^4 + f_2\alpha^2 + f_0 = 0. \tag{4.10}$$

By (4.8) and (4.10), observe that $128c_3^4\beta^6 ns_0$ is not divisible by α^2, which implies that

$$s_i = 0. \tag{4.11}$$

By putting $s_i = 0$ into (4.2), we obtain

$$g_5\alpha^5 + g_4\alpha^4 + g_3\alpha^3 + g_2\alpha^2 + g_1\alpha = 0, \tag{4.12}$$

where

$$g_5 = -96c_3^3\beta b^2nr_0 - 33c_2c_3r_{00} + 64\beta r_0c_2^2c_3 - 248c_3nc_2^2r_{00}b^2 - 320\beta r_0c_1c_3^2$$
$$- 216nc_1c_2^2r_{00} - 192\beta b^2r_0c_3^3 + 288c_3^3b^4nr_{00} - 160c_3^2\beta nr_0c_1 + 32c_3\beta nr_0c_2^2$$
$$+ 66c_3nc_2r_{00} + 448c_3^2nc_1r_{00}b^2 - 32c_3nc_1^2r_{00}, \tag{4.13}$$
$$g_4 = -312c_3\beta nc_1c_2r_{00} + 56c_3^2\beta^2nr_0c_2 + 112\beta^2r_0c_2c_3^2 + 24c_3^2\beta nr_{00} + 12\beta nc_2^3r_{00}$$
$$- 400c_3^2\beta nc_2r_{00}b^2, \tag{4.14}$$
$$g_3 = 64\beta^3r_0c_3^3 - 192c_3^3\beta^2nr_{00}b^2 + 32c_3^3\beta^3nr_0 + 62c_3\beta^2nc_2^2r_{00} - 112c_3^2\beta^2nc_1r_{00}, \tag{4.15}$$
$$g_2 = 100c_3^2\beta^3nc_2r_{00} \tag{4.16}$$
$$g_1 = +48c_3^3\beta^4nr_{00}. \tag{4.17}$$

By (4.12), we get

$$g_4\alpha^4 + g_2\alpha^2 = 0, \tag{4.18}$$
$$g_5\alpha^4 + g_3\alpha^2 + g_1 = 0. \tag{4.19}$$

By (4.17) and (4.19), one can deduce that $48c_3^3\beta^4nr_{00}$ is not divisible by α^2. This implies that

$$r_{ij} = 0. \tag{4.20}$$

On the other hand, the **S**-curvature of $F = c_1\alpha + c_2\beta + c_3\frac{\beta^2}{\alpha}$ is given by

$$\mathbf{S} := \Big\{\frac{c_3(c_1 + c_2s + c_3s^2)}{A} - \frac{f'(b)}{bf(b)}\Big\}(r_0 + s_0) + \frac{1}{8\alpha A^2}\Big\{10nb^2c_2^2c_1^2c_3s^2$$
$$- 14nb^2c_2^2c_1c_3^2s^4 + 28nb^2c_2c_1^2c_3^2s^3 - 44nb^2c_2c_1c_3^3s^5 - 4nb^2c_3sc_1^3c_2 - c_2c_1^3$$
$$+ 6nc_1^2c_2c_3s^2 + 10nc_1c_2c_3^2s^4 + 6nb^2c_2^2c_3^3s^6 + 20nb^2c_2c_3^4s^7 + 16nb^2c_3^3s^4c_1^2$$
$$- 32nb^2c_3^4s^6c_1 - 10ns^4c_2^2c_1^2c_3 + 14ns^6c_2^2c_1c_3^2 - 28ns^5c_2c_1^2c_3^2 + 44ns^7c_2c_1c_3^3$$
$$+ 4ns^3c_3c_1^3c_2 + 8nc_2^2s^3c_1c_3 - 20b^2c_1c_2c_3^2s^2 - 2c_3b^2c_2^2sc_1 + 5c_2c_1^2c_3s^2$$
$$+ 8nc_1^2c_3^2s^3 - 14nc_3^3s^6c_2 - 2nc_2^2sc_1^2 - 6nc_2^2s^5c_3^2 - 2nb^2c_2^2c_1^3 + 16nb^2c_3^5s^8$$
$$+ 2ns^2c_2^2c_1^3 - 6ns^8c_2^2c_3^3 - 20ns^9c_2c_3^4 - 16ns^6c_3^3c_1^2 + 32ns^8c_3^4c_1 + 25c_1c_2c_3^2s^4$$
$$+ 6c_2^2s^3c_1c_3 - 12b^2c_1^2c_3^2s - 16b^2c_1c_3^3s^3 - 2c_3b^2c_1^2c_2 - 10b^2c_2s^4c_3^3 - 6b^2c_2^2s^3c_3^2$$
$$+ 16c_3^3s^5c_1 - 2nc_1^3c_2 - 8nc_3^4s^7 - 16ns^10c_3^5 + 3c_3^3s^6c_2 + 3c_2^2s^5c_3^2$$
$$- 4b^2c_3^4s^5\Big\}(r_{00} - \frac{2(c_2 + 2c_3s)}{c_3s^2 - c_1}s_0), \tag{4.21}$$

where

$$A = (2s^3c_3c_1 - c_1 - c_3s^2 - c_2s - b^2c_2c_1 + b^2c_2c_3s^2 - 2b^2c_3sc_1 + 2b^2c_3^2s^3 + s^2c_2c_1 - s^4c_2c_3 - 2s^5c_3^2)(-c_1 + c_3s^2),$$

$$f(b) = \frac{\int_0^\pi sin^{n-2}tT(bcost)dt}{\int_0^\pi sin^{n-2}tdt},$$

$$T = (c_1 + C_2s + c_3s^2)^2(c_1 + c_2s + c_3s^2)^{n-2}.$$

By putting (4.11) and (4.20) into (4.21) , we obtain $\mathbf{S} = 0$. □

Proof of Theorem 1.2: By the same method used in Theorem (1.1), one can prove Theorem (1.2). □

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