Journal of Finsler Geometry and its Applications
Vol. 4, No. 2 (2023), pp 74-91
https://doi.org/10.22098/jfga.2023.13750.1102

# On class of square Finsler metrics

Fatemeh Barati*

Department of Mathematics, Faculty of Science,
University of Qom, Qom, Iran

E-mail: ftbrtir@gmail.com

**Abstract.** In this paper, we remark some of the well-known curvature properties of square Finsler metrics. Then, we study weakly stretch square Finsler metrics.

**Keywords:** Square metric, stretch curvature, mean stretch curvature.

## 1. Introduction

The well-known Hilbert's Fourth Problem is to characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths. It turns out that there are lots of solutions to the problem. For example, in [4], Blaschke discusses 2-dimensional solutions to the problem. Then, Ambartzumian [1] and Alexander [2] independently give all 2-dimensional solutions. In [8], Pogorelov discusses smooth solutions in 3-dimensional case. Then, Szabó investigates several problems left by Pogorelov and constructs continuous solutions to the problem in high dimensions [12]. See [5] on related issue.

The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called projectively flat Finsler metrics or projective Finsler metrics. Hamel first characterizes projective Finsler metrics by a system of

---

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

PDE's [6]. Then, Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [9].

For an $n$-dimensional Finsler manifold $(M, F)$, a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0 := TM - \{0\}$, which in a standard coordinates $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are called spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}. \tag{1.1}$$

$\mathbf{G}$ is called the spray associated to $F$. $F$ is projectively flat if only if there exists scalar homogeneous function $P : T\mathcal{U} \to \mathbb{R}$ such that the its spray coefficients satisfy

$$G^i(x,y) = P(x,y)y^i. \tag{1.2}$$

In this case, $P = P(x,y)$ is called the projective factor.

In Finsler Geometry, there is an interesting class of projectively flat metrics on the unit ball $\mathbb{B}^n$ which is given by

$$F = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}. \tag{1.3}$$

This class of metrics is called square metrics which can be expressed as

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.4}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{(1-|x|^2)^2} \quad \beta = \frac{\langle x, y\rangle}{(1-|x|^2)^2}. \tag{1.5}$$

That $\alpha$ is a Riemannian metric and $\beta$ is a 1-form with $\|\beta\|_\alpha < 1$. L. Berwald first constructed a special projectively flat square metric of zero flag curvature on the unit ball in $\mathbb{R}^n$ (see [3]).

Then the flag curvature of $F$ is a function $\mathbf{K} = \mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. $F$ is called of scalar curvature if the flag curvature $\mathbf{K} = \mathbf{K}(x,y)$ is a scalar function on the slit tangent bundle $TM_0$, for any $y \in T_xM$. Recently, Shen-Yildirim determine the local structure of all locally projectively flat square metrics of constant flag curvature. Later on, L.Zhou shows that a square metric of constant flag curvature must be locally projectively flat. In [10], Shen-Yang proved the following.

**Theorem A.** ([10]) Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on a $(n \geq 3)$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form on $M$. Then $F$ is of scalar flag curvature if and only if it is locally projectively flat.

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a coordinate system $(x^i)$ in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a $C^\infty$ scalar function on $TM_0 = TM \setminus \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$.

**Theorem B.** ([7]) Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on an open subset $U \subseteq \mathbb{R}^n$ with $n \geq 3$. Then $F$ is dually flat if and only if one of the following holds:
(i) $F$ is a dually flat Riemannian metric.
(ii) $F$ is of Minkowski-type. Moreover, $F$ can be expressed in the following form.

$$F = \frac{(|y| + \langle v, y \rangle)^2}{|y|}, \tag{1.6}$$

where $v \in \mathbb{R}^n$ is a non zero constant vector.

Let $(M, F)$ be an $n$-dimensional manifold Finsler manifold. Then $F$ is called an Einstein metric if its Ricci curvature $Ric$ is isotropic,

$$\mathbf{Ric} = (n-1)\lambda F^2,$$

where $\lambda = \lambda(x)$ is a scalar function on $M$. In [11], Shen-Yu proved the following.

**Theorem C.** ([11]) Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on a $n$-dimensional manifold $M$, Then $F$ is an Einstein metric if and only if it is Ricci flat and

$$\begin{aligned} {}^{\alpha}Ric &= k^2(1-b^2)^2 - [5(n-1) + 2(2n-5)b^2]\alpha^2 + 6(n-2)\beta^2, \\ b_{i|j} &= k(1-b^2)(1+2b^2)a_{ij} - 3b_i b_j. \end{aligned} \tag{1.7}$$

Then, they determined the local structure of Einstein square metrics as follows.

**Theorem D.** ([11]) Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on a $n$-dimensional manifold $M$. Then the following are equivalent:
(1)$F$ is an Einstein metric.

(2) The Riemaninnain metric $\tilde{\alpha} := (1-b^2)\alpha$ and the 1-form $\tilde{\beta} := \sqrt{1-b^2}\beta$ satisfy

$$^{\tilde{\alpha}}Ric = -(n-1)k^2\tilde{\alpha}, \quad \tilde{b}_{i|j} = k\sqrt{1+\tilde{b}^2}\tilde{a}_{ij}, \tag{1.8}$$

where $k$ is a constant number, $\tilde{b} = \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\tilde{b}_{i|j}$ is the covariant derivation of $\tilde{\beta}$ with respect to $\tilde{\alpha}$. In this case, $F$ is given in the following form

$$F = \frac{(\sqrt{1+\tilde{b}^2}\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}} \tag{1.9}$$

with $(1+\tilde{b}^2)(1-b^2) = 1$.
(3) The Riemannian metric $\bar{\alpha} := (1-b^2)^{\frac{3}{2}}\sqrt{\alpha^2-\beta^2}$ and the 1-form $\bar{\beta} := (1-b^2)\beta$ satisfy $\bar{\alpha}_{Ric} = 0$ and $\bar{b}_{i|j} = k\bar{a}_{ij}$ where $k$ is a constant number, $\bar{b} = \|\bar{\beta}\|_{\bar{\alpha}}$ and $\bar{b}_{i|j}$ is the covariant derivation of $\bar{\beta}$ with respect $\bar{\alpha}$. In this case, $F$ is given in the following form

$$F = \frac{(\sqrt{(1-\bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2} + \bar{\beta})^2}{(1-\bar{b}^2)^2\sqrt{(1-\bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2}}, \tag{1.10}$$

with $\bar{b} = b$.

Also, they provide a new description for square metrics with constant flag curvature.

**Theorem E.** ([11]) The Finsler metric $F = (\alpha+\beta)^2/\alpha$ is of constant flag curvature if and only if under the expression (1.10) of $F$, $\bar{\alpha}$ is locally Euclidean, $\bar{\beta}$ is closed and s homothety with respect to $\bar{\alpha}$. In a suitable local coordinate, $F$ can be expressed by

$$F = \frac{(\sqrt{(1-|\bar{x}|^2)|y|^2 + \langle\bar{x}, y\rangle^2} + \langle\bar{x}, y\rangle)^2}{(1-|\bar{x}|^2)^2\sqrt{(1-|\bar{x}|^2)|y|^2 + \langle\bar{x}, y\rangle^2}}, \tag{1.11}$$

where $\bar{x} := cx + a$ for some constant number $c$ and constant vector $a$. In particular, $F$ must be locally projectively flat with zero flag curvature.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \longrightarrow \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := \frac{-1}{2}g_y(\mathbf{B}_y(u,v,w), y).$$

A Finsler metric $F$ is called a Landsberg metric if $\mathbf{L}_y = 0$.

For $y \in T_xM$, define $J_y : T_xM \longrightarrow \mathbb{R}$ by $J_y(u) := J_i(y)u^i$. $J$ is called the mean Landsberg curvature. A Finsler metric $F$ is called a weakly Landsberg metric if $J_y = 0$. For $y \in T_xM$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \longrightarrow \mathbb{R}$, by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}), \tag{1.12}$$

and $'|'$ denotes the horizontal derivation with respect to the Berwald connection of $F$. A Finsler metric $F$ is said to be a stretch metric if $\mathbf{\Sigma} = 0$. Also, one can define Mean stretch curvature $\overline{\Sigma}_y : T_xM \longrightarrow \mathbb{R}$ by $\overline{\Sigma}_y(u, v) := \overline{\Sigma}_{ij}(y)u^iv^j$, where

$$\overline{\Sigma}_{ij} := 2(J_{i|j} - J_{j|i}). \tag{1.13}$$

A Finsler metric $F$ is said to be weakly stretch metric if $\overline{\Sigma} = 0$. It is easy to see that every Landsberg metric or stretch metric is a weakly stretch metric.

For an $(\alpha, \beta)$-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_{00} := r_{ij}y^iy^j.$$

In this paper, we prove the following.

**Theorem 1.1.** *Every weakly stretch square Finsler metric $F = (\alpha + \beta)^2/\alpha$ satisfies following*

$$A_1r_{00}^2 + B_1r_{00} + C_1 = 0, \tag{1.14}$$

$$A_2r_{00}^2 + B_2r_{00} + C_2 = 0, \tag{1.15}$$

*where $A_1, A_2, B_1, B_2, C_1, C_2$ are functions in terms of $b^2$ and $s$, and are given in Appendix.*

## 2. Proof of Theorem 1.1

The mean Landsberg curvature of $(\alpha, \beta)$ metrics is as follows:

$$\begin{aligned} J_i = \frac{-1}{2\Delta\alpha^4}\Big[&\frac{2\alpha^2}{b^2 - s^2}(r_0 + s_0)h_i\Big(\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big) \\ &+ \frac{\alpha}{b^2 - s^2}(\Psi_1 + s\frac{\Phi}{\Delta})(r_{00} - 2\alpha Q s_0)h_i \\ &+ \frac{\alpha\Phi}{\Delta}\Big( - \alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) \\ &+ \alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\Big)\Big]. \end{aligned} \tag{2.1}$$

where

$$\Psi_1 = \sqrt{b^2-s^2}\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]',$$

$$\Phi = -(Q-sQ')(n\Delta+1+sQ)-(b^2-s^2)(1+sQ)Q'',$$

$$Q = \frac{\phi'}{\phi-s\phi'},$$

$$\Delta = 1+sQ+(b^2-s^2)Q',$$

$$h_i = b_i-\alpha^{-1}sy_i$$

For a square metric $F=\alpha\phi(s)$, we have $\phi(s)=(1+s)^2$. Then

$$Q = \frac{2}{1-s} \tag{2.2}$$

$$\Delta = \frac{1-s^2+2(b^2-s^2)}{(1-s)^2} \tag{2.3}$$

$$\Phi = (n+1)\Big(\frac{4s^2+2s-2}{(1-s)^3}\Big)-4(b^2-s^2)\Big(\frac{n(1-2s)+s+1}{(1-s)^4}\Big) \tag{2.4}$$

$$\begin{aligned}\Psi_1 = {}& \frac{-s\Phi}{\Delta}+\frac{9s(b^2-s^2)}{(1-s)^2}\frac{\Phi}{\Delta^2}-\frac{4(b^2-s^2)^2(2-n)}{(1-s)^5\Delta}\\ &+\frac{(b^2-s^2)}{\Delta}\frac{-24ns^2+14ns+14s+2n+2}{(1-s)^4}.\end{aligned} \tag{2.5}$$

By substituting (2.2), (2.3), (2.4) and (2.5) in (2.1) we have:

$$\begin{aligned}J_i = {}& (r_0+s_0)h_iK_1+\Big((1-s)r_{00}-4\alpha s_0\Big)h_iK_2\\ &+\Big[-2\alpha s_0h_i+2\alpha(1-s)(\alpha^2s_i-y_is_0)+\alpha^2(1-s^2+2(b^2-s^2))s_{i0}\\ &+\alpha^2(1-s)((1-s)r_{i0}-4\alpha s_i)-(1-s)((1-s)r_{00}-4\alpha s_0)y_i\Big]K_3,\end{aligned} \tag{2.6}$$

where

$$K_1 = \frac{12s}{\alpha^2\Big(1-s^2+2(b^2-s^2)\Big)^2},$$

$$K_2 = \frac{M_2}{2\alpha^3(1-s)\Big(1-s^2+2(b^2-s^2)\Big)^3},$$

$$\begin{aligned}M_2 = {}& 2(b^2-s^2)\Big(6(2n-3)s^3-2(5n-1)s^2+6(n-1)s-4nb^2-2(2n-2)\Big)\\ &-12(n+3)s^5+8(2n+5)s^4+6(n+5)s^3-2(7n+19)s^2+2(n+1)(3s-1),\end{aligned}$$

$$K_3 = \frac{M_3}{2\alpha^3(1-s)^2\Big(1-s^2+2(b^2-s^2)\Big)^2},$$

$$M_3 = 4(b^2-s^2)\Big(n(1-2s)+s+1\Big)+2(n+1)\Big(2s^3-s^2-2s+1\Big).$$

And the geodesic spray coefficients of square Finsler metric $F = \alpha\phi(s)$ is as follows

$$G^i = G^i_\alpha + Py^i + Q^i \tag{2.7}$$

where $G^i_\alpha$ is the spray coefficient of Reimanian metric, and $P$ and $Q^i$ are given by

$$P = \frac{1-2s}{\alpha(1+2b^2-3s^2)}\Big(r_{00} - \frac{4\alpha}{1-s}s_0\Big)y^i, \tag{2.8}$$

$$Q^i = \frac{b^i}{1+2b^2-3s^2}\Big(r_{00} - \frac{4\alpha}{1-s}s_0\Big) + \frac{2\alpha}{1-s}s^i_0. \tag{2.9}$$

On the other hand we have:

$$J_{i;k}y^k = J_{i|k}y^k - 2\frac{\partial J_i}{\partial y^m}\Big(G^m - \bar{G}^m\Big) - J_m\Big(N^m_i - \bar{N}^m_i\Big), \tag{2.10}$$

where

$$G^m - \bar{G}^m = Py^m + Q^m, \tag{2.11}$$

$$N^m_i - \bar{N}^m_i = P_i y^m + P\delta^m_i + Q^m_i. \tag{2.12}$$

First, we calculate the equation $J_{i|k}y^k$

$$\begin{aligned} J_{i|k}y^k &= \Big((r_0+s_0)K_1h_i\Big)_{|k}y^k + \Big(((1-s)r_{00} - 4\alpha s_0)K_2h_i\Big)_{|k}y^k \\ &+ \Big[\Big(-2\alpha s_0h_i + 2\alpha(1-s)(\alpha^2 s_i - y_is_0) + \alpha^2(1-s^2+2(b^2-s^2))s_{i0} \\ &+ \alpha^2(1-s)((1-s)r_{i0} - 4\alpha s_i) - (1-s)((1-s)r_{00} - 4\alpha s_0)y_i\Big)K_3\Big]_{|k}y^k. \end{aligned} \tag{2.13}$$

Let us put

$$\begin{aligned} F_i &:= \Big(-2\alpha s_0h_i + 2\alpha(1-s)(\alpha^2 s_i - y_is_0) + \alpha^2(1-s^2+2(b^2-s^2))s_{i0} \\ &+ \alpha^2(1-s)((1-s)r_{i0} - 4\alpha s_i) - (1-s)((1-s)r_{00} - 4\alpha s_0)y_i\Big)K_3. \end{aligned}$$

Then we get

$$J_{i|k}y^k = \Big((r_0+s_0)K_1h_i\Big)_{|k}y^k + \Big(((1-s)r_{00} - 4\alpha s_0)K_2h_i\Big)_{|k}y^k + F_{i|k}y^k \tag{2.14}$$

and

$$\begin{aligned} \Big((r_0+s_0)K_1h_i\Big)_{|k}y^k &= K_1\Big\{(r_0+s_0)\Big(\Big[\frac{b_{0|0}}{\alpha s} - \frac{2}{\alpha(1-s^2+2(b^2-s^2)}\\ &(2\alpha a^{tj}b_{j|0}b_t - 6sb_{0|0} + 2\alpha b^tb_{t|0})\Big]h_i + b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}\Big) + (r_{0|0}+s_{0|0})h_i\Big\}, \end{aligned} \tag{2.15}$$

$$\Big(((1-s)r_{00}-4\alpha s_0)K_2h_i\Big)_{|k}y^k = K_2\Bigg[\Big(\frac{-1}{\alpha}b_{0|0}r_{00}+(1-s)r_{00|0}-4\alpha s_{0|0}\Big)h_i$$
$$+((1-s)r_{00}-4\alpha s_0)(b_{i|0}-\frac{y_i}{\alpha^2}b_{0|0})+((1-s)r_{00}-4\alpha s_0)\Big(\frac{T_2}{M_2}+\frac{1}{1-s}$$
$$+\frac{3(6s-\frac{4bb_{|0}}{s_{|0}})}{1-s^2+2(b^2-s^2)}\Big)s_{|0}h_i\Bigg], \tag{2.16}$$

where

$$T_2 = (\frac{4bb_{|0}}{s_{|0}}-4s)\Big(6(2n-3)s^3-2(5n-1)s^2+6(n-1)s-4nb^2+2(2n-2)\Big)$$
$$+2(b^2-s^2)\Big(18(2n-3)s^2-4(5n-1)s+6(n-1)-\frac{8(n-2)bb_{|0}}{s_{|0}}\Big)$$
$$-60(n+3)s^4-32(2n+5)s^3+18(n+5)s^2-4(7n+19)s+6(n+1),$$

and

$$F_{i|k}y^k = K_3\Big[-2\alpha s_0(b_{i|0}-\frac{y_i}{\alpha^2}b_{0|0})-2b_{0|0}(\alpha^2 s_i-y_is_0)+2\alpha(1-s)(\alpha^2 s_{i|0}-y_is_{0|0})$$
$$+\alpha^2(-6s\frac{b_{0|0}}{\alpha}+4bb_{|0})s_{i0}+\alpha^2(1-s^2+2(b^2-s^2))s_{i0|0}-\alpha b_{0|0}((1-s)r_{i0}-4\alpha s_i)$$
$$+\alpha^2(1-s)\Big(\frac{-b_{0|0}}{\alpha}r_{i0}+(1-s)r_{i0|0}-4\alpha s_{i|0}\Big)-2\alpha s_{0|0}h_i$$
$$-y_i\Big(\frac{-b_{0|0}}{\alpha}((1-s)r_{00}-4\alpha s_0)+(1-s)(\frac{-b_{0|0}}{\alpha}r_{00}+(1-s)r_{00|0}-4\alpha s_{0|0})\Big)$$
$$+\Big(\frac{T_3}{M_3}+\frac{2}{1-s}+\frac{12s-8\frac{bb_{|0}}{s_{|0}}}{1-s^2+2(b^2-s^2)}\Big)\Big(-2\alpha s_0h_i+2\alpha(1-s)(\alpha^2 s_i-y_is_0)$$
$$+\alpha^2(1-s^2+2(b^2-s^2))s_{i0}+\alpha^2(1-s)((1-s)r_{i0}-4\alpha s_i)$$
$$-(1-s)((1-s)r_{00}-4\alpha s_0)y_i\Big)s_{|0}\Big]. \tag{2.17}$$

Here, we define

$$T_3 := 8\Big(\frac{bb_{|0}}{s_{|0}}-s\Big)\Big(n(1-2s)+s+1\Big)+4(b^2-s^2)(-2n+1)-(n+1)(12s^2+4s+4).$$

By putting (2.15), (2.16) and (2.17) in (2.14) we have

$$
\begin{aligned}
J_{i|k}y^k = K_1\Big[&(r_{0|0}+s_{0|0})h_i + (r_0+s_0)\Big(\Big(\frac{b_{0|0}}{\alpha s} - \frac{2}{\alpha(1-s^2+2(b^2-s^2)}(-6sb_{0|0}\\
&+2\alpha a^{tj}b_{j|0}b_t + 2\alpha b^t b_{t|0})\Big)h_i + b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}\Big)\Big] + K_2\Big[\Big(\frac{-1}{\alpha}b_{0|0}r_{00} + (1-s)r_{00|0}\\
&-4\alpha s_{0|0}\Big)h_i + ((1-s)r_{00} - 4\alpha s_0)(b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0})\\
&+((1-s)r_{00} - 4\alpha s_0)\Big(\frac{T_2}{M_2} + \frac{1}{1-s} + \frac{3(6s - \frac{4bb_{|0}}{s_{|0}})}{1-s^2+2(b^2-s^2)}\Big)s_{|0}h_i\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+K_3\Big[-2\alpha s_{0|0}h_i - 2\alpha s_0(b_{i|0} - \frac{y_i}{\alpha^2}b_{0|0}) - 2b_{0|0}(\alpha^2 s_i - y_i s_0) + 2\alpha(1-s)(\alpha^2 s_{i|0} - y_i s_{0|0})\\
&+\alpha^2(-6s\frac{b_{0|0}}{\alpha} + 4bb_{|0})s_{i0} + \alpha^2(1-s^2+2(b^2-s^2))s_{i0|0} - \alpha b_{0|0}((1-s)r_{i0} - 4\alpha s_i)\\
&+\alpha^2(1-s)\Big(\frac{-b_{0|0}}{\alpha}r_{i0} + (1-s)r_{i0|0} - 4\alpha s_{i|0}\Big)\\
&-y_i\Big(\frac{-b_{0|0}}{\alpha}((1-s)r_{00} - 4\alpha s_0) + (1-s)(\frac{-b_{0|0}}{\alpha}r_{00} + (1-s)r_{00|0} - 4\alpha s_{0|0})\Big)\\
&+\Big(\frac{T_3}{M_3} + \frac{2}{1-s} + \frac{12s - 8\frac{bb_{|0}}{s_{|0}}}{1-s^2+2(b^2-s^2)}\Big)\Big(-2\alpha s_0 h_i + 2\alpha(1-s)(\alpha^2 s_i - y_i s_0)\\
&+\alpha^2(1-s^2+2(b^2-s^2))s_{i0} + \alpha^2(1-s)((1-s)r_{i0} - 4\alpha s_i)\\
&-(1-s)((1-s)r_{00} - 4\alpha s_0)y_i\Big)s_{|0}\Big]. \qquad (2.18)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
2\frac{\partial J_i}{\partial y^m}\Big(G^m-\bar{G}^m\Big) = \Big(&\frac{2(1-2s)}{\alpha(1+2b^2-3s^2)}(r_{00}-\frac{4\alpha}{1-s}s_0)y^i+\frac{2b^i}{1+2b^2-3s^2}(r_{00}-\frac{4\alpha}{1-s}s_0)\\
&+\frac{4}{1-s}\alpha s_0^i\Big)\Big(K_1\Big[(r_m+s_m)h_i\\
&+(r_0+s_0)\Big(\frac{-\alpha^2b_m+2y_m\beta}{\alpha^4}y_i-\frac{s}{\alpha}a_{im}+h_i\Big(\frac{1}{s}(s)_{y^m}+\frac{2y_m}{\alpha^2}+\frac{12s(s)_{y^m}}{1-s^2+2(b^2-s^2)}\Big)\Big)\Big]\\
&+K_2\Big[\Big((-s)_{y^m}r_{00}+2(1-s)r_{m0}-\frac{4y_ms_0}{\alpha}-4\alpha s_m\Big)h_i\\
&+((1-s)r_{00}-4\alpha s_0)\Big(\frac{-\alpha^2b_m+2y_m\beta}{\alpha^4}y_i-\frac{s}{\alpha}a_{im}\\
&+h_i\Big(\frac{B_2}{M_2}(s)_{y^m}-\frac{3y_m}{\alpha^2}+\frac{(s)_{y^m}}{1-s}+\frac{18s(s)_{y^m}}{1-s^2+2(b^2-s^2)}\Big)\Big)\Big]\\
&+K_3\Big[\frac{-2y_m}{\alpha}s_0h_i-2\alpha s_mh_i-2\alpha s_0(h_i)_{y^m}+(\alpha^2s_i-y_is_0)(2(1-s)\frac{y_m}{\alpha}-2\alpha(s)_{y^m})\\
&+2\alpha(1-s)(2y_ms_i-y_is_m-a_{im}s_0)+(2y_ms_{i0}+\alpha^2s_{im})(1-s^2+2(b^2-s^2))\\
&-6s\alpha^2s_{i0}(s)_{y^m}+((1-s)r_{i0}-4\alpha s_i)(2(1-s)y_m-\alpha^2(s)_{y^m})\\
&+\alpha^2(1-s)(-(s)_{y^m}r_{i0}+(1-s)r_{mi}-\frac{4y_m}{\alpha}s_i)+((1-s)r_{00}-4\alpha s_0)\\
&(y_i(s)_{y^m}-(1-s)a_{im})-(1-s)\Big(-(s)_{y^m}r_{00}+2(1-s)r_{m0}-\frac{4y_m}{\alpha}s_0-4\alpha s_m\Big)y_i\\
&+\Big(\frac{B_3}{M_3}(s)_{y^m}-\frac{3y_m}{\alpha^2}-\frac{2}{1-s}(s)_{y^m}+\frac{12s}{1-s^2+2(b^2-s^2)}(s)_{y^m}\Big)\\
&\Big(-2\alpha s_0h_i+2\alpha(1-s)(\alpha^2s_i-y_is_0)+\alpha^2(1-s^2+2(b^2-s^2))s_{i0}\\
&+\alpha^2(1-s)((1-s)r_{i0}-4\alpha s_i)-(1-s)((1-s)r_{00}-4\alpha s_0)y_i\Big)\Big]\Big),
\end{aligned}
\tag{2.19}
$$

where

$$
\begin{aligned}
B_2 :=\ &-24(2n-3)s^4+8(5n-1)s^3-24(n-1)s^2+16(n-2)b^2s+8(2n-2)s\\
&+2(b^2-s^2)\Big(18(2n-3)s^2-4(5n-1)s+6(n-1)\Big)-60(n+3)s^4\\
&+32(2n+5)s^3+18(n+5)s^2-4(7n+19)s+6(n-1),
\end{aligned}
$$

$$
B_3 := -8(1-2s)ns-8s^2-8s+4(b^2-s^2)(-2n+1)-(n+1)(-12s^2+4s+4),
$$

Also, we have

$$
\begin{aligned}
J_m\Big(N^m_i-\bar{N}^m_i\Big)=&\frac{1-2s}{\alpha(1-s)(1-s^2+2(b^2-s^2))}\Big[K_1(r_0+s_0)((1-s)r_{00}-4\alpha s_0)h_i\\
&+((1-s)r_{00}-4\alpha s_0)^2K_2h_i+K_3((1-s)r_{00}-4\alpha s_0)\Big(-2\alpha s_0h_i\\
&+2\alpha(\alpha^2s_i-y_is_0)+\alpha^2(1-s^2+2(b^2-s^2))s_{i0}+\alpha^2(1-s)((1-s)r_{i0}-4\alpha s_i)\\
&-(1-s)((1-s)r_{00}-4\alpha s_0)y_i\Big)\Big]+\Big[(r_0+s_0)K_1h_m+(r_{00}-4\alpha s_0)K_2h_m\\
&+K_3\Big(-2\alpha s_0h_m+2\alpha(1-s)(\alpha^2s_m-y_ms_0)+\alpha^2(1-s^2+2(b^2-s^2))s_{m0}\\
&+\alpha^2((1-s)r_{m0}-4\alpha s_m)-(1-s)((1-s)r_{00}-4\alpha s_0)y_m\Big)\Big]\\
&\Big[\frac{6s((1-s)r_{00}-4\alpha s_0)(s)_{y^i}}{(1-s)(1-s^2+2(b^2-s^2))}b^m\\
&+\frac{2\alpha(1-s)^2r_{i0}-4(1-s)y_is_0-4\alpha^2s_0(s)_{y^i}-4\alpha^2(1-s)s_i}{\alpha(1-s)^2(1-s^2+2(b^2-s^2))}b^m\\
&+\frac{2\alpha(s)_{y^i}}{(1-s)^2}s^m_0+\frac{2}{1-s}(\frac{y_i}{\alpha}s^m_0+\alpha s^m_i)\Big]. \qquad (2.20)
\end{aligned}
$$

The following hold

$$
\begin{aligned}
(s)_{y^m}&=\frac{b_m}{\alpha}-\frac{\beta}{\alpha^3}y_m,\\
(h_i)_m&=\frac{-\alpha^2b_m+2\beta y_m}{\alpha^4}y_i-\frac{s}{\alpha}a_{im},\\
b_{0|0}&=r_{00},\\
b^jb_{j|0}&=s_0+r_0,\\
bb_{|0}&=r_0+s_0,\\
h_ib^i&=b^2-s^2, \qquad (2.21)\\
s_ib^i&=0,
\end{aligned}
$$

$$
\begin{aligned}
s_{i|0}b^i&=-s_i(r^i_0+s^i_0),\\
s_{i0|0}b^i&=s_{0|0}-s_{i0}\Big(r^i_0+s^i_0\Big),\\
s_{|0}&=\frac{r_{00}}{\alpha},\\
h_{i|0}&=s_{i0}+r_{i0}-\frac{y_i}{\alpha^2}r_{00}.
\end{aligned}
$$

By substitutions (2.18) , (2.19), (2.20) and (2.21) in (2.10) we get $J_{i;k}y^k$.

Now, by computing $J_{i;k}y^k = 0$ we get

$$
\begin{aligned}
&\Big[(1-s)(b^2-s^2)q_2\acute{K_2} + 2s(1-s)\acute{K_3}\alpha + s(1-s)^2q_3\acute{K_3}\alpha + \frac{2(1-2s)(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\\
&- \frac{12s(1-s)^2(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{2(b^2-s^2)^2}{1+2b^2-3s^2}\acute{K_2} - \frac{4s^3(1-s)}{1+2b^2-3s^2}\acute{K_2} - s(1-s)\acute{K_1}\alpha\\
&+ \frac{4sb^2(1-s)}{1+2b^2-3s^2}\acute{K_2} - \frac{2(1-s)(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2} + \frac{6s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\\
&- \frac{4s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha - \frac{2b^2(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{2s(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&- \frac{6s^2(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha - \frac{(1-2s)(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2} + \frac{s(1-2s)(1-s)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&- \frac{6s(1-s)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_2} + \frac{6s^2(1-s)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\Big]r_{00}^2\\
&+ \Big[\frac{(r_0+s_0)(b^2-s^2)}{\beta}\acute{K_1}\alpha^2 + \frac{12s(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha - s(r_0+s_0)\acute{K_1}\alpha - (b^2-s^2)\acute{K_2}\alpha\\
&+ (1-s)(r_0+s_0)\acute{K_2}\alpha + 4ss_0\acute{K_2}\alpha + 4(1-s)(b^2-s^2)(r_0+s_0)\acute{q_2}\acute{K_2}\alpha\\
&- \frac{16(n-2)(1-s)(b^2-s^2)^2(r_0+s_0)}{M_2}\acute{K_2}\alpha - \frac{6(1-2s)(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\acute{K_1}\alpha\\
&- 4s_0(b^2-s^2)q_2\acute{K_2}\alpha - 2ss_0\acute{K_3}\alpha - 8\beta - 2s_0(b^2-s^2)q_3\acute{K_3}\alpha s_0\acute{K_3}\alpha - 2r_0(1-s)\acute{K_3}\alpha^2\\
&+ 2\beta s_0(1-s)q_3\acute{K_3}\alpha + s_0(1-s^2+2(b^2-s^2))q_3\acute{K_3}\alpha^2 + r_0(1-s)^2q_3\acute{K_3}\alpha^2\\
&- \frac{16s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha + \frac{8s_0(b^2-s^2)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{28ss_0(1-s)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2\\
&- \frac{24ss_0\beta(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{16ss_0(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2 + 2s_0(1-s)\acute{K_3}\alpha^2 + 8s(1-s)^2(r_0+s_0)\acute{q_3}\acute{K_3}\alpha^2\\
&- \frac{12s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{36s_0\beta(1-s)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{4s(r_0+s_0)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_1}\alpha\\
&- \frac{2b^2(b^2-s^2)(r_0+s_0)}{\beta(1+2b^2-3s^2)}\acute{K_1}\alpha^2 - \frac{2s(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\acute{K_1}\alpha - \frac{24s(b^2-s^2)^2(r_0+s_0)}{(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))}\acute{K_1}\alpha\\
&- \frac{4r_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha - \frac{40ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha - \frac{32ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha - \frac{2r_0(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2\\
&- \frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2}\alpha - \frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha + \frac{4s_0s^2(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2 - \frac{12s_0b^2(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{24s^2s_0\beta}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{2s_0\beta(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2}\acute{K_3}\alpha-\frac{(1-2s)(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha\\
&-\frac{2r_0\beta(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{8ss_0\beta}{1+2b^2-3s^2}\acute{K_3}\alpha+8ss_0\acute{K_2}\alpha+\frac{4s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_3\acute{K_3}\alpha\\
&-\frac{4ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2-\frac{2s_0(b^2-s^2)(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2-\frac{4\beta s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&-\frac{8s_0(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}\acute{K_2}\alpha+\frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2}\alpha+\frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha^2\\
&+\frac{8s_0b^2(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2-\frac{8ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha^2+\frac{4s_0(b^2-s^2)}{1-s}\acute{K_2}\alpha+\frac{36ss_0\beta(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&-4s_0(b^2-s^2)P_2\acute{K_2}\alpha-8ss_0(1-s)\acute{K_3}\alpha^2-4s_0(1-s)\acute{K_3}\alpha^2+4s_0\beta(1-s)\acute{K_3}\alpha\\
&+\frac{8s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha+\frac{2s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-\frac{6s_0\beta(1-s)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&-s_0(1-2s)\acute{K_3}\alpha^2-\frac{r_0(1-s)^2(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^2-\frac{6s(r_0+s_0)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha\\
&+\frac{48ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha-\frac{2r_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha+\frac{4ss_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha\\
&+\frac{4s_0(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K_2}\alpha-\frac{2s_0(b^2-s^2)}{1-s}\acute{K_2}\alpha+\frac{12ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&-\frac{12ss_0\beta(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-6ss_0(b^2-s^2)\acute{K_3}\alpha^2+\frac{2\beta r_0(1-s)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&-\frac{4ss_0\beta(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-\frac{6sr_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^2\Big]r_{00}+(r_{0|0}+s_{0|0})(b^2-s^2)\acute{K_1}\alpha^2\\
&-\frac{8(r_0+s_0)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha^2+(r_0+s_0)^2\acute{K_1}\alpha^2+(1-s)(b^2-s^2)r_{00|0}\acute{K_2}\alpha\\
&-4s_{0|0}(b^2-s^2)\acute{K_2}\alpha^2-4s_0(r_0+s_0)\acute{K_2}\alpha^2-16s_0(b^2-s^2)(r_0+s_0)\acute{q_2}\acute{K_2}\alpha^2\\
&+\frac{64s_0(n-2)(b^2-s^2)^2(r_0+s_0)}{M_2}\acute{K_2}\alpha^2+4s_0(r_0+s_0)\acute{K_3}\alpha^3+8r_0(1-s)^2(r_0+s_0)\acute{q_3}\acute{K_3}\alpha^3\\
&-2(b^2-s^2)s_{0|0}\acute{K_3}\alpha^2-2s_0(r_0+s_0)\acute{K_3}\alpha^2+2ss_0(1-s)\acute{K_3}\alpha^3+2(1-s)s_m(r_0^m+s_0^m)\acute{K_3}\alpha^4\\
&+(1-s)^2r_{m0|0}b^m\acute{K_3}\alpha^3-\beta(1-s)^2r_{00|0}\acute{K_3}\alpha+4\beta(1-s)s_{0|0}\acute{K_3}\alpha^2-16s_0(r_0+s_0)(b^2-s^2)\acute{q_3}\acute{K_3}\alpha^2\\
&+16\beta s_0(1-s)(r_0+s_0)\acute{q_3}\acute{K_3}\alpha^2+8s_0(r_0+s_0)(1-s^2+2(b^2-s^2))\acute{q_3}\acute{K_3}\alpha^3\\
&+\frac{24s_0(b^2-s^2)(r_0+s_0)(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K_1}\alpha^2+\frac{32s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K_2}\alpha^2\\
&+\frac{96s^2s_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K_3}\alpha^3-\frac{32ss_0^2(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha^3-\frac{64ss_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K_3}\alpha^3\\
&-\frac{16ss_0(r_0+s_0)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K_1}\alpha^2+\frac{8s_0b^2(b^2-s^2)(r_0+s_0)}{s(1-s)(1+2b^2-3s^2)}\acute{K_1}\alpha^2+\frac{8ss_0(b^2-s^2)(r_0+s_0)}{(1-s)(1+2b^2-3s^2)}\acute{K_1}\alpha^2
\end{aligned}
$$

$$+\frac{96ss_0b^2(b^2-s^2)^2(r_0+s_0)}{(1-s)(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2-\frac{64s^3s_0^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2$$
$$+\frac{16r_0s_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K}_2\alpha+\frac{8r_0s_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K}_3\alpha^3-\frac{4ss_0(b^2-s^2)(r_0+s_0)}{1-s}\acute{K}_1\alpha^2$$
$$+\frac{64ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2+\frac{64ss_0^2b^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2-\frac{32s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}P_2\acute{K}_2\alpha^2$$
$$+\frac{32ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^2-\frac{32s_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^2-\frac{16s^2s_0^2}{1+2b^2-3s^2}\acute{K}_3\alpha^3$$
$$+\frac{48s_0^2b^2}{1+2b^2-3s^2}\acute{K}_3\alpha^3+\frac{96s^3s_0^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3-\frac{8ss_0^2(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3$$
$$+\frac{8sr_0s_0(1-s)}{1+2b^2-3s^2}\acute{K}_3\alpha^3+\frac{32s^2s_0^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3-\frac{16s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}P_3\acute{K}_3\alpha^2$$
$$+\frac{16ss_0^2(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K}_3\alpha^3+\frac{8s_0^2(b^2-s^2)(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)}P_3\acute{K}_3\alpha^3$$
$$-\frac{48s^2s_0^2}{1+2b^2-3s^2}\acute{K}_3\alpha^3-\frac{4(b^2-s^2)(r_ms_0^m+s_ms_0^m)}{1-s}\acute{K}_1\alpha^3+\frac{8ss_0(r_0+s_0)}{1-s}\acute{K}_1\alpha^2$$
$$-\frac{48ss_0(b^2-s^2)(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2-8(b^2-s^2)r_{m0}s_0^m\acute{K}_2\alpha^2+\frac{16s_ms_0^m(b^2-s^2)}{1-s}\acute{K}_2\alpha^3$$
$$-\frac{32ss_0^2}{1-s}\acute{K}_2\alpha^2+\frac{8s_0^2(b^2-s^2)}{1-s}P_3\acute{K}_3\alpha-\frac{4s_ms_0^m(1-s^2+2(b^2-s^2))}{1-s}\acute{K}_3\alpha^4-4(1-s)r_ms_0^m\acute{K}_3\alpha^4$$
$$+\frac{16s_0^2(b^2-s^2)}{1-s}P_2\acute{K}_2\alpha^2+\frac{8(b^2-s^2)s_ms_0^m}{1-s}\acute{K}_3\alpha^3-\frac{16ss_0^2}{1-s}\acute{K}_3\alpha^2-24s_0^2\acute{K}_3\alpha^3+8ss_ms_0^m\acute{K}_3\alpha^4$$
$$-8s(1-s)s_0^mr_{m0}\acute{K}_3\alpha^3+\frac{16ss_ms_0^m}{1-s}\acute{K}_3\alpha^4+\frac{4s_0(r_0+s_0)(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2$$
$$-8ss_0^2P_3\acute{K}_3\alpha^2-\frac{4s_0^2(1-s^2+2(b^2-s^2))}{1-s}P_3\acute{K}_3\alpha^3-4r_0s_0(1-s)P_3\acute{K}_3\alpha^3+\frac{4s_0^2}{1-s}\acute{K}_3\alpha^3$$
$$-\frac{16s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2-\frac{8s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2+\frac{8ss_0^2(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^3$$
$$+\frac{4s_0r_0(1-s)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^3+\frac{24ss_0(b^2-s^2)^2(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2-\frac{2r_0(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)}\acute{K}_1\alpha^2$$
$$+\frac{4ss_0(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_1\alpha^2+\frac{4s_0(b^2-s^2)^2(r_0+s_0)}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2-\frac{2s_0(r_0+s_0)(b^2-s^2)}{(1-s)^2}\acute{K}_1\alpha^2$$
$$-\frac{96ss_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha+\frac{8r_0s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_2\alpha^2-\frac{16ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2$$
$$-\frac{16s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2-\frac{48ss_0^2(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2+\frac{4r_0s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^2$$
$$-\frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2-\frac{8s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2+\frac{4s_0^2(b^2-s^2)}{(1-s)^2}\acute{K}_3\alpha^2$$
$$-\frac{48s^2s_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^3+\frac{4ss_0^2}{1-s}\acute{K}_3\alpha^3-\frac{2s(1-s^2+2(b^2-s^2))}{1-s}s_{m0}s_0^m\acute{K}_3\alpha^3$$
$$+\frac{4sr_0s_0(1-s)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^3-\frac{8s^2s_0^2}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^3-\frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^3$$

$$
\begin{aligned}
&-\frac{4s_m s_0^m(b^2-s^2)}{1-s}\acute{K_3}\alpha^4-4ss_m s_0^m\acute{K_3}\alpha^4-4s_m s^m\acute{K_3}\alpha^5+4s_0^2\acute{K_3}\alpha^3+\frac{24ss_0^2(b^2-s^2)}{1-s}\acute{K_3}\alpha^3\\
&-2r_0s_0\acute{K_3}\alpha^3+\frac{4s_0^2(b^2-s^2)}{(1-s)^2}\acute{K_3}\alpha^3-\frac{2(b^2-s^2)(1-s^2+2(b^2-s^2))}{(1-s)^2}s_{m0}s_0^m\acute{K_3}\alpha^3\\
&+\frac{4ss_0r_0}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-\frac{2(1-s^2+2(b^2-s^2))}{1-s}s_{m0}s_0^m\acute{K_3}\alpha^4+\frac{24ss_0r_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3\\
&-\frac{2r_0(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-2sr_{m0}s_0^m\acute{K_3}\alpha^3+\frac{16s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K_3}\alpha^2\\
&+\frac{4r_0s_0(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K_3}\alpha^3-\frac{2(b^2-s^2)}{1-s}r_{m0}s_0^m\acute{K_3}\alpha^3+\frac{8(b^2-s^2)}{(1-s)^2}s_m s_0^m\acute{K_3}\alpha^4\\
&+\frac{8ss_m s_0^m}{1-s}\acute{K_3}\alpha^4-2s^m r_{m0}\acute{K_3}\alpha^4+\frac{8s_m s^m}{1-s}\acute{K_3}\alpha^5+\frac{96s^2s_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-8s_0^2\acute{K_3}\alpha^3\\
&+\frac{8ss_0r_0(1-s)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{16s^2s_0^2(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{16ss_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3\\
&+(1-s^2+2(b^2-s^2))\Big(s_{0|0}-s_{m0}(s_0^m+r_0^m)\Big)\acute{K_3}\alpha^3=0.
\end{aligned}
$$

The above relationship is equivalent to (1.14) and (1.15). □

## 3. Appendix

$$
\begin{aligned}
A_1&=(1-s)(b^2-s^2)q_2\acute{K_2}+\frac{2(1-2s)(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}+\frac{2(b^2-s^2)^2}{1+2b^2-3s^2}\acute{K_2}\\
&+\frac{4s^3(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}-\frac{6s(1-s)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_2}-\frac{2(1-s)(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2}\\
&+\frac{6s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}-\frac{(1-2s)(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2}
\end{aligned}
$$

$$
\begin{aligned}
A_2&=s(1-s)^2q_3\acute{K_3}\alpha-s(1-s)\acute{K_1}\alpha-\frac{12s(1-s)^2(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha-\frac{4s(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&+\frac{s(1-s)^2(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-\frac{2b^2(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{2s(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&-\frac{6s^2(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{6s^2(1-s)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha+2s(1-s)\acute{K_3}\alpha
\end{aligned}
$$

$$
\begin{aligned}
B_1&=\frac{(r_0+s_0)(b^2-s^2)}{\beta}\acute{K_1}\alpha^2+s_0(1-s^2+2(b^2-s^2))q_3\acute{K_3}\alpha^2+r_0(1-s)^2q_3\acute{K_3}\alpha^2\\
&+\frac{28ss_0(1-s)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2-2r_0(1-s)\acute{K_3}\alpha^2-\frac{r_0(1-s)^2(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^2\\
&+\frac{16ss_0(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2+2s_0(1-s)\acute{K_3}\alpha^2-\frac{2b^2(b^2-s^2)(r_0+s_0)}{\beta(1+2b^2-3s^2)}\acute{K_1}\alpha^2+\frac{4s^2s_0(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{12s_0b^2(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2-\frac{4ss_0(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2-\frac{2s_0(b^2-s^2)(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2\\
&-\frac{2r_0(1-s)^2(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^2+\frac{16ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha^2+\frac{8s_0b^2(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha^2\\
&-\frac{8ss_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha^2-8ss_0(1-s)\acute{K_3}\alpha^2-4s_0(1-s)\acute{K_3}\alpha^2-s_0(1-2s)\acute{K_3}\alpha^2\\
&-6ss_0(b^2-s^2)\acute{K_3}\alpha^2-\frac{6sr_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^2+8s(1-s)^2(r_0+s_0)\acute{q_3}\acute{K_3}\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
B_2=&\frac{12s(b^2-s^2)(r_0+s_0)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha-s(r_0+s_0)\acute{K_1}\alpha-(b^2-s^2)\acute{K_2}\alpha+(1-s)(r_0+s_0)\acute{K_2}\alpha\\
&+4ss_0\acute{K_2}\alpha-8\beta s_0\acute{K_3}\alpha-2s_0(b^2-s^2)q_3\acute{K_3}\alpha-\frac{24ss_0\beta(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha-2ss_0\acute{K_3}\alpha\\
&-\frac{16(n-2)(1-s)(b^2-s^2)^2(r_0+s_0)}{M_2}\acute{K_2}\alpha-4s_0(b^2-s^2)q_2\acute{K_2}\alpha-\frac{72ss_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha\\
&-\frac{6(1-2s)(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\acute{K_1}\alpha-\frac{16s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha+\frac{8s_0(b^2-s^2)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&-\frac{12s_0(1-2s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{36s_0\beta(1-s)(1-2s)}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{4s(r_0+s_0)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_1}\alpha\\
&+\frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2}\alpha-\frac{16ss_0}{1+2b^2-3s^2}\acute{K_3}\alpha-\frac{24s_0s^2\alpha\beta}{1+2b^2-3s^2}\acute{K_3}+\frac{2s_0^2(1-s^2+2(b^2-s^2))}{1+2b^2-3s^2}\acute{K_3}\alpha\\
&-\frac{2r_0\beta(1-s)^2}{1+2b^2-3s^2}\acute{K_3}\alpha+\frac{8ss_0\beta}{1+2b^2-3s^2}\acute{K_3}\alpha+8ss_0\acute{K_2}\alpha+\frac{4s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_3\acute{K_3}\alpha\\
&+4(1-s)(b^2-s^2)(r_0+s_0)\acute{q_2}\acute{K_2}\alpha+2\beta s_0(1-s)q_3\acute{K_3}\alpha-\frac{4\beta s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha\\
&+\frac{36ss_0\beta(1-s)}{1+2b^2-3s^2}\acute{K_3}\alpha-\frac{8s_0(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}\acute{K_2}\alpha-\frac{(1-2s)(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha\\
&+\frac{8s_0(b^2-s^2)^2}{1+2b^2-3s^2}P_2\acute{K_2}\alpha+\frac{4s_0(b^2-s^2)}{1-s}\acute{K_2}\alpha-4s_0(b^2-s^2)P_2\acute{K_2}\alpha+4s_0\beta(1-s)\acute{K_3}\alpha\\
&-\frac{6s_0\beta(1-s)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-\frac{6s(r_0+s_0)(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha+\frac{48ss_0(b^2-s^2)^2}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha\\
&-\frac{2r_0(1-s)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha+\frac{4ss_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha+\frac{4s_0(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K_2}\alpha\\
&-\frac{2s_0(b^2-s^2)\alpha}{1-s}\acute{K_2}+\frac{12ss_0(b^2-s^2)^2\alpha}{1-s^2+2(b^2-s^2)}\acute{K_3}-\frac{12ss_0\beta\alpha}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha+\frac{2r_0\beta(1-s)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\\
&-\frac{4ss_0\beta(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha-\frac{2s(b^2-s^2)(r_0+s_0)}{1+2b^2-3s^2}\acute{K_1}\alpha-\frac{24s(b^2-s^2)^2(r_0+s_0)}{(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))}\acute{K_1}\alpha\\
&-\frac{4r_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}\acute{K_2}\alpha+\frac{8s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_2}\alpha+\frac{2s_0(b^2-s^2)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha
\end{aligned}
$$

$$
\begin{aligned}
C_1=&(r_{0|0}+s_{0|0})(b^2-s^2)\acute{K_1}\alpha^2-\frac{8(r_0+s_0)^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_1}\alpha^2+(r_0+s_0)^2\acute{K_1}\alpha^2-4s_{0|0}(b^2-s^2)\acute{K_2}\alpha^2\\
&-4s_0(r_0+s_0)\acute{K_2}\alpha^2-16s_0(b^2-s^2)(r_0+s_0)\acute{q_2}\acute{K_2}\alpha^2+\frac{64s_0(n-2)(b^2-s^2)^2(r_0+s_0)}{M_2}\acute{K_2}\alpha^2\\
&-2(b^2-s^2)s_{0|0}\acute{K_3}\alpha^2+\frac{96ss_0b^2(b^2-s^2)^2(r_0+s_0)}{(1-s)(1+2b^2-3s^2)(1-s^2+2(b^2-s^2))}\acute{K_1}\alpha^2-2s_0(r_0+s_0)\acute{K_3}\alpha^2\\
&+2(1-s)s_m(r_0^m+s_0^m)\acute{K_3}\alpha^4+4\beta(1-s)s_{0|0}\acute{K_3}\alpha^2-16s_0(r_0+s_0)(b^2-s^2)\acute{q_3}\acute{K_3}\alpha^2\\
&+16s_0\beta(1-s)(r_0+s_0)\acute{q_3}\acute{K_3}\alpha^2+\frac{24s_0(b^2-s^2)(r_0+s_0)(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K_1}\alpha^2+\frac{32s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K_2}\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8ss_0(r_0+s_0)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_1\alpha^2+\frac{8s_0b^2(b^2-s^2)(r_0+s_0)}{s(1-s)(1+2b^2-3s^2)}\acute{K}_1\alpha^2+\frac{16s_0^2(1-2s)(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^2\\
&-\frac{64s_0^2s^3}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2+\frac{64ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2+\frac{64ss_0^2b^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_2\alpha^2\\
&-\frac{32s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}P_2\acute{K}_2\alpha^2+\frac{32ss_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^2-\frac{32s_0^2(b^2-s^2)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^2\\
&-\frac{16s_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}P_3\acute{K}_3\alpha^2+\frac{8ss_0(r_0+s_0)}{1-s}\acute{K}_1\alpha^2-\frac{4ss_0(b^2-s^2)(r_0+s_0)}{1-s}\acute{K}_1\alpha^2\\
&-\frac{48ss_0(b^2-s^2)(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2-8(b^2-s^2)r_{m0}s_0^m\acute{K}_2\alpha^2-\frac{32ss_0^2}{1-s}\acute{K}_2\alpha^2+\frac{16s_0^2(b^2-s^2)}{1-s}P_2\acute{K}_2\alpha^2\\
&-\frac{16ss_0^2}{1-s}\acute{K}_3\alpha^2+8ss_ms_0^m\acute{K}_3\alpha^4-\frac{4s_ms_0^m(1-s^2+2(b^2-s^2))}{1-s}\acute{K}_3\alpha^4-4(1-s)r_ms_0^m\acute{K}_3\alpha^4\\
&+\frac{16ss_ms_0^m}{1-s}\acute{K}_3\alpha^4-8ss_0^2P_3\acute{K}_3\alpha^2+\frac{4s_0(r_0+s_0)(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2\\
&-\frac{16s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2-2s^mr_{m0}\acute{K}_3\alpha^4\\
&-\frac{8s_0^2(b^2-s^2)(1-2s)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2+\frac{24ss_0(b^2-s^2)^2(r_0+s_0)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2+\frac{8ss_ms_0^m}{1-s}\acute{K}_3\alpha^4\\
&-\frac{2r_0(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_1\alpha^2+\frac{4ss_0(r_0+s_0)(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_1\alpha^2+\frac{4s_0(b^2-s^2)^2(r_0+s_0)}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_1\alpha^2\\
&-\frac{2s_0(r_0+s_0)(b^2-s^2)}{(1-s)^2}\acute{K}_1\alpha^2+\frac{8s_0r_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_2\alpha^2-\frac{16ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2\\
&-\frac{16s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_2\alpha^2-\frac{48ss_0^2(b^2-s^2)^2}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2+\frac{4r_0s_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K}_3\alpha^2\\
&+\frac{4s_0^2(b^2-s^2)}{(1-s)^2}\acute{K}_3\alpha^2-\frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2-\frac{8s_0^2(b^2-s^2)^2}{(1-s)^2(1-s^2+2(b^2-s^2))}\acute{K}_3\alpha^2\\
&-\frac{4s_ms_0^m(b^2-s^2)}{1-s}\acute{K}_3\alpha^4-4ss_ms_0^m\acute{K}_3\alpha^4-\frac{2(1-s^2+2(b^2-s^2))}{1-s}s_{m0}s_0^m\acute{K}_3\alpha^4+\frac{8(b^2-s^2)}{(1-s)^2}s_ms_0^m\acute{K}_3\alpha^4
\end{aligned}
$$

$$
\begin{aligned}
C_2=\;&(1-s)(b^2-s^2)r_{00|0}\acute{K}_2\alpha+2ss_0(1-s)\acute{K}_3\alpha^3+4s_0(r_0+s_0)\acute{K}_3\alpha^3+(1-s)^2r_{m0|0}b^m\acute{K}_3\alpha^3\\
&-\beta(1-s)^2r_{00|0}\acute{K}_3\alpha+8s_0(r_0+s_0)(1-s^2+2(b^2-s^2))\acute{q}_3\acute{K}_3\alpha^3+8r_0(1-s)^2(r_0+s_0)\acute{q}_3\acute{K}_3\alpha^3\\
&-\frac{32ss_0^2(1-2s)}{1+2b^2-3s^2}\acute{K}_3\alpha^3-\frac{64ss_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3+\frac{16r_0s_0(b^2-s^2)}{1+2b^2-3s^2}\acute{K}_2\alpha\\
&+\frac{96s^2s_0^2(1-2s)}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3-\frac{16s^2s_0^2}{1+2b^2-3s^2}\acute{K}_3\alpha^3+\frac{48s_0^2b^2}{1+2b^2-3s^2}\acute{K}_3\alpha^3\\
&+\frac{96s^3s_0^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3+\frac{8r_0s_0(1-s)(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K}_3\alpha^3+\frac{8(b^2-s^2)s_ms_0^m}{1-s}\acute{K}_3\alpha^3\\
&-\frac{8ss_0^2(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3+\frac{8sr_0s_0(1-s)}{1+2b^2-3s^2}\acute{K}_3\alpha^3+\frac{32s^2s_0^2}{(1-s)(1+2b^2-3s^2)}\acute{K}_3\alpha^3
\end{aligned}
$$

$$+\frac{16ss_0^2(b^2-s^2)}{1+2b^2-3s^2}P_3\acute{K_3}\alpha^3+\frac{8s_0^2(b^2-s^2)(1-s^2+2(b^2-s^2))}{(1-s)(1+2b^2-3s^2)}P_3\acute{K_3}\alpha^3$$
$$-\frac{48s^2s_0^2}{1+2b^2-3s^2}\acute{K_3}\alpha^3-\frac{4(b^2-s^2)(r_ms_0^m+s_ms_0^m)}{1-s}\acute{K_1}\alpha^3+\frac{16s_ms_0^m(b^2-s^2)}{1-s}\acute{K_2}\alpha^3$$
$$-24s_0^2\acute{K_3}\alpha^3-8s(1-s)s_0^mr_{m0}\acute{K_3}\alpha^3+\frac{8s_0^2(b^2-s^2)}{1-s}P_3\acute{K_3}\alpha-\frac{4s_0^2(1-s^2+2(b^2-s^2))}{1-s}P_3\acute{K_3}\alpha^3$$
$$-4r_0s_0(1-s)P_3\acute{K_3}\alpha^3+\frac{8ss_0^2(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{4s_0^2}{1-s}\acute{K_3}\alpha^3+\frac{4r_0s_0(1-s)(1-2s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3$$
$$-\frac{96ss_0^2(b^2-s^2)^2}{(1-s)(1+2b^2-3s^2)}\acute{K_2}\alpha-\frac{48s^2s_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{4ss_0r_0(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3$$
$$-\frac{8s^2s_0^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-\frac{8ss_0^2(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K_3}\alpha^3+\frac{4ss_0^2}{1-s}\acute{K_3}\alpha^3-4s^ms_m\acute{K_3}\alpha^5$$
$$+4s_0^2\acute{K_3}\alpha^3+\frac{24ss_0^2(b^2-s^2)}{1-s}\acute{K_3}\alpha^3-2r_0s_0\acute{K_3}\alpha^3+\frac{4s_0^2(b^2-s^2)}{(1-s)^2}\acute{K_3}\alpha^3-8s_0^2\acute{K_3}\alpha^3$$
$$-\frac{2r_0(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-\frac{2(b^2-s^2)(1-s^2+2(b^2-s^2))}{(1-s)^2}s_{m0}s_0^m\acute{K_3}\alpha^3$$
$$+\frac{4ss_0r_0}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{24ss_0r_0(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{4r_0s_0(b^2-s^2)}{(1-s)(1-s^2+2(b^2-s^2))}\acute{K_3}\alpha^3$$
$$-\frac{2(b^2-s^2)}{1-s}r_{m0}s_0^m\acute{K_3}\alpha^3-2sr_{m0}s_0^m\acute{K_3}\alpha^3+\frac{8s_ms^m}{1-s}\acute{K_3}\alpha^5+\frac{96s^2s_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3$$
$$+\frac{8ss_0r_0(1-s)^2}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\frac{16s^2s_0^2(1-s)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3-\frac{2s(1-s^2+2(b^2-s^2))}{1-s}s_{m0}s_0^m\acute{K_3}\alpha^3$$
$$+\frac{16ss_0^2(b^2-s^2)}{1-s^2+2(b^2-s^2)}\acute{K_3}\alpha^3+\Big(1-s^2+2(b^2-s^2)\Big)\Big(s_{0|0}-s_{m0}(s_0^m+r_0^m)\Big)\acute{K_3}\alpha^3$$

and

$$\vartheta(s,b^2)=6(2n-3)s^3-2(5n-1)s^2+6(n-1)s-4(n-2)b^2+4(n-1)$$
$$N_2(s,b^2)=2(b^2-s^2)\Big(18(2n-3)s^2-4(5n-1)s+6(n-1)\Big)-60(n+3)s^4$$
$$-32(2n+5)s^3+18(n+5)s^2$$
$$N_3(s,b^2)=-8s(n(1-2s)+s+1)+4(b^2-s^2)(-2n+1)-(n+1)(-12s^2+4s+4)$$
$$M_3(s,b^2)=4(b^2-s^2)(n(1-2s)+s+1)-(n+1)(-4s^3+2s^2+4s-2)$$
$$q_2(s,b^2)=\frac{N_2(s,b^2)}{M_2(s,b^2)}+\frac{1}{1-s}+\frac{18s}{1-s^2+2(b^2-s^2)}$$
$$\acute{q_2}(s,b^2)=\frac{\vartheta(s,b^2)}{M_2(s,b^2)}-\frac{3}{1-s^2+2(b^2-s^2)}$$
$$q_3(s,b^2)=\frac{N_3(s,b^2)}{M_3(s,b^2)}+\frac{2}{1-s}+\frac{12s}{1-s^2+2(b^2-s^2)}$$
$$\acute{q_3}(s,b^2)=\frac{n(1-2s)+s+1}{M_3}-\frac{1}{1-s^2+2(b^2-s^2)}$$

$$P_2(s,b^2)=\frac{1}{1-s}+\frac{18s}{1-s^2+2(b^2-s^2)}$$
$$P_3(s,b^2)=\frac{-2}{1-s}+\frac{12s}{1-s^2+2(b^2-s^2)}$$
$$\acute{K}_1(s,b^2)=\alpha^2K_1(s,b^2),\quad \acute{K}_2=\alpha^3K_2,\quad \acute{K}_3=\alpha^3K_3$$

## References


1. R.V. Ambartzumian, *A note on pseudo-metrics on the plane*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **37**(1976), 145-155.
2. R. Alexander, *Planes for which the lines are the shortest paths between points*, Illinois J. of Math., **22**(1978), 177-190.
3. L. Berwald, *On Finsler and Cartan geometries III, Two-dimensional Finsler spaces with rectilinear extremals*, Ann. of Math. **42**(1941), 84-112.
4. W. Blaschke, *Integral geometrie 11: Zur Variationsrechnung*, Abh. Math. Sem. Univ. Hamburg, **11**(1936), 359-366.
5. H. Busemann, *Problem IV: Desarguesian spaces*, in Mathematical Developments arising from Hilbert Problems, Proc. Symp. Pure Math. **28**(1976), Amer. Math. Soc., Providence, RI, 131-141.
6. G. Hamel, *Über die Geometrien, in denen die Geraden die Kürtzesten sind*, Math. Ann. **57**(1903), 231-264.
7. Y. Li and X. Mo, *On dually flat square metrics*, Differ. Geom. Appl. **62**(2019), 60-71.
8. A.V. Pogorelov, *Hilbert's Fourth Problem*, Winston & Wiley, New York, 1982.
9. A. Rapcsák, *Über die bahntreuen Abbildungen metrisher Räume*,
10. Z. Shen and G. Yang, *On square metrics of scalar flag curvature*, arXiv:1302.3156v1.
11. Y. Shen and C. Yu, *On Einstein square metrics*, Publ. Math. Debrecen. **85**(3-4) (2014), 413-424.
12. Z.I. Szabó, *Hilbert's fourth problem*, Adv. Math. **59**(1986), 185-301.


Received: 01.10.2023
Accepted: 08.12.2023