

## On class of Square Finsler Metrics

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**Abstract.** In this paper, we remark some of the well-known curvature properties of square Finsler metrics. Then, we study weakly stretch square Finsler metrics.

**Keywords:** Square metric, stretch curvature, mean stretch curvature.

### 1. Introduction

The well-known Hilbert's Fourth Problem is to characterize the distance functions on an open subset in  $\mathbb{R}^n$  such that straight lines are shortest paths. It turns out that there are lots of solutions to the problem. For example, in [4], Blaschke discusses 2-dimensional solutions to the problem. Then, Ambartzumian [1] and Alexander [2] independently give all 2-dimensional solutions. In [8], Pogorelov discusses smooth solutions in 3-dimensional case. Then, Szabó investigates several problems left by Pogorelov and constructs continuous solutions to the problem in high dimensions [12]. See [5] on related issue.

The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in  $\mathbb{R}^n$  whose geodesics are straight lines. Such Finsler metrics are called projectively flat Finsler metrics or projective Finsler metrics. Hamel first characterizes projective Finsler metrics by a system of PDE's [6]. Then, Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [9].

For an  $n$ -dimensional Finsler manifold  $(M, F)$ , a global vector field  $\mathbf{G}$  is induced by  $F$  on  $TM_0 := TM - \{0\}$ , which in a standard coordinates  $(x^i, y^i)$

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for  $TM_0$  is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

where  $G^i = G^i(x, y)$  are called spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}. \quad (1.1)$$

$\mathbf{G}$  is called the spray associated to  $F$ .  $F$  is projectively flat if only if there exists scalar homogeneous function  $P : TU \rightarrow \mathbb{R}$  such that the its spray coefficients satisfy

$$G^i(x, y) = P(x, y) y^i. \quad (1.2)$$

In this case,  $P = P(x, y)$  is called the projective factor.

In Finsler Geometry, there is an interesting class of projectively flat metrics on the unit ball  $\mathbb{B}^n$  which is given by

$$F = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y \rangle^2} + \langle x, y \rangle)^2}{(1-|x|^2)^2 \sqrt{(1-|x|^2)|y|^2 + \langle x, y \rangle^2}}. \quad (1.3)$$

This class of metrics is called square metrics which can be expressed as

$$F = \frac{(\alpha + \beta)^2}{\alpha}, \quad (1.4)$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y \rangle^2}}{(1-|x|^2)^2} \quad \beta = \frac{\langle x, y \rangle}{(1-|x|^2)^2}. \quad (1.5)$$

That  $\alpha$  is a Riemannian metric and  $\beta$  is a 1-form with  $\|\beta\|_\alpha < 1$ . L. Berwald first constructed a special projectively flat square metric of zero flag curvature on the unit ball in  $\mathbb{R}^n$  (see [3]).

Then the flag curvature of  $F$  is a function  $\mathbf{K} = \mathbf{K}(P, y)$  of tangent planes  $P \subset T_x M$  and directions  $y \in P$ .  $F$  is called of scalar curvature if the flag curvature  $\mathbf{K} = \mathbf{K}(x, y)$  is a scalar function on the slit tangent bundle  $TM_0$ , for any  $y \in T_x M$ . Recently, Shen-Yildirim determine the local structure of all locally projectively flat square metrics of constant flag curvature. Later on, L.Zhou shows that a square metric of constant flag curvature must be locally projectively flat. In [10], Shen-Yang proved the following.

**Theorem A.** ([10]) Let  $F = (\alpha + \beta)^2/\alpha$  be a square metric on a  $(n \geq 3)$ -dimensional manifold  $M$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is Riemannian and  $\beta = b_i(x)y^i$  is a 1-form on  $M$ . Then  $F$  is of scalar flag curvature if and only if it is locally projectively flat.

A Finsler metric  $F = F(x, y)$  on a manifold  $M$  is said to be locally dually flat if at any point there is a coordinate system  $(x^i)$  in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j},$$

where  $H = H(x, y)$  is a  $C^\infty$  scalar function on  $TM_0 = TM \setminus \{0\}$  satisfying  $H(x, \lambda y) = \lambda^3 H(x, y)$  for all  $\lambda > 0$ .

**Theorem B.** ([7]) Let  $F = (\alpha + \beta)^2/\alpha$  be a square metric on an open subset  $U \subseteq \mathbb{R}^n$  with  $n \geq 3$ . Then  $F$  is dually flat if and only if one of the following holds:

- (i)  $F$  is a dually flat Riemannian metric.
- (ii)  $F$  is of Minkowski-type. Moreover,  $F$  can be expressed in the following form.

$$F = \frac{(|y| + \langle v, y \rangle)^2}{|y|}, \quad (1.6)$$

where  $v \in \mathbb{R}^n$  is a non zero constant vector.

Let  $(M, F)$  be an  $n$ -dimensional manifold Finsler manifold. Then  $F$  is called an Einstein metric if its Ricci curvature  $Ric$  is isotropic,

$$\mathbf{Ric} = (n - 1)\lambda F^2,$$

where  $\lambda = \lambda(x)$  is a scalar function on  $M$ . In [11], Shen-Yu proved the following.

**Theorem C.** ([11]) Let  $F = (\alpha + \beta)^2/\alpha$  be a square metric on a  $n$ -dimensional manifold  $M$ , Then  $F$  is an Einstein metric if and only if it is Ricci flat and

$$\begin{aligned} {}^\alpha Ric &= k^2(1 - b^2)^2 - [5(n - 1) + 2(2n - 5)b^2]\alpha^2 + 6(n - 2)\beta^2, \\ b_{i|j} &= k(1 - b^2)(1 + 2b^2)a_{ij} - 3b_i b_j. \end{aligned} \quad (1.7)$$

Then, they determined the local structure of Einstein square metrics as follows.

**Theorem D.** ([11]) Let  $F = (\alpha + \beta)^2/\alpha$  be a square metric on a  $n$ -dimensional manifold  $M$ . Then the following are equivalent:

- (1)  $F$  is an Einstein metric.
- (2) The Riemannian metric  $\tilde{\alpha} := (1 - b^2)\alpha$  and the 1-form  $\tilde{\beta} := \sqrt{1 - b^2}\beta$  satisfy

$${}^{\tilde{\alpha}} Ric = -(n - 1)k^2\tilde{\alpha}, \quad \tilde{b}_{i|j} = k\sqrt{1 + \tilde{b}^2}\tilde{a}_{ij}, \quad (1.8)$$

where  $k$  is a constant number,  $\tilde{b} = \|\tilde{\beta}\|_{\tilde{\alpha}}$  and  $\tilde{b}_{i|j}$  is the covariant derivation of  $\tilde{\beta}$  with respect to  $\tilde{\alpha}$ . In this case,  $F$  is given in the following form

$$F = \frac{(\sqrt{1 + \tilde{b}^2} \tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}} \quad (1.9)$$

with  $(1 + \tilde{b}^2)(1 - b^2) = 1$ .

(3) The Riemannian metric  $\bar{\alpha} := (1 - b^2)^{\frac{3}{2}} \sqrt{\alpha^2 - \beta^2}$  and the 1-form  $\bar{\beta} := (1 - b^2)\beta$  satisfy  $\bar{\alpha}_{Ric} = 0$  and  $\bar{b}_{i|j} = k\bar{a}_{ij}$  where  $k$  is a constant number,  $\bar{b} = \|\bar{\beta}\|_{\bar{\alpha}}$  and  $\bar{b}_{i|j}$  is the covariant derivation of  $\bar{\beta}$  with respect to  $\bar{\alpha}$ . In this case,  $F$  is given in the following form

$$F = \frac{(\sqrt{(1 - \bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2} + \bar{\beta})^2}{(1 - \bar{b}^2)^2 \sqrt{(1 - \bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2}}, \quad (1.10)$$

with  $\bar{b} = b$ .

Also, they provide a new description for square metrics with constant flag curvature.

**Theorem E.** ([11]) The Finsler metric  $F = (\alpha + \beta)^2/\alpha$  is of constant flag curvature if and only if under the expression (1.10) of  $F$ ,  $\bar{\alpha}$  is locally Euclidean,  $\bar{\beta}$  is closed and s homothety with respect to  $\bar{\alpha}$ . In a suitable local coordinate,  $F$  can be expressed by

$$F = \frac{(\sqrt{(1 - |\bar{x}|^2)|y|^2 + \langle \bar{x}, y \rangle^2} + \langle \bar{x}, y \rangle)^2}{(1 - |\bar{x}|^2)^2 \sqrt{(1 - |\bar{x}|^2)|y|^2 + \langle \bar{x}, y \rangle^2}}, \quad (1.11)$$

where  $\bar{x} := cx + a$  for some constant number  $c$  and constant vector  $a$ . In particular,  $F$  must be locally projectively flat with zero flag curvature.

For  $y \in T_x M$ , define the Landsberg curvature  $\mathbf{L}_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$  by

$$\mathbf{L}_y(u, v, w) := \frac{-1}{2} g_y(\mathbf{B}_y(u, v, w), y).$$

A Finsler metric  $F$  is called a Landsberg metric if  $\mathbf{L}_y = 0$ .

For  $y \in T_x M$ , define  $J_y : T_x M \rightarrow \mathbb{R}$  by  $J_y(u) := J_i(y)u^i$ .  $J$  is called the mean Landsberg curvature. A Finsler metric  $F$  is called a weakly Landsberg metric if  $J_y = 0$ . For  $y \in T_x M$ , define the stretch curvature  $\Sigma_y : T_x M \otimes T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ , by  $\Sigma_y(u, v, w, z) := \Sigma_{ijkl}(y)u^i v^j w^k z^l$ , where

$$\Sigma_{ijkl} := 2(L_{ij|k|l} - L_{ijl|k}), \quad (1.12)$$

and  $'|'$  denotes the horizontal derivation with respect to the Berwald connection of  $F$ . A Finsler metric  $F$  is said to be a stretch metric if  $\Sigma = 0$ . Also, one can define Mean stretch curvature  $\bar{\Sigma}_y : T_x M \rightarrow \mathbb{R}$  by  $\bar{\Sigma}_y(u, v) := \bar{\Sigma}_{ij}(y)u^i v^j$ , where

$$\bar{\Sigma}_{ij} := 2(J_{i|j} - J_{j|i}). \quad (1.13)$$

A Finsler metric  $F$  is said to be weakly stretch metric if  $\bar{\Sigma} = 0$ . It is easy to see that every Landsberg metric or stretch metric is a weakly stretch metric.

For an  $(\alpha, \beta)$ -metric, let us define  $b_{i|j}$  by  $b_{i|j}\theta^j := db_i - b_j\theta_i^j$ , where  $\theta^i := dx^i$  and  $\theta_i^j := \Gamma_{ik}^j dx^k$  denote the Levi-Civita connection form of  $\alpha$ . Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_{00} := r_{ij}y^i y^j.$$

In this paper, we prove the following.

**Theorem 1.1.** *Every weakly stretch square Finsler metric  $F = (\alpha + \beta)^2/\alpha$  satisfies following*

$$A_1 r_{00}^2 + B_1 r_{00} + C_1 = 0, \quad (1.14)$$

$$A_2 r_{00}^2 + B_2 r_{00} + C_2 = 0, \quad (1.15)$$

where  $A_1, A_2, B_1, B_2, C_1, C_2$  are functions in terms of  $b^2$  and  $s$ , and are given in Appendix.

## 2. Proof of Theorem 1.1

The mean Landsberg curvature of  $(\alpha, \beta)$  metrics is as follows:

$$\begin{aligned} J_i &= \frac{-1}{2\Delta\alpha^4} \left[ \frac{2\alpha^2}{b^2 - s^2} (r_0 + s_0) h_i \left( \frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right) \right. \\ &\quad + \frac{\alpha}{b^2 - s^2} \left( \Psi_1 + s \frac{\Phi}{\Delta} \right) (r_{00} - 2\alpha Q s_0) h_i \\ &\quad + \frac{\alpha\Phi}{\Delta} \left( -\alpha Q' s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0) \right. \\ &\quad \left. \left. + \alpha^2 \Delta s_{i0} + \alpha^2 (r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0) y_i \right) \right]. \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} \Psi_1 &= \sqrt{b^2 - s^2} \Delta^{\frac{1}{2}} \left[ \frac{\sqrt{b^2 - s^2} \Phi}{\Delta^{\frac{3}{2}}} \right]', \\ \Phi &= -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \\ Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Delta &= 1 + sQ + (b^2 - s^2)Q', \\ h_i &= b_i - \alpha^{-1} s y_i \end{aligned}$$

For a square metric  $F = \alpha\phi(s)$ , we have  $\phi(s) = (1 + s)^2$ . Then

$$Q = \frac{2}{1-s} \quad (2.2)$$

$$\Delta = \frac{1-s^2+2(b^2-s^2)}{(1-s)^2} \quad (2.3)$$

$$\Phi = (n+1)\left(\frac{4s^2+2s-2}{(1-s)^3}\right) - 4(b^2-s^2)\left(\frac{n(1-2s)+s+1}{(1-s)^4}\right) \quad (2.4)$$

$$\begin{aligned} \Psi_1 = & \frac{-s\Phi}{\Delta} + \frac{9s(b^2-s^2)}{(1-s)^2} \frac{\Phi}{\Delta^2} - \frac{4(b^2-s^2)^2(2-n)}{(1-s)^5\Delta} \\ & + \frac{(b^2-s^2)-24ns^2+14ns+14s+2n+2}{\Delta} \frac{1}{(1-s)^4}. \end{aligned} \quad (2.5)$$

By substituting (2.2), (2.3), (2.4) and (2.5) in (2.1) we have:

$$\begin{aligned} J_i = & (r_0 + s_0)h_iK_1 + \left((1-s)r_{00} - 4\alpha s_0\right)h_iK_2 \\ & + \left[-2\alpha s_0h_i + 2\alpha(1-s)(\alpha^2s_i - y_i s_0) + \alpha^2(1-s^2 + 2(b^2-s^2))s_{i0}\right. \\ & \left. + \alpha^2(1-s)((1-s)r_{i0} - 4\alpha s_i) - (1-s)((1-s)r_{00} - 4\alpha s_0)y_i\right]K_3, \end{aligned} \quad (2.6)$$

where

$$K_1 = \frac{12s}{\alpha^2(1-s^2+2(b^2-s^2))^2},$$

$$K_2 = \frac{M_2}{2\alpha^3(1-s)(1-s^2+2(b^2-s^2))^3},$$

$$\begin{aligned} M_2 = & 2(b^2-s^2)\left(6(2n-3)s^3 - 2(5n-1)s^2 + 6(n-1)s - 4nb^2 - 2(2n-2)\right) \\ & - 12(n+3)s^5 + 8(2n+5)s^4 + 6(n+5)s^3 - 2(7n+19)s^2 + 2(n+1)(3s-1), \end{aligned}$$

$$K_3 = \frac{M_3}{2\alpha^3(1-s)^2(1-s^2+2(b^2-s^2))^2},$$

$$M_3 = 4(b^2-s^2)\left(n(1-2s)+s+1\right) + 2(n+1)\left(2s^3-s^2-2s+1\right).$$

And the geodesic spray coefficients of square Finsler metric  $F = \alpha\phi(s)$  is as follows

$$G^i = G_\alpha^i + Py^i + Q^i \quad (2.7)$$

where  $G_\alpha^i$  is the spray coefficient of Reimannian metric, and  $P$  and  $Q^i$  are given by

$$P = \frac{1-2s}{\alpha(1+2b^2-3s^2)}\left(r_{00} - \frac{4\alpha}{1-s}s_0\right)y^i, \quad (2.8)$$

$$Q^i = \frac{b^i}{1+2b^2-3s^2}\left(r_{00} - \frac{4\alpha}{1-s}s_0\right) + \frac{2\alpha}{1-s}s_0^i. \quad (2.9)$$

On the other hand we have:

$$J_{i;k}y^k = J_{i|k}y^k - 2\frac{\partial J_i}{\partial y^m}\left(G^m - \bar{G}^m\right) - J_m\left(N_i^m - \bar{N}_i^m\right), \quad (2.10)$$

where

$$G^m - \bar{G}^m = Py^m + Q^m, \quad (2.11)$$

$$N_i^m - \bar{N}_i^m = P_iy^m + P\delta_i^m + Q_i^m. \quad (2.12)$$

First, we calculate the equation  $J_{i|k}y^k$

$$\begin{aligned} J_{i|k}y^k &= \left((r_0 + s_0)K_1h_i\right)_{|k}y^k + \left(\left((1-s)r_{00} - 4\alpha s_0\right)K_2h_i\right)_{|k}y^k \\ &+ \left[\left(-2\alpha s_0h_i + 2\alpha(1-s)(\alpha^2s_i - y_is_0) + \alpha^2(1-s^2 + 2(b^2 - s^2))s_{i0}\right.\right. \\ &\left.\left.+ \alpha^2(1-s)\left((1-s)r_{i0} - 4\alpha s_i\right) - (1-s)\left((1-s)r_{00} - 4\alpha s_0\right)y_i\right)K_3\right]_{|k}y^k. \end{aligned} \quad (2.13)$$

Let us put

$$\begin{aligned} F_i &:= \left(-2\alpha s_0h_i + 2\alpha(1-s)(\alpha^2s_i - y_is_0) + \alpha^2(1-s^2 + 2(b^2 - s^2))s_{i0}\right. \\ &\left.+ \alpha^2(1-s)\left((1-s)r_{i0} - 4\alpha s_i\right) - (1-s)\left((1-s)r_{00} - 4\alpha s_0\right)y_i\right)K_3. \end{aligned}$$

Then we get

$$J_{i|k}y^k = \left((r_0 + s_0)K_1h_i\right)_{|k}y^k + \left(\left((1-s)r_{00} - 4\alpha s_0\right)K_2h_i\right)_{|k}y^k + F_{i|k}y^k \quad (2.14)$$

and

$$\begin{aligned} \left((r_0 + s_0)K_1h_i\right)_{|k}y^k &= K_1\left\{(r_0 + s_0)\left(\left[\frac{b_{0|0}}{\alpha s} - \frac{2}{\alpha(1-s^2 + 2(b^2 - s^2))}\right.\right.\right. \\ &\left.\left.\left.(2\alpha a^{tj}b_{j|0}b_t - 6sb_{0|0} + 2\alpha b^tb_{t|0})\right]h_i + b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}\right) + (r_{0|0} + s_{0|0})h_i\right\}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \left(\left((1-s)r_{00} - 4\alpha s_0\right)K_2h_i\right)_{|k}y^k &= K_2\left[\left(\frac{-1}{\alpha}b_{0|0}r_{00} + (1-s)r_{00|0} - 4\alpha s_{0|0}\right)h_i\right. \\ &+ \left.\left((1-s)r_{00} - 4\alpha s_0\right)\left(b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}\right) + \left((1-s)r_{00} - 4\alpha s_0\right)\left(\frac{T_2}{M_2} + \frac{1}{1-s}\right.\right. \\ &\left.\left.+ \frac{3(6s - \frac{4bb_{10}}{s_{10}})}{1-s^2 + 2(b^2 - s^2)}\right)s_{10}h_i\right], \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} T_2 = & \left( \frac{4bb_{|0}}{s_{|0}} - 4s \right) \left( 6(2n-3)s^3 - 2(5n-1)s^2 + 6(n-1)s - 4nb^2 + 2(2n-2) \right) \\ & + 2(b^2 - s^2) \left( 18(2n-3)s^2 - 4(5n-1)s + 6(n-1) - \frac{8(n-2)bb_{|0}}{s_{|0}} \right) \\ & - 60(n+3)s^4 - 32(2n+5)s^3 + 18(n+5)s^2 - 4(7n+19)s + 6(n+1), \end{aligned}$$

and

$$\begin{aligned} F_{i|k}y^k = & K_3 \left[ -2\alpha s_0(b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}) - 2b_{0|0}(\alpha^2 s_i - y_i s_0) + 2\alpha(1-s)(\alpha^2 s_{i|0} - y_i s_{0|0}) \right. \\ & + \alpha^2 \left( -6s \frac{b_{0|0}}{\alpha} + 4bb_{|0} \right) s_{i0} + \alpha^2(1-s^2 + 2(b^2 - s^2))s_{i0|0} - \alpha b_{0|0}((1-s)r_{i0} - 4\alpha s_i) \\ & + \alpha^2(1-s) \left( \frac{-b_{0|0}}{\alpha} r_{i0} + (1-s)r_{i0|0} - 4\alpha s_{i|0} \right) - 2\alpha s_{0|0} h_i \\ & - y_i \left( \frac{-b_{0|0}}{\alpha} ((1-s)r_{00} - 4\alpha s_0) + (1-s) \left( \frac{-b_{0|0}}{\alpha} r_{00} + (1-s)r_{00|0} - 4\alpha s_{0|0} \right) \right) \\ & + \left( \frac{T_3}{M_3} + \frac{2}{1-s} + \frac{12s - 8 \frac{bb_{|0}}{s_{|0}}}{1-s^2 + 2(b^2 - s^2)} \right) \left( -2\alpha s_0 h_i + 2\alpha(1-s)(\alpha^2 s_i - y_i s_0) \right. \\ & + \alpha^2(1-s^2 + 2(b^2 - s^2))s_{i0} + \alpha^2(1-s)((1-s)r_{i0} - 4\alpha s_i) \\ & \left. - (1-s)((1-s)r_{00} - 4\alpha s_0)y_i \right) s_{|0} \Big]. \quad (2.17) \end{aligned}$$

Here, we define

$$T_3 := 8 \left( \frac{bb_{|0}}{s_{|0}} - s \right) \left( n(1-2s) + s + 1 \right) + 4(b^2 - s^2)(-2n+1) - (n+1)(12s^2 + 4s + 4).$$

By putting (2.15), (2.16) and (2.17) in (2.14) we have

$$\begin{aligned} J_{i|k}y^k = & K_1 \left[ (r_{0|0} + s_{0|0})h_i + (r_0 + s_0) \left( \left( \frac{b_{0|0}}{\alpha s} - \frac{2}{\alpha(1-s^2 + 2(b^2 - s^2))} \right) (-6sb_{0|0} \right. \right. \\ & \left. \left. + 2\alpha a^{tj} b_{j|0} b_t + 2\alpha b^t b_{t|0} \right) h_i + b_{i|0} - \frac{y_i}{\alpha^2} b_{0|0} \right] + K_2 \left[ \left( \frac{-1}{\alpha} b_{0|0} r_{00} + (1-s)r_{00|0} \right. \right. \\ & \left. \left. - 4\alpha s_{0|0} \right) h_i + ((1-s)r_{00} - 4\alpha s_0) \left( b_{i|0} - \frac{y_i}{\alpha^2} b_{0|0} \right) \right. \\ & \left. + ((1-s)r_{00} - 4\alpha s_0) \left( \frac{T_2}{M_2} + \frac{1}{1-s} + \frac{3(6s - \frac{4bb_{|0}}{s_{|0}})}{1-s^2 + 2(b^2 - s^2)} \right) s_{|0} h_i \right] \end{aligned}$$



$$\begin{aligned}
& + K_3 \left[ -2\alpha s_{0|0} h_i - 2\alpha s_0 (b_{i|0} - \frac{y_i}{\alpha^2} b_{0|0}) - 2b_{0|0} (\alpha^2 s_i - y_i s_0) + 2\alpha (1-s) (\alpha^2 s_{i|0} - y_i s_{0|0}) \right. \\
& + \alpha^2 (-6s \frac{b_{0|0}}{\alpha} + 4bb_{|0}) s_{i0} + \alpha^2 (1-s^2 + 2(b^2 - s^2)) s_{i0|0} - \alpha b_{0|0} ((1-s)r_{i0} - 4\alpha s_i) \\
& + \alpha^2 (1-s) \left( \frac{-b_{0|0}}{\alpha} r_{i0} + (1-s)r_{i0|0} - 4\alpha s_{i|0} \right) \\
& - y_i \left( \frac{-b_{0|0}}{\alpha} ((1-s)r_{00} - 4\alpha s_0) + (1-s) \left( \frac{-b_{0|0}}{\alpha} r_{00} + (1-s)r_{00|0} - 4\alpha s_{0|0} \right) \right) \\
& + \left( \frac{T_3}{M_3} + \frac{2}{1-s} + \frac{12s - 8\frac{bb_{|0}}{s_{|0}}}{1-s^2 + 2(b^2 - s^2)} \right) \left( -2\alpha s_0 h_i + 2\alpha (1-s) (\alpha^2 s_i - y_i s_0) \right. \\
& + \alpha^2 (1-s^2 + 2(b^2 - s^2)) s_{i0} + \alpha^2 (1-s) ((1-s)r_{i0} - 4\alpha s_i) \\
& \left. - (1-s) ((1-s)r_{00} - 4\alpha s_0) y_i \right) s_{|0} \left. \right]. \tag{2.18}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
2 \frac{\partial J_i}{\partial y^m} (G^m - \bar{G}^m) & = \left( \frac{2(1-2s)}{\alpha(1+2b^2-3s^2)} (r_{00} - \frac{4\alpha}{1-s} s_0) y^i + \frac{2b^i}{1+2b^2-3s^2} (r_{00} - \frac{4\alpha}{1-s} s_0) \right. \\
& \quad \left. + \frac{4}{1-s} \alpha s_0^i \right) \left( K_1 \left[ (r_m + s_m) h_i \right. \right. \\
& \quad \left. \left. + (r_0 + s_0) \left( \frac{-\alpha^2 b_m + 2y_m \beta}{\alpha^4} y_i - \frac{s}{\alpha} a_{im} + h_i \left( \frac{1}{s} (s)_{y^m} + \frac{2y_m}{\alpha^2} + \frac{12s(s)_{y^m}}{1-s^2 + 2(b^2 - s^2)} \right) \right) \right] \right. \\
& \quad \left. + K_2 \left[ \left( (-s)_{y^m} r_{00} + 2(1-s)r_{m0} - \frac{4y_m s_0}{\alpha} - 4\alpha s_m \right) h_i \right. \right. \\
& \quad \left. \left. + ((1-s)r_{00} - 4\alpha s_0) \left( \frac{-\alpha^2 b_m + 2y_m \beta}{\alpha^4} y_i - \frac{s}{\alpha} a_{im} \right. \right. \right. \\
& \quad \left. \left. \left. + h_i \left( \frac{B_2}{M_2} (s)_{y^m} - \frac{3y_m}{\alpha^2} + \frac{(s)_{y^m}}{1-s} + \frac{18s(s)_{y^m}}{1-s^2 + 2(b^2 - s^2)} \right) \right) \right] \right. \\
& \quad \left. + K_3 \left[ \frac{-2y_m}{\alpha} s_0 h_i - 2\alpha s_m h_i - 2\alpha s_0 (h_i)_{y^m} + (\alpha^2 s_i - y_i s_0) (2(1-s) \frac{y_m}{\alpha} - 2\alpha (s)_{y^m}) \right. \right. \\
& \quad \left. \left. + 2\alpha (1-s) (2y_m s_i - y_i s_m - a_{im} s_0) + (2y_m s_{i0} + \alpha^2 s_{im}) (1-s^2 + 2(b^2 - s^2)) \right. \right. \\
& \quad \left. \left. - 6s\alpha^2 s_{i0} (s)_{y^m} + ((1-s)r_{i0} - 4\alpha s_i) (2(1-s)y_m - \alpha^2 (s)_{y^m}) \right. \right. \\
& \quad \left. \left. + \alpha^2 (1-s) (-s)_{y^m} r_{i0} + (1-s)r_{mi} - \frac{4y_m}{\alpha} s_i + ((1-s)r_{00} - 4\alpha s_0) \right. \right. \\
& \quad \left. \left. (y_i (s)_{y^m} - (1-s)a_{im}) - (1-s) \left( (-s)_{y^m} r_{00} + 2(1-s)r_{m0} - \frac{4y_m}{\alpha} s_0 - 4\alpha s_m \right) y_i \right. \right. \\
& \quad \left. \left. + \left( \frac{B_3}{M_3} (s)_{y^m} - \frac{3y_m}{\alpha^2} - \frac{2}{1-s} (s)_{y^m} + \frac{12s}{1-s^2 + 2(b^2 - s^2)} (s)_{y^m} \right) \right. \right. \\
& \quad \left. \left. \left( -2\alpha s_0 h_i + 2\alpha (1-s) (\alpha^2 s_i - y_i s_0) + \alpha^2 (1-s^2 + 2(b^2 - s^2)) s_{i0} \right. \right. \right. \\
& \quad \left. \left. \left. + \alpha^2 (1-s) ((1-s)r_{i0} - 4\alpha s_i) - (1-s) ((1-s)r_{00} - 4\alpha s_0) y_i \right) \right] \right], \tag{2.19}
\end{aligned}$$

where

$$\begin{aligned} B_2 := & -24(2n-3)s^4 + 8(5n-1)s^3 - 24(n-1)s^2 + 16(n-2)b^2s + 8(2n-2)s \\ & + 2(b^2-s^2)\left(18(2n-3)s^2 - 4(5n-1)s + 6(n-1)\right) - 60(n+3)s^4 \\ & + 32(2n+5)s^3 + 18(n+5)s^2 - 4(7n+19)s + 6(n-1), \end{aligned}$$

$$B_3 := -8(1-2s)ns - 8s^2 - 8s + 4(b^2-s^2)(-2n+1) - (n+1)(-12s^2+4s+4),$$

Also, we have

$$\begin{aligned} J_m\left(N_i^m - \bar{N}_i^m\right) = & \frac{1-2s}{\alpha(1-s)(1-s^2+2(b^2-s^2))} \left[ K_1(r_0+s_0)((1-s)r_{00}-4\alpha s_0)h_i \right. \\ & + ((1-s)r_{00}-4\alpha s_0)^2 K_2 h_i + K_3((1-s)r_{00}-4\alpha s_0) \left( -2\alpha s_0 h_i \right. \\ & + 2\alpha(\alpha^2 s_i - y_i s_0) + \alpha^2(1-s^2+2(b^2-s^2))s_{i0} + \alpha^2(1-s)((1-s)r_{i0}-4\alpha s_i) \\ & \left. \left. - (1-s)((1-s)r_{00}-4\alpha s_0)y_i \right) \right] + \left[ (r_0+s_0)K_1 h_m + (r_{00}-4\alpha s_0)K_2 h_m \right. \\ & + K_3 \left( -2\alpha s_0 h_m + 2\alpha(1-s)(\alpha^2 s_m - y_m s_0) + \alpha^2(1-s^2+2(b^2-s^2))s_{m0} \right. \\ & \left. \left. + \alpha^2((1-s)r_{m0}-4\alpha s_m) - (1-s)((1-s)r_{00}-4\alpha s_0)y_m \right) \right] \\ & \left[ \frac{6s((1-s)r_{00}-4\alpha s_0)(s)y_i}{(1-s)(1-s^2+2(b^2-s^2))} b^m \right. \\ & + \frac{2\alpha(1-s)^2 r_{i0} - 4(1-s)y_i s_0 - 4\alpha^2 s_0 (s)y_i - 4\alpha^2(1-s)s_i}{\alpha(1-s)^2(1-s^2+2(b^2-s^2))} b^m \\ & \left. + \frac{2\alpha(s)y_i}{(1-s)^2} s_0^m + \frac{2}{1-s} \left( \frac{y_i}{\alpha} s_0^m + \alpha s_i^m \right) \right]. \quad (2.20) \end{aligned}$$

The following hold

$$\begin{aligned} (s)_{y^m} &= \frac{b_m}{\alpha} - \frac{\beta}{\alpha^3} y_m, \\ (h_i)_m &= \frac{-\alpha^2 b_m + 2\beta y_m}{\alpha^4} y_i - \frac{s}{\alpha} a_{im}, \\ b_{0|0} &= r_{00}, \\ b^j b_{j|0} &= s_0 + r_0, \\ bb_{|0} &= r_0 + s_0, \\ h_i b^i &= b^2 - s^2, \\ s_i b^i &= 0, \end{aligned} \quad (2.21)$$

$$\begin{aligned}
s_{i|0}b^i &= -s_i(r_0^i + s_0^i), \\
s_{i0|0}b^i &= s_{0|0} - s_{i0}\left(r_0^i + s_0^i\right), \\
s_{|0} &= \frac{r_{00}}{\alpha}, \\
h_{i|0} &= s_{i0} + r_{i0} - \frac{y_i}{\alpha^2}r_{00}.
\end{aligned}$$

By substitutions (2.18) , (2.19), (2.20) and (2.21) in (2.10) we get  $J_{i;k}y^k$ .

Now, by computing  $J_{i;k}y^k = 0$  we get

$$\begin{aligned}
&\left[(1-s)(b^2-s^2)q_2\dot{K}_2 + 2s(1-s)\dot{K}_3\alpha + s(1-s)^2q_3\dot{K}_3\alpha + \frac{2(1-2s)(1-s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2\right. \\
&- \frac{12s(1-s)^2(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{2(b^2-s^2)^2}{1+2b^2-3s^2}\dot{K}_2 - \frac{4s^3(1-s)}{1+2b^2-3s^2}\dot{K}_2 - s(1-s)\dot{K}_1\alpha \\
&+ \frac{4sb^2(1-s)}{1+2b^2-3s^2}\dot{K}_2 - \frac{2(1-s)(b^2-s^2)^2}{1+2b^2-3s^2}P_2\dot{K}_2 + \frac{6s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2 \\
&- \frac{4s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_3\alpha - \frac{2b^2(1-s)^2}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{2s(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_3\alpha \\
&- \frac{6s^2(1-s)^2}{1+2b^2-3s^2}\dot{K}_3\alpha - \frac{(1-2s)(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\dot{K}_2 + \frac{s(1-2s)(1-s)^2}{1-s^2+2(b^2-s^2)}\dot{K}_3\alpha \\
&- \left.\frac{6s(1-s)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\dot{K}_2 + \frac{6s^2(1-s)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\dot{K}_3\alpha\right]r_{00}^2 \\
&+ \left[\frac{(r_0+s_0)(b^2-s^2)}{\beta}\dot{K}_1\alpha^2 + \frac{12s(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)}\dot{K}_1\alpha - s(r_0+s_0)\dot{K}_1\alpha - (b^2-s^2)\dot{K}_2\alpha\right. \\
&+ (1-s)(r_0+s_0)\dot{K}_2\alpha + 4ss_0\dot{K}_2\alpha + 4(1-s)(b^2-s^2)(r_0+s_0)q_2\dot{K}_2\alpha \\
&- \frac{16(n-2)(1-s)(b^2-s^2)^2(r_0+s_0)}{M_2}\dot{K}_2\alpha - \frac{6(1-2s)(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\dot{K}_1\alpha \\
&- 4s_0(b^2-s^2)q_2\dot{K}_2\alpha - 2ss_0\dot{K}_3\alpha - 8\beta - 2s_0(b^2-s^2)q_3\dot{K}_3\alpha s_0\dot{K}_3\alpha - 2r_0(1-s)\dot{K}_3\alpha^2 \\
&+ 2\beta s_0(1-s)q_3\dot{K}_3\alpha + s_0(1-s^2+2(b^2-s^2))q_3\dot{K}_3\alpha^2 + r_0(1-s)^2q_3\dot{K}_3\alpha^2 \\
&- \frac{16s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2\alpha + \frac{8s_0(b^2-s^2)(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{28ss_0(1-s)(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha^2 \\
&- \frac{24ss_0\beta(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{16ss_0(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha^2 + 2s_0(1-s)\dot{K}_3\alpha^2 + 8s(1-s)^2(r_0+s_0)q_3\dot{K}_3\alpha^2 \\
&- \frac{12s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{36s_0\beta(1-s)(1-2s)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{4s(r_0+s_0)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_1\alpha \\
&- \frac{2b^2(b^2-s^2)(r_0+s_0)}{\beta(1+2b^2-3s^2)}\dot{K}_1\alpha^2 - \frac{2s(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\dot{K}_1\alpha - \frac{24s(b^2-s^2)^2(r_0+s_0)}{(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))}\dot{K}_1\alpha \\
&- \frac{4r_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2\alpha - \frac{40ss_0(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2\alpha - \frac{32ss_0(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_2\alpha - \frac{2r_0(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}P_3\dot{K}_3\alpha^2 \\
&- \frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_2\dot{K}_2\alpha - \frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2}\dot{K}_3\alpha + \frac{4s_0s^2(1-s)}{1+2b^2-3s^2}\dot{K}_3\alpha^2 - \frac{12s_0b^2(1-s)}{1+2b^2-3s^2}\dot{K}_3\alpha^2
\end{aligned}$$

$$\begin{aligned}
& - \frac{24s^2s_0\beta}{1+2b^2-3s^2} \dot{K}_3\alpha + \frac{2s_0\beta(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2} \dot{K}_3\alpha - \frac{(1-2s)(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha \\
& - \frac{2r_0\beta(1-s)^2}{1+2b^2-3s^2} \dot{K}_3\alpha + \frac{8ss_0\beta}{1+2b^2-3s^2} \dot{K}_3\alpha + 8ss_0\dot{K}_2\alpha + \frac{4s_0(b^2-s^2)^2}{1+2b^2-3s^2} P_3\dot{K}_3\alpha \\
& - \frac{4ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2} P_3\dot{K}_3\alpha^2 - \frac{2s_0(b^2-s^2)(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2} P_3\dot{K}_3\alpha^2 - \frac{4\beta s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
& - \frac{8s_0(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)} \dot{K}_2\alpha + \frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2} P_2\dot{K}_2\alpha + \frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 \\
& + \frac{8s_0b^2(1-s)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 - \frac{8ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 + \frac{4s_0(b^2-s^2)}{1-s} \dot{K}_2\alpha + \frac{36ss_0\beta(1-s)}{1+2b^2-3s^2} \dot{K}_3\alpha \\
& - 4s_0(b^2-s^2)P_2\dot{K}_2\alpha - 8ss_0(1-s)\dot{K}_3\alpha^2 - 4s_0(1-s)\dot{K}_3\alpha^2 + 4s_0\beta(1-s)\dot{K}_3\alpha \\
& + \frac{8s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha + \frac{2s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha - \frac{6s_0\beta(1-s)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
& - s_0(1-2s)\dot{K}_3\alpha^2 - \frac{r_0(1-s)^2(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha^2 - \frac{6s(r_0+s_0)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha \\
& + \frac{48ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha - \frac{2r_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha + \frac{4ss_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha \\
& + \frac{4s_0(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))} \dot{K}_2\alpha - \frac{2s_0(b^2-s^2)}{1-s} \dot{K}_2\alpha + \frac{12ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
& - \frac{12ss_0\beta(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha - 6ss_0(b^2-s^2)\dot{K}_3\alpha^2 + \frac{2\beta r_0(1-s)^2}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
& - \frac{4ss_0\beta(1-s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha - \frac{6sr_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha^2 \Big] r_{00} + (r_{0|0} + s_{0|0})(b^2-s^2)\dot{K}_1\alpha^2 \\
& - \frac{8(r_0+s_0)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha^2 + (r_0+s_0)^2\dot{K}_1\alpha^2 + (1-s)(b^2-s^2)r_{00|0}\dot{K}_2\alpha \\
& - 4s_{0|0}(b^2-s^2)\dot{K}_2\alpha^2 - 4s_0(r_0+s_0)\dot{K}_2\alpha^2 - 16s_0(b^2-s^2)(r_0+s_0)q_2\dot{K}_2\alpha^2 \\
& + \frac{64s_0(n-2)(b^2-s^2)^2(r_0+s_0)}{M_2} \dot{K}_2\alpha^2 + 4s_0(r_0+s_0)\dot{K}_3\alpha^3 + 8r_0(1-s)^2(r_0+s_0)q_3\dot{K}_3\alpha^3 \\
& - 2(b^2-s^2)s_{0|0}\dot{K}_3\alpha^2 - 2s_0(r_0+s_0)\dot{K}_3\alpha^2 + 2ss_0(1-s)\dot{K}_3\alpha^3 + 2(1-s)s_m(r_0^m + s_0^m)\dot{K}_3\alpha^4 \\
& + (1-s)^2r_{m0|0}b^m\dot{K}_3\alpha^3 - \beta(1-s)^2r_{00|0}\dot{K}_3\alpha + 4\beta(1-s)s_{0|0}\dot{K}_3\alpha^2 - 16s_0(r_0+s_0)(b^2-s^2)q_3\dot{K}_3\alpha^2 \\
& + 16\beta s_0(1-s)(r_0+s_0)q_3\dot{K}_3\alpha^2 + 8s_0(r_0+s_0)(1-s^2+2(b^2-s^2))q_3\dot{K}_3\alpha^3 \\
& + \frac{24s_0(b^2-s^2)(r_0+s_0)(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_1\alpha^2 + \frac{32s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_2\alpha^2 \\
& + \frac{96s^2s_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3\alpha^3 - \frac{32ss_0^2(1-2s)}{1+2b^2-3s^2} \dot{K}_3\alpha^3 - \frac{64ss_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3\alpha^3 \\
& - \frac{16ss_0(r_0+s_0)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_1\alpha^2 + \frac{8s_0b^2(b^2-s^2)(r_0+s_0)}{s(1-s)(1+2b^2-3s^2)} \dot{K}_1\alpha^2 + \frac{8ss_0(b^2-s^2)(r_0+s_0)}{(1-s)(1+2b^2-3s^2)} \dot{K}_1\alpha^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{96ss_0b^2(b^2-s^2)^2(r_0+s_0)}{(1-s)(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))} K_1\alpha^2 - \frac{64s^3s_0^2}{(1-s)(1+2b^2-3s^2)} K_2\alpha^2 \\
& + \frac{16r_0s_0(b^2-s^2)}{1+2b^2-3s^2} K_2\alpha + \frac{8r_0s_0(1-s)(b^2-s^2)}{1+2b^2-3s^2} P_3K_3\alpha^3 - \frac{4ss_0(b^2-s^2)(r_0+s_0)}{1-s} K_1\alpha^2 \\
& + \frac{64ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} K_2\alpha^2 + \frac{64ss_0^2b^2}{(1-s)(1+2b^2-3s^2)} K_2\alpha^2 - \frac{32s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)} P_2K_2\alpha^2 \\
& + \frac{32ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} K_3\alpha^2 - \frac{32s_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} K_3\alpha^2 - \frac{16s^2s_0^2}{1+2b^2-3s^2} K_3\alpha^3 \\
& + \frac{48s_0^2b^2}{1+2b^2-3s^2} K_3\alpha^3 + \frac{96s^3s_0^2}{(1-s)(1+2b^2-3s^2)} K_3\alpha^3 - \frac{8ss_0^2(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)} K_3\alpha^3 \\
& + \frac{8sr_0s_0(1-s)}{1+2b^2-3s^2} K_3\alpha^3 + \frac{32s^2s_0^2}{(1-s)(1+2b^2-3s^2)} K_3\alpha^3 - \frac{16s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)} P_3K_3\alpha^2 \\
& + \frac{16ss_0^2(b^2-s^2)}{1+2b^2-3s^2} P_3K_3\alpha^3 + \frac{8s_0^2(b^2-s^2)(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)} P_3K_3\alpha^3 \\
& - \frac{48s^2s_0^2}{1+2b^2-3s^2} K_3\alpha^3 - \frac{4(b^2-s^2)(r_ms_0^m+s_ms_0^m)}{1-s} K_1\alpha^3 + \frac{8ss_0(r_0+s_0)}{1-s} K_1\alpha^2 \\
& - \frac{48ss_0(b^2-s^2)(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))} K_1\alpha^2 - 8(b^2-s^2)r_ms_0^m K_2\alpha^2 + \frac{16s_ms_0^m(b^2-s^2)}{1-s} K_2\alpha^3 \\
& - \frac{32ss_0^2}{1-s} K_2\alpha^2 + \frac{8s_0^2(b^2-s^2)}{1-s} P_3K_3\alpha - \frac{4s_ms_0^m(1-s^2+2(b^2-s^2))}{1-s} K_3\alpha^4 - 4(1-s)r_ms_0^m K_3\alpha^4 \\
& + \frac{16s_0^2(b^2-s^2)}{1-s} P_2K_2\alpha^2 + \frac{8(b^2-s^2)s_ms_0^m}{1-s} K_3\alpha^3 - \frac{16ss_0^2}{1-s} K_3\alpha^2 - 24s_0^2K_3\alpha^3 + 8ss_ms_0^m K_3\alpha^4 \\
& - 8s(1-s)s_0^m r_ms_0 K_3\alpha^3 + \frac{16ss_ms_0^m}{1-s} K_3\alpha^4 + \frac{4s_0(r_0+s_0)(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))} K_1\alpha^2 \\
& - 8ss_0^2P_3K_3\alpha^2 - \frac{4s_0^2(1-s^2+2(b^2-s^2))}{1-s} P_3K_3\alpha^3 - 4r_0s_0(1-s)P_3K_3\alpha^3 + \frac{4s_0^2}{1-s} K_3\alpha^3 \\
& - \frac{16s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))} K_2\alpha^2 - \frac{8s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))} K_3\alpha^2 + \frac{8ss_0^2(1-2s)}{1-s^2+2(b^2-s^2)} K_3\alpha^3 \\
& + \frac{4s_0r_0(1-s)(1-2s)}{1-s^2+2(b^2-s^2)} K_3\alpha^3 + \frac{24ss_0(b^2-s^2)^2(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))} K_1\alpha^2 - \frac{2r_0(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)} K_1\alpha^2 \\
& + \frac{4ss_0(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)} K_1\alpha^2 + \frac{4s_0(b^2-s^2)^2(r_0+s_0)}{(1-s)^2(1-s^2+2(b^2-s^2))} K_1\alpha^2 - \frac{2s_0(r_0+s_0)(b^2-s^2)}{(1-s)^2} K_1\alpha^2 \\
& - \frac{96ss_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)} K_2\alpha + \frac{8r_0s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} K_2\alpha^2 - \frac{16ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))} K_2\alpha^2 \\
& - \frac{16s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))} K_2\alpha^2 - \frac{48ss_0^2(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))} K_3\alpha^2 + \frac{4r_0s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} K_3\alpha^2 \\
& - \frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))} K_3\alpha^2 - \frac{8s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))} K_3\alpha^2 + \frac{4s_0^2(b^2-s^2)}{(1-s)^2} K_3\alpha^2 \\
& - \frac{48s^2s_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)} K_3\alpha^3 + \frac{4ss_0^2}{1-s} K_3\alpha^3 - \frac{2s(1-s^2+2(b^2-s^2))}{1-s} s_ms_0^m K_3\alpha^3 \\
& + \frac{4sr_0s_0(1-s)}{1-s^2+2(b^2-s^2)} K_3\alpha^3 - \frac{8s^2s_0^2}{1-s^2+2(b^2-s^2)} K_3\alpha^3 - \frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))} K_3\alpha^3
\end{aligned}$$

$$\begin{aligned}
& - \frac{4s_m s_0^m (b^2 - s^2)}{1 - s} \dot{K}_3 \alpha^4 - 4s s_m s_0^m \dot{K}_3 \alpha^4 - 4s_m s^m \dot{K}_3 \alpha^5 + 4s_0^2 \dot{K}_3 \alpha^3 + \frac{24s s_0^2 (b^2 - s^2)}{1 - s} \dot{K}_3 \alpha^3 \\
& - 2r_0 s_0 \dot{K}_3 \alpha^3 + \frac{4s_0^2 (b^2 - s^2)}{(1 - s)^2} \dot{K}_3 \alpha^3 - \frac{2(b^2 - s^2)(1 - s^2 + 2(b^2 - s^2))}{(1 - s)^2} s_{m0} s_0^m \dot{K}_3 \alpha^3 \\
& + \frac{4s s_0 r_0}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - \frac{2(1 - s^2 + 2(b^2 - s^2))}{1 - s} s_{m0} s_0^m \dot{K}_3 \alpha^4 + \frac{24s s_0 r_0 (b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 \\
& - \frac{2r_0 (1 - s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - 2s r_{m0} s_0^m \dot{K}_3 \alpha^3 + \frac{16s_0^2 (1 - 2s)(b^2 - s^2)}{(1 - s)(1 + 2b^2 - 3s^2)} \dot{K}_3 \alpha^2 \\
& + \frac{4r_0 s_0 (b^2 - s^2)}{(1 - s)(1 - s^2 + 2(b^2 - s^2))} \dot{K}_3 \alpha^3 - \frac{2(b^2 - s^2)}{1 - s} r_{m0} s_0^m \dot{K}_3 \alpha^3 + \frac{8(b^2 - s^2)}{(1 - s)^2} s_m s_0^m \dot{K}_3 \alpha^4 \\
& + \frac{8s s_m s_0^m}{1 - s} \dot{K}_3 \alpha^4 - 2s^m r_{m0} \dot{K}_3 \alpha^4 + \frac{8s_m s^m}{1 - s} \dot{K}_3 \alpha^5 + \frac{96s^2 s_0^2 (b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - 8s_0^2 \dot{K}_3 \alpha^3 \\
& + \frac{8s s_0 r_0 (1 - s)^2}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{16s^2 s_0^2 (1 - s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{16s s_0^2 (b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 \\
& + (1 - s^2 + 2(b^2 - s^2)) (s_{0|0} - s_{m0} (s_0^m + r_0^m)) \dot{K}_3 \alpha^3 = 0.
\end{aligned}$$

The above relationship is equivalent to (1.14) and (1.15).  $\square$

### 3. Appendix

$$\begin{aligned}
A_1 &= (1 - s)(b^2 - s^2) q_2 \dot{K}_2 + \frac{2(1 - 2s)(1 - s)(b^2 - s^2)}{1 + 2b^2 - 3s^2} \dot{K}_2 + \frac{2(b^2 - s^2)^2}{1 + 2b^2 - 3s^2} \dot{K}_2 \\
&+ \frac{4s^3 (1 - s)(b^2 - s^2)}{1 + 2b^2 - 3s^2} \dot{K}_2 - \frac{6s(1 - s)(b^2 - s^2)^2}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_2 - \frac{2(1 - s)(b^2 - s^2)^2}{1 + 2b^2 - 3s^2} P_2 \dot{K}_2 \\
&+ \frac{6s(1 - s)(b^2 - s^2)}{1 + 2b^2 - 3s^2} \dot{K}_2 - \frac{(1 - 2s)(1 - s)(b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_2
\end{aligned}$$

$$\begin{aligned}
A_2 &= s(1 - s)^2 q_3 \dot{K}_3 \alpha - s(1 - s) \dot{K}_1 \alpha - \frac{12s(1 - s)^2 (1 - 2s)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha - \frac{4s(1 - s)(b^2 - s^2)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha \\
&+ \frac{s(1 - s)^2 (1 - 2s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha - \frac{2b^2 (1 - s)^2}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha + \frac{2s(1 - s)^2 (b^2 - s^2)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha \\
&- \frac{6s^2 (1 - s)^2}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha + \frac{6s^2 (1 - s)^2 (b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha + 2s(1 - s) \dot{K}_3 \alpha
\end{aligned}$$

$$\begin{aligned}
B_1 &= \frac{(r_0 + s_0)(b^2 - s^2)}{\beta} \dot{K}_1 \alpha^2 + s_0(1 - s^2 + 2(b^2 - s^2)) q_3 \dot{K}_3 \alpha^2 + r_0(1 - s)^2 q_3 \dot{K}_3 \alpha^2 \\
&+ \frac{28s s_0 (1 - s)(1 - 2s)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha^2 - 2r_0(1 - s) \dot{K}_3 \alpha^2 - \frac{r_0(1 - s)^2 (1 - 2s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^2 \\
&+ \frac{16s s_0 (1 - 2s)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha^2 + 2s_0(1 - s) \dot{K}_3 \alpha^2 - \frac{2b^2 (b^2 - s^2)(r_0 + s_0)}{\beta(1 + 2b^2 - 3s^2)} \dot{K}_1 \alpha^2 + \frac{4s^2 s_0 (1 - s)}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha^2
\end{aligned}$$

$$\begin{aligned}
& - \frac{12s_0b^2(1-s)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 - \frac{4ss_0(b^2-s^2)}{1+2b^2-3s^2} P_3\dot{K}_3\alpha^2 - \frac{2s_0(b^2-s^2)(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2} P_3\dot{K}_3\alpha^2 \\
& - \frac{2r_0(1-s)^2(b^2-s^2)}{1+2b^2-3s^2} P_3\dot{K}_3\alpha^2 + \frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 + \frac{8s_0b^2(1-s)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 \\
& - \frac{8ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_3\alpha^2 - 8ss_0(1-s)\dot{K}_3\alpha^2 - 4s_0(1-s)\dot{K}_3\alpha^2 - s_0(1-2s)\dot{K}_3\alpha^2 \\
& - 6ss_0(b^2-s^2)\dot{K}_3\alpha^2 - \frac{6sr_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha^2 + 8s(1-s)^2(r_0+s_0)q_3\dot{K}_3\alpha^2 \\
B_2 = & \frac{12s(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha - s(r_0+s_0)\dot{K}_1\alpha - (b^2-s^2)\dot{K}_2\alpha + (1-s)(r_0+s_0)\dot{K}_2\alpha \\
& + 4ss_0\dot{K}_2\alpha - 8\beta s_0\dot{K}_3\alpha - 2s_0(b^2-s^2)q_3\dot{K}_3\alpha - \frac{24ss_0\beta(1-2s)}{1+2b^2-3s^2} \dot{K}_3\alpha - 2ss_0\dot{K}_3\alpha \\
& - \frac{16(n-2)(1-s)(b^2-s^2)^2(r_0+s_0)}{M_2} \dot{K}_2\alpha - 4s_0(b^2-s^2)q_2\dot{K}_2\alpha - \frac{72ss_0(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_2\alpha \\
& - \frac{6(1-2s)(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2} \dot{K}_1\alpha - \frac{16s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_2\alpha + \frac{8s_0(b^2-s^2)(1-2s)}{1+2b^2-3s^2} \dot{K}_3\alpha \\
& - \frac{12s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_3\alpha + \frac{36s_0\beta(1-s)(1-2s)}{1+2b^2-3s^2} \dot{K}_3\alpha + \frac{4s(r_0+s_0)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_1\alpha \\
& + \frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2} P_2\dot{K}_2\alpha - \frac{16ss_0}{1+2b^2-3s^2} \dot{K}_3\alpha - \frac{24s_0s^2\alpha\beta}{1+2b^2-3s^2} \dot{K}_3 + \frac{2s_0^2(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2} \dot{K}_3\alpha \\
& - \frac{2r_0\beta(1-s)^2}{1+2b^2-3s^2} \dot{K}_3\alpha + \frac{8ss_0\beta}{1+2b^2-3s^2} \dot{K}_3\alpha + 8ss_0\dot{K}_2\alpha + \frac{4s_0(b^2-s^2)^2}{1+2b^2-3s^2} P_3\dot{K}_3\alpha \\
& + 4(1-s)(b^2-s^2)(r_0+s_0)q_2\dot{K}_2\alpha + 2\beta s_0(1-s)q_3\dot{K}_3\alpha - \frac{4\beta s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
& + \frac{36ss_0\beta(1-s)}{1+2b^2-3s^2} \dot{K}_3\alpha - \frac{8s_0(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)} \dot{K}_2\alpha - \frac{(1-2s)(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha \\
& + \frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2} P_2\dot{K}_2\alpha + \frac{4s_0(b^2-s^2)}{1-s} \dot{K}_2\alpha - 4s_0(b^2-s^2)P_2\dot{K}_2\alpha + 4s_0\beta(1-s)\dot{K}_3\alpha \\
& - \frac{6s_0\beta(1-s)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha - \frac{6s(r_0+s_0)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha + \frac{48ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha \\
& - \frac{2r_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha + \frac{4ss_0(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha + \frac{4s_0(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))} \dot{K}_2\alpha \\
& - \frac{2s_0(b^2-s^2)\alpha}{1-s} \dot{K}_2 + \frac{12ss_0(b^2-s^2)^2\alpha}{1-s^2+2(b^2-s^2)} \dot{K}_3 - \frac{12ss_0\beta\alpha}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha + \frac{2r_0\beta(1-s)^2}{1-s^2+2(b^2-s^2)} \dot{K}_3 \\
& - \frac{4ss_0\beta(1-s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha - \frac{2s(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2} \dot{K}_1\alpha - \frac{24s(b^2-s^2)^2(r_0+s_0)}{(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))} \dot{K}_1\alpha \\
& - \frac{4r_0(1-s)(b^2-s^2)}{1+2b^2-3s^2} \dot{K}_2\alpha + \frac{8s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_2\alpha + \frac{2s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)} \dot{K}_3\alpha \\
C_1 = & (r_0|_0 + s_0|_0)(b^2-s^2)\dot{K}_1\alpha^2 - \frac{8(r_0+s_0)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)} \dot{K}_1\alpha^2 + (r_0+s_0)^2\dot{K}_1\alpha^2 - 4s_0|_0(b^2-s^2)\dot{K}_2\alpha^2 \\
& - 4s_0(r_0+s_0)\dot{K}_2\alpha^2 - 16s_0(b^2-s^2)(r_0+s_0)q_2\dot{K}_2\alpha^2 + \frac{64s_0(n-2)(b^2-s^2)^2(r_0+s_0)}{M_2} \dot{K}_2\alpha^2 \\
& - 2(b^2-s^2)s_0|_0\dot{K}_3\alpha^2 + \frac{96ss_0b^2(b^2-s^2)^2(r_0+s_0)}{(1-s)(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))} \dot{K}_1\alpha^2 - 2s_0(r_0+s_0)\dot{K}_3\alpha^2 \\
& + 2(1-s)s_m(r_0^m + s_0^m)\dot{K}_3\alpha^4 + 4\beta(1-s)s_0|_0\dot{K}_3\alpha^2 - 16s_0(r_0+s_0)(b^2-s^2)q_3\dot{K}_3\alpha^2 \\
& + 16s_0\beta(1-s)(r_0+s_0)q_3\dot{K}_3\alpha^2 + \frac{24s_0(b^2-s^2)(r_0+s_0)(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_1\alpha^2 + \frac{32s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_2\alpha^2
\end{aligned}$$

$$\begin{aligned}
& - \frac{8ss_0(r_0 + s_0)(b^2 - s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_1 \alpha^2 + \frac{8s_0b^2(b^2 - s^2)(r_0 + s_0)}{s(1-s)(1+2b^2-3s^2)} \dot{K}_1 \alpha^2 + \frac{16s_0^2(1-2s)(b^2 - s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^2 \\
& - \frac{64s_0^2s^3}{(1-s)(1+2b^2-3s^2)} \dot{K}_2 \alpha^2 + \frac{64ss_0^2(b^2 - s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_2 \alpha^2 + \frac{64ss_0^2b^2}{(1-s)(1+2b^2-3s^2)} \dot{K}_2 \alpha^2 \\
& - \frac{32s_0^2(b^2 - s^2)^2}{(1-s)(1+2b^2-3s^2)} P_2 \dot{K}_2 \alpha^2 + \frac{32ss_0^2(b^2 - s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^2 - \frac{32s_0^2(b^2 - s^2)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^2 \\
& - \frac{16s_0^2(b^2 - s^2)^2}{(1-s)(1+2b^2-3s^2)} P_3 \dot{K}_3 \alpha^2 + \frac{8ss_0(r_0 + s_0)}{1-s} \dot{K}_1 \alpha^2 - \frac{4ss_0(b^2 - s^2)(r_0 + s_0)}{1-s} \dot{K}_1 \alpha^2 \\
& - \frac{48ss_0(b^2 - s^2)(r_0 + s_0)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_1 \alpha^2 - 8(b^2 - s^2)r_{m0}s_0^m \dot{K}_2 \alpha^2 - \frac{32ss_0^2}{1-s} \dot{K}_2 \alpha^2 + \frac{16s_0^2(b^2 - s^2)}{1-s} P_2 \dot{K}_2 \alpha^2 \\
& - \frac{16ss_0^2}{1-s} \dot{K}_3 \alpha^2 + 8ss_ms_0^m \dot{K}_3 \alpha^4 - \frac{4s_ms_0^m(1-s^2+2(b^2 - s^2))}{1-s} \dot{K}_3 \alpha^4 - 4(1-s)r_ms_0^m \dot{K}_3 \alpha^4 \\
& + \frac{16ss_ms_0^m}{1-s} \dot{K}_3 \alpha^4 - 8ss_0^2 P_3 \dot{K}_3 \alpha^2 + \frac{4s_0(r_0 + s_0)(b^2 - s^2)(1-2s)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_1 \alpha^2 \\
& - \frac{16s_0^2(b^2 - s^2)(1-2s)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_2 \alpha^2 - 2s^m r_{m0} \dot{K}_3 \alpha^4 \\
& - \frac{8s_0^2(b^2 - s^2)(1-2s)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_3 \alpha^2 + \frac{24ss_0(b^2 - s^2)^2(r_0 + s_0)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_1 \alpha^2 + \frac{8ss_ms_0^m}{1-s} \dot{K}_3 \alpha^4 \\
& - \frac{2r_0(r_0 + s_0)(b^2 - s^2)}{1-s^2+2(b^2 - s^2)} \dot{K}_1 \alpha^2 + \frac{4ss_0(r_0 + s_0)(b^2 - s^2)}{1-s^2+2(b^2 - s^2)} \dot{K}_1 \alpha^2 + \frac{4s_0(b^2 - s^2)^2(r_0 + s_0)}{(1-s)^2(1-s^2+2(b^2 - s^2))} \dot{K}_1 \alpha^2 \\
& - \frac{2s_0(r_0 + s_0)(b^2 - s^2)}{(1-s)^2} \dot{K}_1 \alpha^2 + \frac{8s_0r_0(b^2 - s^2)}{1-s^2+2(b^2 - s^2)} \dot{K}_2 \alpha^2 - \frac{16ss_0^2(b^2 - s^2)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_2 \alpha^2 \\
& - \frac{16s_0^2(b^2 - s^2)^2}{(1-s)^2(1-s^2+2(b^2 - s^2))} \dot{K}_2 \alpha^2 - \frac{48ss_0^2(b^2 - s^2)^2}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_3 \alpha^2 + \frac{4r_0s_0(b^2 - s^2)}{1-s^2+2(b^2 - s^2)} \dot{K}_3 \alpha^2 \\
& + \frac{4s_0^2(b^2 - s^2)}{(1-s)^2} \dot{K}_3 \alpha^2 - \frac{8ss_0^2(b^2 - s^2)}{(1-s)(1-s^2+2(b^2 - s^2))} \dot{K}_3 \alpha^2 - \frac{8s_0^2(b^2 - s^2)^2}{(1-s)^2(1-s^2+2(b^2 - s^2))} \dot{K}_3 \alpha^2 \\
& - \frac{4s_ms_0^m(b^2 - s^2)}{1-s} \dot{K}_3 \alpha^4 - 4ss_ms_0^m \dot{K}_3 \alpha^4 - \frac{2(1-s^2+2(b^2 - s^2))}{1-s} s_{m0}s_0^m \dot{K}_3 \alpha^4 + \frac{8(b^2 - s^2)}{(1-s)^2} s_ms_0^m \dot{K}_3 \alpha^4 \\
C_2 = & (1-s)(b^2 - s^2)r_{00|0} \dot{K}_2 \alpha + 2ss_0(1-s) \dot{K}_3 \alpha^3 + 4s_0(r_0 + s_0) \dot{K}_3 \alpha^3 + (1-s)^2 r_{m0|0} b^m \dot{K}_3 \alpha^3 \\
& - \beta(1-s)^2 r_{00|0} \dot{K}_3 \alpha + 8s_0(r_0 + s_0)(1-s^2+2(b^2 - s^2)) q_3 \dot{K}_3 \alpha^3 + 8r_0(1-s)^2(r_0 + s_0) q_3 \dot{K}_3 \alpha^3 \\
& - \frac{32ss_0^2(1-2s)}{1+2b^2-3s^2} \dot{K}_3 \alpha^3 - \frac{64ss_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^3 + \frac{16r_0s_0(b^2 - s^2)}{1+2b^2-3s^2} \dot{K}_2 \alpha \\
& + \frac{96s^2s_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^3 - \frac{16s^2s_0^2}{1+2b^2-3s^2} \dot{K}_3 \alpha^3 + \frac{48s_0^2b^2}{1+2b^2-3s^2} \dot{K}_3 \alpha^3 \\
& + \frac{96s^3s_0^2}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^3 + \frac{8r_0s_0(1-s)(b^2 - s^2)}{1+2b^2-3s^2} P_3 \dot{K}_3 \alpha^3 + \frac{8(b^2 - s^2)s_ms_0^m}{1-s} \dot{K}_3 \alpha^3 \\
& - \frac{8ss_0^2(1-s^2+2(b^2 - s^2))}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^3 + \frac{8sr_0s_0(1-s)}{1+2b^2-3s^2} \dot{K}_3 \alpha^3 + \frac{32s^2s_0^2}{(1-s)(1+2b^2-3s^2)} \dot{K}_3 \alpha^3
\end{aligned}$$



$$\begin{aligned}
& + \frac{16ss_0^2(b^2 - s^2)}{1 + 2b^2 - 3s^2} P_3 \dot{K}_3 \alpha^3 + \frac{8s_0^2(b^2 - s^2)(1 - s^2 + 2(b^2 - s^2))}{(1 - s)(1 + 2b^2 - 3s^2)} P_3 \dot{K}_3 \alpha^3 \\
& - \frac{48s^2 s_0^2}{1 + 2b^2 - 3s^2} \dot{K}_3 \alpha^3 - \frac{4(b^2 - s^2)(r_m s_0^m + s_m s_0^m)}{1 - s} \dot{K}_1 \alpha^3 + \frac{16s_m s_0^m (b^2 - s^2)}{1 - s} \dot{K}_2 \alpha^3 \\
& - 24s_0^2 \dot{K}_3 \alpha^3 - 8s(1 - s)s_0^m r_{m0} \dot{K}_3 \alpha^3 + \frac{8s_0^2(b^2 - s^2)}{1 - s} P_3 \dot{K}_3 \alpha - \frac{4s_0^2(1 - s^2 + 2(b^2 - s^2))}{1 - s} P_3 \dot{K}_3 \alpha^3 \\
& - 4r_0 s_0(1 - s) P_3 \dot{K}_3 \alpha^3 + \frac{8ss_0^2(1 - 2s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{4s_0^2}{1 - s} \dot{K}_3 \alpha^3 + \frac{4r_0 s_0(1 - s)(1 - 2s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 \\
& - \frac{96ss_0^2(b^2 - s^2)^2}{(1 - s)(1 + 2b^2 - 3s^2)} \dot{K}_2 \alpha - \frac{48s^2 s_0^2(b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{4ss_0 r_0(1 - s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 \\
& - \frac{8s^2 s_0^2}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - \frac{8ss_0^2(b^2 - s^2)}{(1 - s)(1 - s^2 + 2(b^2 - s^2))} \dot{K}_3 \alpha^3 + \frac{4ss_0^2}{1 - s} \dot{K}_3 \alpha^3 - 4s^m s_m \dot{K}_3 \alpha^5 \\
& + 4s_0^2 \dot{K}_3 \alpha^3 + \frac{24ss_0^2(b^2 - s^2)}{1 - s} \dot{K}_3 \alpha^3 - 2r_0 s_0 \dot{K}_3 \alpha^3 + \frac{4s_0^2(b^2 - s^2)}{(1 - s)^2} \dot{K}_3 \alpha^3 - 8s_0^2 \dot{K}_3 \alpha^3 \\
& - \frac{2r_0(1 - s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - \frac{2(b^2 - s^2)(1 - s^2 + 2(b^2 - s^2))}{(1 - s)^2} s_{m0} s_0^m \dot{K}_3 \alpha^3 \\
& + \frac{4ss_0 r_0}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{24ss_0 r_0(b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{4r_0 s_0(b^2 - s^2)}{(1 - s)(1 - s^2 + 2(b^2 - s^2))} \dot{K}_3 \alpha^3 \\
& - \frac{2(b^2 - s^2)}{1 - s} r_{m0} s_0^m \dot{K}_3 \alpha^3 - 2sr_{m0} s_0^m \dot{K}_3 \alpha^3 + \frac{8s_m s^m}{1 - s} \dot{K}_3 \alpha^5 + \frac{96s^2 s_0^2(b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 \\
& + \frac{8ss_0 r_0(1 - s)^2}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \frac{16s^2 s_0^2(1 - s)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 - \frac{2s(1 - s^2 + 2(b^2 - s^2))}{1 - s} s_{m0} s_0^m \dot{K}_3 \alpha^3 \\
& + \frac{16ss_0^2(b^2 - s^2)}{1 - s^2 + 2(b^2 - s^2)} \dot{K}_3 \alpha^3 + \left(1 - s^2 + 2(b^2 - s^2)\right) \left(s_{0|0} - s_{m0}(s_0^m + r_0^m)\right) \dot{K}_3 \alpha^3
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(s, b^2) &= 6(2n - 3)s^3 - 2(5n - 1)s^2 + 6(n - 1)s - 4(n - 2)b^2 + 4(n - 1) \\
N_2(s, b^2) &= 2(b^2 - s^2) \left(18(2n - 3)s^2 - 4(5n - 1)s + 6(n - 1)\right) - 60(n + 3)s^4 \\
&\quad - 32(2n + 5)s^3 + 18(n + 5)s^2 \\
N_3(s, b^2) &= -8s(n(1 - 2s) + s + 1) + 4(b^2 - s^2)(-2n + 1) - (n + 1)(-12s^2 + 4s + 4) \\
M_3(s, b^2) &= 4(b^2 - s^2)(n(1 - 2s) + s + 1) - (n + 1)(-4s^3 + 2s^2 + 4s - 2) \\
q_2(s, b^2) &= \frac{N_2(s, b^2)}{M_2(s, b^2)} + \frac{1}{1 - s} + \frac{18s}{1 - s^2 + 2(b^2 - s^2)} \\
\acute{q}_2(s, b^2) &= \frac{\vartheta(s, b^2)}{M_2(s, b^2)} - \frac{3}{1 - s^2 + 2(b^2 - s^2)} \\
q_3(s, b^2) &= \frac{N_3(s, b^2)}{M_3(s, b^2)} + \frac{2}{1 - s} + \frac{12s}{1 - s^2 + 2(b^2 - s^2)} \\
\acute{q}_3(s, b^2) &= \frac{n(1 - 2s) + s + 1}{M_3} - \frac{1}{1 - s^2 + 2(b^2 - s^2)}
\end{aligned}$$

$$\begin{aligned}
P_2(s, b^2) &= \frac{1}{1-s} + \frac{18s}{1-s^2+2(b^2-s^2)} \\
P_3(s, b^2) &= \frac{-2}{1-s} + \frac{12s}{1-s^2+2(b^2-s^2)} \\
\acute{K}_1(s, b^2) &= \alpha^2 K_1(s, b^2), \quad \acute{K}_2 = \alpha^3 K_2, \quad \acute{K}_3 = \alpha^3 K_3
\end{aligned}$$

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