Journal of Finsler Geometry and its Applications
Vol. 4, No. 2 (2023), pp 64-73
https://doi.org/10.22098/jfga.2023.13783.1105

On non-Riemannian quantities in Finsler geometry

Nasrin Sadeghzadeh*

Department of Mathematics, Faculty of Science,
University of Qom, Qom, Iran

E-mail: nsadeghzadeh@qom.ac.ir

Abstract. This paper introduces new non-Riemannian quantities and classes of Finsler metrics. The study focuses on the class of generalized Douglas-Weyl metrics, which is contained in the class of Finsler metrics. The paper constructs the new sub-classes of generalized Douglas-Weyl metrics and presents illustrative examples.

Keywords: Douglas metric, generalized Douglas-Weyl metric, generalized Berwald-projective Weyl metric, $\bar{D}$-metric.

1. **Introduction**

Projective Finsler geometry studies equivalent Finsler metrics on the same manifold with the same geodesics as points. Two regular metrics on a manifold M are called projectively related if they have the same geodesics as the point sets. In this context, a geodesic curve in a Finsler space is defined by a second-order system of differential equations. A geodesic curve $c(t)$ in a Finsler space (M, F), is defined by the second order system of differential equations

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0,$$

where G^i are local functions on TM.

---

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

Within the context of Finsler metrics, well-known projective invariants are represented by Douglas curvature, Weyl curvature [1], and the generalized Douglas-Weyl curvature [3].

The C-projective invariant H-curvature was brought forth by Akbar-Zadeh. C-projective Weyl curvature ($\widetilde{W}$-curvature), the new C-projectively invariant quantity which characterizes Finsler metrics of constant flag curvature is presented in [9].

The Finsler metrics satisfying,

$$D_j{}^i{}_{kl|m}y^m = T_{jkl}y^i,$$

for some tensor T_{jkl}, where $D_j{}^i{}_{kl|m}$ denotes the horizontal covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connection of F, are called GDW-metrics [3]. Although, all Douglas metrics are of GDW type, there are many GDW Finsler metrics which are not of Douglas type. The following example presents a GDW-metric which is not of Douglas type.

Example 1.1. ([6]) Put

$$\Omega = \{(x,y,z) \in R^3 | x^2+y^2+z^2 < 1\}, \quad p=(x,y,z)\in\Omega, \quad y=(u,v,w)\in T_p\Omega.$$

Define the Randers metric $F = \alpha + \beta$ by

$$\alpha = \frac{\sqrt{(-yu+xv)^2 + (u^2+v^2+w^2)(1-x^2-y^2)}}{1-x^2-y^2}, \quad \beta = \frac{-yu+xv}{1-x^2-y^2}.$$

The above Randers metric has vanishing flag curvature $K = 0$ and S-curvature $\mathbf{S} = 0$. F has zero Weyl curvature then F is of GDW metric. But β is not closed then F is not of Douglas type.

On the other hands, the class of Douglas metrics contains all Riemannian metrics and the locally projectively flat Finsler metrics. But, there are many Douglas metrics which are not Riemannian. There are many Douglas metrics which are not locally projectively flat, too.
The following example presents a Douglas metric which is not locally projectively flat.

Example 1.2. ([16]) Define α and β by

$$\widetilde{\alpha} = \eta^{\frac{m}{1-m}}\alpha, \qquad \widetilde{\beta} = \eta^{-1}\beta,$$

for some $\eta = \eta(x)$ and $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$ where $\widetilde{\alpha}$ and $\widetilde{\beta}$

$$\widetilde{\alpha} = \sqrt{\frac{|y|^2}{|u|^2}}, \qquad \widetilde{\beta} = \frac{<x,y>}{|u|^2},$$

and $u = (u^1(x), ..., u^n(x))$ is a vector satisfying the following

$$u^i = -2(\lambda + t <f,x>)x^i + t|x|^2 f^i + f^i,$$

where t is a constant and f is a constant vector satisfying $tf \neq 0$ and $\lambda^2 + t|f|^2 \neq 0$. Then the m-Kropina metric $F = \alpha^m \beta^{1-m}$ is Douglasian but not locally projectively flat, where $m \neq 0, 1$.

Based on Douglas curvature, a new class of Finsler metrics so called $\bar{D}$-metrics is introduced which includes all the Douglas metrics. A Finsler metric F is called $\bar{D}$-metric if $D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = 0$ or equivalently $D_j{}^i{}_{kl|m} y^m = 0$ which has been introduced in [10]. Clearly, the class of $\bar{D}$-metrics contains all Douglas metrics but there are many $\bar{D}$-metrics which are not Douglas. The paper will delve into the class of $\bar{D}(M)$ and explore the fascinating metrics it encompasses.

R-quadratic Finsler metrics are another interesting class of Finsler metrics which are the subset of the class of GDW-metrics. The Riemann curvature is one of the important quantities, in Finsler geometry. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations

$$\mathbf{R}_y : T_x M \to T_x M,$$

where $y \in T_x M$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, for every $\lambda > 0$. R-quadratic Finsler spaces form a rich class of Finsler spaces which were introduced by Z. Shen and could be considered as a generalization of Berwald metrics. A Finsler metric (M, F) is called R-quadratic if its Riemann curvature R_y is quadratic in $y \in T_x M$. In [8], it is proved that every R-quadratic Finsler metric is a GDW-metric.

This paper also studies a new quantity in Finsler geometry, so-called generalized Berwald projective Weyl ($GB\widetilde{W}$) curvature, which is a C-projective invariant. For a manifold M, let GB$\widetilde{W}(M)$ denotes the class of all Finsler metrics satisfying

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl} y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$; where $\beta_j{}^i{}_{kl|m} y^m = 0$. An investigation has been undertaken, providing detailed information as presented in [11].

A natural question that could be raised is: "How big is the class of GB$\widetilde{W}(M)$ and what sorts of intriguing metrics does it feature?" It is clear that all Berwald metrics belong to this class. However, the Berwald metrics are not C-projective invariants. It is shown that the class of $GB\widetilde{W}$-metrics is the proper subset of the class of GDW-metrics.

In the paper, the vertical and horizontal derivatives with respect to the Berwald connection are denoted by "$_.$" and "$_|$", respectively and for convenience, we use the following notations. For Finsler manifold (M, F),
(i) $D(M)$ denotes the class of all Douglas metrics,
(ii) $B(M)$ denotes the class of all Berwald metrics,
(iii) $\bar{D}(M)$ denotes the class of all $\bar{D}$-metrics,
(vi) $GB\widetilde{W}(M)$ denotes the class of all Generalized Berwald Projective Weyl

$(GB\widetilde{W})$-metrics,
(v) $GDW(M)$ denotes the class of all Generalized Douglas Weyl (GDW)-metrics on the manifold M.

2. **Preliminaries**

A Finsler metric on a manifold M is a nonnegative function F on TM with the following properties

(1) F is C^∞ on $TM \setminus \{0\}$;
(2) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(3) For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big] \mid_{s,t=0}, \qquad u, v \in T_xM. \tag{2.1}$$

At each point $x \in M$, $F_x := F \mid_{T_xM}$, is an Euclidean norm, if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big] \mid_{t=0}, \qquad u, v, w \in T_xM. \tag{2.2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion.* A curve $c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(\dot{c}(t)) = 0, \tag{2.3}$$

where, $G^i(y)$ denotes local functions on TM given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}, \quad y \in T_xM. \tag{2.4}$$

G^is called the associated spray to (M, F).

F is called a *Berwald metric* if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$. Define

$$B_y : T_xM \times T_xM \times T_xM \to T_xM$$
$$B_y(u,v,w) = B_j{}^i{}_{kl}u^jv^kw^l\frac{\partial}{\partial x^i},$$

where

$$B^i_{j\,kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l},$$

and

$$E_y : T_xM \times T_xM \to \mathbb{R}$$
$$E_y(u,v) = E_{jk}u^jv^k,$$

where

$$E_{jk} = \frac{1}{2}B_j{}^m{}_{km},$$

$u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and the mean Berwald curvature, respectively. F is called a Berwald metric and weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [13].

By means of E-curvature, we can define $\bar{E}$-curvature as follows

$$\bar{E}_y : T_xM \times T_xM \times T_xM \longrightarrow \mathbb{R}$$

$$\bar{E}_y(u, v, w) := \bar{E}_{jkl}(y)u^iv^jw^k = E_{jk|l}u^jv^kw^l.$$

It is worth noting that $\bar{E}_{ijk}$ is not completely symmetric with respect to all three indices.

To define the H-curvature, we take the covariant derivative of E along geodesics. Specifically, $H_{ij} = E_{ij|m}y^m$,

$$H_y : T_xM \times T_xM \longrightarrow \mathbb{R}$$

$$H_y(u, v) := H_{ij}u^iv^j$$

Let

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{1}{n+1}\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}(\frac{\partial G^m}{\partial y^m}y^i).$$

It is easy to verify that $D := D_j{}^i{}_{kl}dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call D the Douglas tensor, which is a non-Riemannian projective invariant.

For two Finsler metrics, F and $\bar{F}$, with the geodesic coefficients G^i and $\bar{G}^i$, respectively, the diffeomorphism $\phi : F \to \bar{F}$ is called projective mapping if there exists a positive homogeneous scalar function $P = P(x, y)$ of degree one which is satisfying

$$\bar{G^i} = G^i + Py^i,$$

where, $P = P(x, y)$ is positively y-homogeneous of degree one which called projective factor [5] and [13].

A projective transformation with projective factor P would be C-projective if $Q_{ij} = 0$, where

$$Q_{ij} = \frac{\partial Q_j}{\partial y^i} - \frac{\partial Q_i}{\partial y^j},$$

$$Q_i = \frac{\partial P}{\partial x^i} - G^m_i\frac{\partial P}{\partial y^m} - P\frac{\partial P}{\partial y^i}.$$

One could easily show that

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jkl}y^i\}, \tag{2.5}$$

where $E_{jkl} = \frac{\partial E_{jk}}{\partial y^l}$.

A Finsler metric is called $\bar{D}$-metric if $\bar{D}_j{}^i{}_{klm} = 0$, where

$$\bar{D}_j{}^i{}_{klm} = D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l}. \tag{2.6}$$

Clearly, this class of metrics includes all Douglas metrics.

There is another projective invariant equation in Finsler geometry for some tensors T_{jkl}

$$D_j{}^i{}_{kl|m} y^m = T_{jkl} y^i,$$

where $D_j{}^i{}_{kl|m}$ denotes the horizontal covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connection of F. These metrics are called GDW metrics.

In [15], Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extends Weyl's projective invariant to Finsler metrics [7]. Finsler metrics with vanishing projective Weyl curvature are called Weyl metrics or W metrics. In [14], Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature. In [9], a new C-projective invariant quantity is defined, namely $\widetilde{W}$ curvature, as follows

$$\widetilde{W}^i_k = K^i_k - \frac{1}{1-n^2}\{y^i \widetilde{K}_{0k} - \delta^i_k \widetilde{K}_{00}\},$$

where

$$\widetilde{K}_{jk} = nK_{jk} + K_{kj} + y^r \frac{\partial K_{kr}}{\partial y^j}.$$

$\widetilde{W}^i{}_k$ is called projective Weyl curvature or $\widetilde{W}$ curvature which is another candidate for characterizing the Finsler metrics of constant flag curvature.
Now a new C-projective invariant equation is introduced as follows

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl} y^i, \tag{2.7}$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$ where $\beta_j{}^i{}_{kl|m} y^m = 0$, or equivalently,

$$h^i_r B_j{}^r{}_{kl|m} y^m = 0.$$

Finsler metrics satisfying (2.7) are called $GB\widetilde{W}$ metrics.

There are numerous Finsler metrics that belong to the GB$\widetilde{W}$ type. Specifically, all Berwald metrics and $\widetilde{W}$ metrics, as well as Finsler metrics with a constant flag curvature ($n > 2$), are included in this class of metrics [11].

3. Non-Riemannian quantities in Finsler geometry

This section of this research studies the classes of $\bar{D}(M)$ and $GB\widetilde{W}(M)$ on a smooth manifold M and their intriguing metrics. The study aims to answer the natural question: "How big are these classes?" By exploring the properties and characteristics of these classes, here, we attempt to prove the main theorems of this research.

Theorem 3.1. *For a Finsler manifold (M, F), $D(M)$ is a proper subset of $\bar{D}(M)$ and $\bar{D}(M)$ is a proper subset of $GDW(M)$.*

Proof. The condition (2.6) ensures the fulfillment of every Douglas metric, $D_j{}^i{}_{kl} = 0$, making them qualify as $\bar{D}$-metrics. Now, consider a Finsler metric F in $\bar{D}(M)$, so that we obtain

$$D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = 0.$$

The contraction of the aforementioned equation with y^m leads to the result $D_j{}^i{}_{kl|0} = 0$, signifying that F qualifies as a GDW-metric.

Two following examples complete the proof of the Theorem. The following example is a $\bar{D}$-metric while it is not a Douglas metric. Considering (2.5) and its vanishing S-curvature and flag curvature, one can conclude that

$$D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l} = R_j{}^i{}_{ml.k} = 0.$$

Example 3.2. ([6]) Put

$$\Omega = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 < 1\},$$

where $p = (x, y, z) \in \Omega$ and $y = (u, v, w) \in T_p\Omega$. Define the Randers metric $F = \alpha + \beta$ by

$$\alpha = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2}, \quad \beta = \frac{-yu + xv}{1 - x^2 - y^2}.$$

The above Randers metric has vanishing flag curvature $K = 0$ and S-curvature $\mathbf{S} = 0$. F has zero Weyl curvature then F is of GDW metric. But β is not closed then F is not of Douglas type.

In the following a GDW-metric is presented which is not a $\bar{D}$-metric. In fact F has constant positive curvature, $\lambda = 1$, which means that has vanishing Weyl curvature. Then it is of GDW-metric. On the other hands, it is not of R-quadratic type ($\lambda \neq 0$). Thus according to the following Ricci identity [13]

$$B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l} = R_j{}^i{}_{lm.k}, \tag{3.1}$$

(2.5) and vanishing S-curvature one finds

$$D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = B_j{}^i{}_{kl|m} - B_j{}^i{}_{km|l} \neq 0.$$

Example 3.3. ([4]) The family of Randers metrics on S^3 constructed by Bao-Shen are weakly Berwald which are not Berwaldian. Denote generic tangent vectors on S^3 as

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x,y,z;u,v,w)=\alpha(x,y,z;u,v,w)+\beta(x,y,z;u,v,w),$$

with

$$\alpha=\frac{\sqrt{\lambda(cu-zv+yw)^2+(zu+cv-xw)^2+(-yu+xv+cw)^2}}{1+x^2+y^2+z^2},$$

$$\beta=\frac{\pm\sqrt{\lambda-1}(cu-zv+yw)}{1+x^2+y^2+z^2},$$

where $\lambda>1$ is a real constant. The above Randers metric has vanishing S-curvature and with positive constant flag curvature 1.

Theorem 3.4. *For a Finsler manifold (M,F), $Rq(M)$ (and then $B(M)$) is a proper subset of $GB\widetilde{W}(M)$ and $GB\widetilde{W}(M)$ is a proper subset of $GDW(M)$.*

Proof. First of all, according to (3.1), every Berwald metric is of R-quadratic type and every R-quadratic Finsler metric, based on (3.1), satisfies

$$B_j{}^i{}_{kl|m}-B_j{}^i{}_{km|l}=0,$$

and then $B_j{}^i{}_{kl|0}=0$. It means that every R-quadratic metric and then every Berwald metric is of $GB\widetilde{W}$ type. The first part of the Theorem above is proven by the completion of the example (3.3), in previous Theorem. The Finsler metric in Example (3.3), is not of R-quadratic type. It is of vanishing E-curvature, then based on (2.5), one has $D_j{}^i{}_{kl}=B_j{}^i{}_{kl}$, which Given that F is a GDW-metric, we have

$$D_j{}^i{}_{kl|0}=B_j{}^i{}_{kl|0}=T_{jkl}y^i,$$

for some tensors T_{jkl}. It means that F is a $GB\widetilde{W}$-metric, too.

Now, we prove the second part of the Theorem. Let F be a Finsler metric of $GB\widetilde{W}$ type; then the Berwald curvature satisfies the following equation

$$B_j{}^i{}_{kl}=\beta_j{}^i{}_{kl}+b_{jkl}y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$ where $\beta_j{}^i{}_{kl|0}=0$. Then

$$B_j{}^i{}_{kl|0}=b_{jkl|0}y^i$$

and contracting it by y^l yields

$$b_{jkl|0}y^l=0.$$

Thus, by the definition of E-curvature, one gets

$$2E_{jk} = \beta_j{}^m{}_{km} + b_{jkm}y^m.$$

Thus, one may conclude that

$$H_{jk} = E_{jk|0} = 0. \tag{3.2}$$

Considering (2.5), one easily gets

$$D_j{}^i{}_{kl} = (\beta_j{}^i{}_{kl} - \frac{2}{n+1}e_j{}^i{}_{kl}) + (b_{jkl} + E_{jkl})y^i \tag{3.3}$$

where,

$$e_j{}^i{}_{kl} = E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j,$$

and

$$E_{jkl} = \frac{\partial E_{jk}}{\partial y^l} = \frac{1}{2}\frac{\partial^3 S}{\partial y^j \partial y^k \partial y^l}.$$

According to (3.3) and (3.2) and considering $\beta_j{}^i{}_{kl|m}y^m = 0$, we have

$$D_j{}^i{}_{kl|0} = (b_{jkl|0} + E_{jkl|0})y^i,$$

meaning that F is GDW metric.
The proof is completed by the example that follows.

Example 3.5. ([12]) Let

$$F = \frac{\sqrt{\Big(|x|^2 < a, y > -2 < a, x >< x, y > \Big)^2 + |y|^2\Big(1 - |a|^2|x|^4\Big)}}{1 - |a|^2|x|^4}$$

$$-\frac{\Big(|x|^2 < a, y > -2 < a, x >< x, y > \Big)}{1 - |a|^2|x|^4},$$

where, a is a constant vector in R^n, which has very important properties. It is of scalar curvature and isotropic S-curvature; however, the flag curvature and the S-curvature are not constant. We have

$$\mathbf{S} = (n+1) < a, x > F,$$

Then by lemma 9.1 in [12], one has

$$E_{ij} = \frac{n+1}{2} < a, x > F_{ij}.$$

Then $H \neq 0$; it is not of $GB\widetilde{W}$ type while it is of GDW metric.

References

1. PL. Antonelli and RS. Ingarden and M. Matsumoto, *The Theorey of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, 1993.
2. H. Akbar-Zadeh, *Sur les espaces de Finsler á courbures sectionnelles constantes*, Bulletins de ÍAcadémie Royale de Belgique, 5e Série - Tome LXXXIV, **10**(1988), 281-322.
3. S. Bácsó and I. Papp, *A note on generalized Douglas space*, Periodica Mathematica Hungarica, **48**(2004), 181-184.
4. D. Bao and Z. Shen, *Finsler metrics of constant positive curvature on the Lie group* S^3, Journal of London Mathematics Society, **66**(2002), 453-467.
5. X. Cheng and Z. Shen, *On Douglas metrics*, Publ. Math. Debrecen. **66** 3-4(2005), 503-512.
6. X. Cheng and Z. Shen, *Randers metrics with special curvature properties*, Osaka Journal of Mathematics, **40** (2003), 87-101.
7. J. Douglas, *The general geometry of paths*, Annals of Mathematics, *29*(1927-28), 143-168.
8. B. Najafi and B. Bidabad and A. Tayebi, *On R-quadratic Finsler metrics*, Iranian Journal of Science and Technology, Transaction A, Science **32** (2008), 439-443.
9. B. Najafi and A. Tayebi, *A new quantity in Finsler geometry*, Comptes Rendus Mathematique. **349**(1) (2011), 81-83.
10. N. Sadeghzadeh, *Beyond the Generalized Douglas-Weyl sapaces*, Publ. Math. Debrecen. Submitted.
11. N. Sadeghzadeh, *Generalized Berwald Projective Weyl* ($GB\widetilde{W}$) *metrics*, Submitted.
12. Z. Shen, *Landsberg curvature, S-curvature, and Riemann curvature*, in A Sampler of Finsler Geometry, Mathematical Sciences Research Institute Publications, 50. Cambridge University Press, Cambridge, (2004), 303-355.
13. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.
14. Z. Szabó, *Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wennseine Weyl sche ProjectivKrümmung verschwindet*, Acta Scientiarum Mathematicarum **39** (1977), 163-168.
15. H. Weyl, *Zur Infinitesimal geometrie ; Einordnung der projektiven und der konformen Auffassung*, Göttinger Nachrichten, (1921), 99-112.
16. G. Yang, *On a class of singular Douglas and projectively flat Finsler metrics*, Differ. Geom. Appl. **32**(2014), 113-129.

Received: 12.10.2023
Accepted: 08.12.2023