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SOFT INTERSECTION ABEL-GRASSMANN'S GROUPS

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Abstract. This paper is a bridging among soft set theory, set theory and AG-groups, in which soft intersection AG-group (abbreviated by soft int-AG-group) is defined and investigated. The concept of soft int-AG-group is further extended to define the notions of conjugates soft int-AG-group, normal soft int-AG-group, e-set and α-inclusion of soft int-AG-groups. Various properties of these notions are investigated and supported by relevant examples that are produced by GAP.

Key Words: Soft set, soft int-AG-subgroup, α-inclusion, conjugate normal soft int-AG-group, normal soft int-AG-subgroup.

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1. Introduction

The researchers face uncertainty in various fields for example physical sciences, medical sciences, social sciences even in computer and in economics etc. To deal with this uncertainty they used various mean of mathematical tools that is not so much effective as required. Therefore, Molodstov [16] was the first researcher who proposed soft set theory as an alternative approach to probability theory, fuzzy set theory, rough set theory and any other mathematical tool to describe uncertainty. Soft set theory is a powerful mathematical tool for dealing with uncertainty, and is a parameterized family of subset of the universal set, and free from all the difficulties present in other existing mathematical tools. Soft set

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theory is easily applicable to daily life problems, and has lot of applications in many fields like: operation research, game theory, analysis and many more. Due to its diverse applications, researchers are actively involved and achieved various results both in theoretical and practical aspects.

In 1971, Rosenfeld [17] defined fuzzy subgroup of a group. Since then, various researchers have studied on fuzzy subgroup theory; and similar results are derived from classical group theory. First study on algebraic structure of soft sets was made by Aktaş and Çağman [2] in 2007. They introduced concept of soft group, and derived some of the basic properties of soft groups. Since then, many papers on soft groups have been published [4-7]. Çağman et al. [7] proposed a new algebraic concept called soft-int group by inspiring from definition of fuzzy subgroups defined by Rosenfeld [17], and obtained some of its properties existing in group theory. A large amount of the literature on algebraic structures of soft sets can be found in [9, 23, 19, 12, 24, 25, 26, 27].

In 1987, concept of AG-groups (LA-groups) was defined by Mushtaq and Kamran [15]. In generally an AG-group is non-associative, so an AG-group is a different structure from classical group structure. Unlike groups and other structures, commutativity and associativity imply each other in AG-groups and thus AG-groups become Abelian group if any one of them is allowed in AG-group. Many researchers studied on algebraic structures of the soft sets constructed by using different algebraic structures such as group, ring, ideal, BCK/BCI algebra, near ring, and LA-semigroup instead of the parameter set in soft sets. With this motivation, in this paper, we define concept of soft int-AG-group as a bridge among soft set theory, set theory and AG-group theory. Basic difference between soft int-AG-group and soft int-group [7] is algebraic structure corresponding to parameter set of soft set. After we obtain some properties of soft int-AG-groups, based on definition and properties of soft int-AG-groups Since an AG-group is a generalization of a group, it can be said that soft int-AG-group is a generalization of soft int-group defined in [7]. Therefore, obtained some properties of soft int-group given in [7] and [10] are available for soft int-AG-group. In this study we point out these properties and obtain some new results related to soft int-AG-groups. We also define some new concepts such as conjugates soft int-AG-group, normal soft int-AG-groups, e-set and

α-inclusion. Furthermore, we obtain some results of α-level sets of soft-int-AG-groups. Also, we support defined new concepts with examples to be more understandable.

2. Preliminaries

In this section, we recall definition and set theoretical operations of soft sets, and present some properties of AG-groups.

Definition 2.1. [16, 8] Let U be the universal set, E be the set of parameters and $P(U)$ be the power set of U. Then a soft set, A is a set of order pairs

$$A = \{(\varepsilon,\, f_A(\varepsilon)) \,:\, \varepsilon \in E\},$$

where f_A is a set valued function from E to $P(U)$ and f_A is called approximate function of soft set A. The subscript A in the function f_A denotes that f_A approximate function of soft set A. $f_A(\varepsilon)$ may be arbitrary i.e for some $\varepsilon \in E$ $f_A(\varepsilon)$ may be empty, some may have nonempty intersection. Also, the set $\{f_A(\varepsilon)|\varepsilon \in E\}$ is called image of A and is denoted by $Im(A)$.

For convenience, if $f_A(\varepsilon) = \emptyset$, $(\varepsilon,\, f_A(\varepsilon))$ will not be appear in the set A. The set of all soft sets over U is denoted by $S(U)$.

Definition 2.2. [8] Let $A, B \in S(U)$. Then,

(1) If $f_A(\varepsilon) = \emptyset$ for all $\varepsilon \in E$, A is said to be a null soft set, denoted by Φ.
(2) If $f_A(\varepsilon) = U$ for all $\varepsilon \in E$, A is said to be absolute soft set, denoted by $\hat{U}$.
(3) A is soft subset of B, denoted by $A\tilde{\subseteq}B$, if $f_A(\varepsilon) \subseteq f_B(\varepsilon)$ for all $\varepsilon \in E$.
(4) $A\tilde{=}B$, if $A\tilde{\subseteq}B$ and $B\tilde{\subseteq}A$.
(5) Soft union of A and B, denoted by $A\tilde{\cup}B$, is a soft set over U and defined by

$$\begin{aligned} A\tilde{\cup}B &= \{(\varepsilon,\, (f_A\tilde{\cup}f_B)\,(\varepsilon)) \,:\, \varepsilon \in E\} \\ &= \{(\varepsilon,\, (f_A(\varepsilon) \cup f_B(\varepsilon))) \,:\, \varepsilon \in E\}. \end{aligned}$$

(6) Soft intersection of A and B, denoted by $A\tilde{\cap}B$, is a soft set over U and defined by

$$\begin{aligned} A\tilde{\cap}B &= \{(\varepsilon,\, (f_A\tilde{\cap}f_B)\,(\varepsilon)) \,:\, \varepsilon \in E\} \\ &= \{(\varepsilon,\, (f_A(\varepsilon) \cap f_B(\varepsilon))) \,:\, \varepsilon \in E\}. \end{aligned}$$

(7) Soft complement of A is denoted by $A^{\tilde{c}}$ and defined by $A^{\tilde{c}} = U \setminus f_A(\varepsilon)$ for all $\varepsilon \in E$.

Definition 2.3. [14] Let $E = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ be a set of parameters. The NOT set of E denoted by $\rceil E$ is defined by $\rceil E = \{\neg\varepsilon_1, \neg\varepsilon_2, ..., \neg\varepsilon_n\}$ where $\neg\varepsilon_i =$not ε_i, $\forall\, 1 \leq i \leq n$.

(It may be noted that $\rceil$ and $\neg$ are different operators also NOT operator is different from the soft complement of A).

Definition 2.4. [8] Let $A, B \in S(U)$. Then, $\wedge$-product and $\vee$-product of A and B, is denoted by $A \wedge B$ and $A \vee B$ respectively, and is defined by the approximate functions as follows:

$$f_{A\wedge B}(\varepsilon, \varepsilon') : E \times E \to P(U), f_{A\wedge B}(\varepsilon, \varepsilon') = f_A(\varepsilon) \cap f_B(\varepsilon'),$$

and

$$f_{A\vee B}(\varepsilon, \varepsilon') : E \times E \to P(U), f_{A\vee B}(\varepsilon, \varepsilon') = f_A(\varepsilon) \cup f_B(\varepsilon').$$

Also, $A \wedge B$ and $A \vee B$ can be written as a set of pairs as follows:

$$\begin{aligned}(A \wedge B) &= \left\{\left((\varepsilon, \varepsilon'), f_{A\wedge B}(\varepsilon, \varepsilon')\right) : \varepsilon, \varepsilon' \in E\right\} \\ &= \left\{\left((\varepsilon, \varepsilon'), f_A(\varepsilon) \cap f_B(\varepsilon')\right) : \varepsilon, \varepsilon' \in E\right\},\end{aligned}$$

and

$$\begin{aligned}(A \vee B) &= \left\{\left((\varepsilon, \varepsilon'), f_{A\vee B}(\varepsilon, \varepsilon')\right) : \varepsilon, \varepsilon' \in E\right\} \\ &= \left\{\left((\varepsilon, \varepsilon'), f_A(\varepsilon) \cup f_B(\varepsilon')\right) : \varepsilon, \varepsilon' \in E\right\},\end{aligned}$$

respectively.

In the rest of this paper, G denotes an AG-group and e denotes the left identity of G unless otherwise stated. An AG-group is a non-associative structure, in which commutativity and associativity imply each other and thus AG-group become an abelian group if any one of the property is allowed in AG-group. AG-group is a generalization of abelian group and a special case of quasi-group. An AG-groupoid (or LA-semigroups) is a non-associative groupoid in general, in which the left invertive law $(ab)c = (cb)a$ holds for all $a, b, c \in G$. An AG-groupoid G is called an AG-group or left almost group (LA-group), if there exists a unique left identity e in G (i.e. $ea = a$ for all $a \in G$), and for all $a \in G$ there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$. Now a day's many researchers are taking interest to fuzzify AG-groupoids and AG-groups and to develop soft set theory for AG-groupoids and AG-groups [9-16].

An AG-group $(G, *)$ can easily be obtained from an abelian group $(G_1, \cdot)$ by:

$$a * b = a^{-1} \cdot b \quad \text{or} \quad a * b = b \cdot a^{-1} \ \forall \ a, b \in G_1.$$

It is easy to prove that in an AG-group G the right identity become the two sided identity, and that an AG-group G with right identity is an abelian group. AG-group posses the property of cancellative like group. A nonempty subset H of G is called an AG-subgroup of G, if H itself is an AG-group under the same binary operation defined in G. In AG-subgroup one can easily obtain the following results. Let $\phi \neq H \subseteq G$, then H is an AG-subgroup of G if and only if for any $h_1, h_2 \in H \Rightarrow h_1h_2 \in H$, and for all $h^{-1} \in H$ for all $h \in H$. Similarly, let $\phi \neq H \subseteq G$, then H is an AG-subgroup of G if and only if $h_1h_2^{-1} \in H \ \forall \ h_1, h_2 \in H$. Let H and K be AG-subgroups of G of order m and n respectively, where $(m, n) = 1$. Then

$$HK = \{hk \ : \ h \in H, \, k \in K\},$$

and has exactly mn elements and is also an AG-subgroup of G. Various properties of AG-groups are explored in [21, 22] such as: if the order of a finite group is prime then it has the trivial subgroup only. While, in AG-group it is not necessary. The product of two AG-subgroups is always an AG-subgroup, although it is not common in groups. Like groups, conjugate classes of AG-groups also form a partition of G, unlike groups the conjugate class of $e \in G$ is not a singleton set. Therefore, if the conjugate class of G contain e, then it is also an AG-group. If the order of G is prime then it has single conjugate.

The following identities can be easily proved in an AG-group G:

Lemma 2.5. [21] *Let $e \in G$, and a, b, c, $d \in G$, then*

(1) $(ab)(cd) = (ac)(bd)$ (medial law),
(2) $a(bc) = b(ac)$,
(3) $(ab)(cd) = (db)(ca)$ (paramedial law),
(4) $(ab)(cd) = (dc)(ba)$,
(5) $(ab)^{-1} = a^{-1}b^{-1}$,

3. Soft Intersection AG-groups

In this section the basic definition of soft intersection AG-group (soft int-AG-group) is given, some of the basic results along with suitable examples are provided. The following definition is similar soft int-group given in [7]. Basic difference of this definition from soft int-group is that parameter set is an AG-group.

Definition 3.1. Let $G \subseteq E$ be a set of parameters, and $A \in S(U)$ be a soft set. Then, A is called **soft int-AG-group** over U if for all $g, g' \in G$, the following conditions are satisfied:

(1) $f_A(gg') \supseteq f_A(g) \cap f_A(g')$,
(2) $f_A(g^{-1}) = f_A(g)$.

Example 3.2. Consider a non-associative AG-group $G = \{0, 1, 2\}$ of order 3 with left identity 0, defined in the following table:

.	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let A be a soft set over $U = \{u_1, u_2, \ldots, u_{10}\}$, defined by

$$A = \{(0, f_A(0)), (1, f_A(1)), (2, f_A(2))\}$$
$$A = \{(0, U), (1, \{u_2, u_4, u_6\}), (2, \{u_2, u_4, u_6\})\}.$$

It can be easily verified that A is a soft int-AG-group over U.

Example 3.3. Consider a non-associative AG-group $G = \{a, b, c, d\}$ of order 4 with left identity d, defined in the following Cayley's table:

.	a	b	c	d
a	d	a	b	c
b	c	d	a	b
c	b	c	d	a
d	a	b	c	d

Let A be a soft set over $U = \mathbb{Z}$, defined by

$$A = \{(a, f_A(a)), (b, f_A(b)), (c, f_A(c)), (d, f_A(d))\}$$
$$A = \{(a, \{1, 2, 4\}), (b, \{1, 2, 3, 4\}), (c, \{1, 2, 4\}), (d, \mathbb{Z})\}.$$

It can be easily verified that A is a soft int-AG-group over U.
The set of all soft int-AG-groups over U is represented by $S_{\cap AG}(U)$.

The following lemma is available for soft int-groups.

Lemma 3.4. *Let $A \in S_{\cap AG}(U)$. Then, $f_A(e) \supseteq f_A(g)$ for all $g \in G$.*

Proof. The proof can be made in similar way to proof of Theorem 1 in [7] □

Lemma 3.5. *Let $A \in S_{\cap AG}(U)$. Then $f_A(gg') = f_A(g'g)$ for all $g, g' \in G$.*

Proof. Let $A \in S_{\cap AG}(U)$. Then for all $g, g' \in G$,

$$\begin{aligned} f_A(gg') &= f_A((eg)g') \\ &= f_A\left((g'g)e\right) && \text{(by the left invertive law)} \\ &\supseteq f_A(g'g) \cap f_A(e) \\ &= f_A(g'g) && \text{(by Lemma 3.4)} \\ \Rightarrow f_A(gg') &\supseteq f_A(g'g). \end{aligned}$$

Similarly, we can show that $f_A(g'g) \supseteq f_A(gg')$. Hence, $f_A(gg') = f_A(g'g)$ for all $g, g' \in G$. □

Note that, this lemma is available in soft int-groups only when parameter set (group) of soft int-group is an Abelian group. However, in a soft int-AG-group, it is not necessary that parameter set is an Abelian group.

The following theorem is available both soft int-groups and soft int-AG-groups.

Theorem 3.6. *A soft set A over U is a soft int-AG group over U if and only if $f_A(gg'^{-1}) \supseteq f_A(g) \cap f_A(g')$ for all $g, g' \in G$.*

Proof. The proof can be made in similar way to proof of Theorem 2 given in [7]. □

Lemma 3.7. *Let $A \in S_{\cap AG}(U)$. Then, for all $g, g' \in G$, $f_A(gg') = f_A(g')$ if and only if $f_A(g) = f_A(e)$.*

Proof. Let $A \in S_{\cap AG}(U)$ and $f_A(gg') = f_A(g')$ for all $g, g' \in G$. By choosing $g' = e$ we get

$$\begin{aligned} f_A(ge) &= f_A(e) \\ \Rightarrow f_A(eg) &= f_A(e) && \text{(by Lemma 3.5)} \\ \Rightarrow f_A(g) &= f_A(e). \end{aligned}$$

Conversely, suppose that $f_A(g) = f_A(e)\ \forall\ g \in G$. Then,

$$\begin{aligned} f_A(gg') &\supseteq f_A(g) \cap f_A(g') \\ &= f_A(e) \cap f_A(g') \\ &= f_A(g') && \text{(by Lemma 3.4)} \end{aligned}$$

This implies that

$$f_A(gg') \supseteq f_A(g'). \tag{3.1}$$

Also,

$$\begin{aligned}
f_A(g') &= f_A(eg') = f_A((g^{-1}g)g') \\
&= f_A\left((g'g)g^{-1}\right) && \text{(by the left invertive law)} \\
&\supseteq f_A(g'g) \cap f_A(g^{-1}) \\
&= f_A(gg') \cap f_A(g) && \text{(by Lemma 3.5)} \\
&= f_A(gg') \cap f_A(e) \\
&= f_A(gg') && \text{(by Lemma 3.4)}
\end{aligned}$$

This implies that

$$f_A(g') \supseteq f_A(gg'). \tag{3.2}$$

Consequently, Equations (3.1) and (3.2) implies that, $f_A(g') \supseteq f_A(gg') \supseteq f_A(g')$. Hence, $f_A(gg') = f_A(g')$. □

Lemma 3.8. *Let $A \in S_{\cap AG}(U)$. If $f_A(gg'^{-1}) = f_A(e)$ then $f_A(g) = f_A(g')$ for all $g, g' \in G$.*

Proof. Let $A \in S_{\cap AG}(U)$ such that $f_A(gg'^{-1}) = f_A(e)$. Therefore, for all $g, g' \in G$

$$\begin{aligned}
f_A(g) &= f_A(e \cdot g) = f_A((g'g'^{-1})g) \\
&= f_A\left((gg'^{-1})g'\right) && \text{(by the left invertive law)} \\
&\supseteq f_A(gg'^{-1}) \cap f_A(g') \\
&= f_A(e) \cap f_A(g') \\
&= f_A(g'). && \text{(by Lemma 3.4)}
\end{aligned}$$

Thus

$$f_A(g) \supseteq f_A(g'). \tag{3.3}$$

And

$$\begin{aligned}
f_A(g') &= f_A(g'^{-1}) = f_A(e \cdot g'^{-1}) = f_A((g^{-1}g)g'^{-1}) \\
&= f_A\left((g'^{-1}g)g^{-1}\right) && \text{(by the left invertive law)} \\
&\supseteq f_A(g'^{-1}g) \cap f_A(g^{-1}) \\
&= f_A(gg'^{-1}) \cap f_A(g) && \text{(by Lemma 3.5)} \\
&= f_A(g). && \text{(by Lemma 3.4)}
\end{aligned}$$

Thus

$$f_A(g') \supseteq f_A(g). \tag{3.4}$$

Hence, by Equations (3.3) and (3.4) we get, $f_A(g) = f_A(g')$ for all $g, g' \in G$. ☐

Note that, Lemma 3.8 is similar Theorem 3.2 in [10]. In [10], since parameter set is a group, proof of the theorem is made by using associative law. But an AG-group is non-associative. Therefore we make proof of lemma by using left invertive law, Lemma (3.5) and (3.4).

The following theorem is available for both soft int-groups and soft int-AG-groups.

Theorem 3.9. *Let $A, B \in S_{\cap AG}(U)$. Then, $A \wedge B \in S_{\cap AG}(U)$.*

Proof. The proof can be made by similar way to proof of Theorem 4. in [7]. ☐

Note that $A \vee B$ of any two soft sets A and B may or may not be a soft int-AG-group as given by the following counter example.

Example 3.10. Assume that $U = \{u_1, u_2, \ldots, u_6\}$ is the universal set. Let G be any AG-group of order 4 as defined in the following table

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	3	2	1	0
3	2	3	0	1

Consider soft sets A, B over U as follows:

$$A = \{(0, U), (1, \{u_2, u_4\}), (2, \{u_2\}), (3, \{u_2\})\},$$

and

$$B = \{(0, U), (1, \{u_1, u_3\}), (2, \{u_3\}), (3, \{u_3\})\}.$$

It is clear that both $A, B \in S_{\cap AG}(U)$. Now, take

$$\begin{aligned}
(f_A \vee f_B)\left((1,1)\cdot(0,2)^{-1}\right) &= (f_A \vee f_B)\left((1,1)\cdot(0,3)\right) \\
& (f_A \vee f_B)(1\cdot 0, 1\cdot 3) \\
&= (f_A \vee f_B)(1,2) \\
&= (f_A)(1) \vee (f_B)(2) = \{u_2, u_3, u_4\},
\end{aligned}$$

and

$$(f_A \vee f_B)(1,1) \cap (f_A \vee f_B)(0,2) = \{u_1, u_2, u_3, u_4\} \cap U = \{u_1, u_2, u_3, u_4\},$$

this implies that

$$(f_A \vee f_B)((1,1)\cdot(0,2)^{-1}) \supsetneq (f_A \vee f_B)(1,1) \cap (f_A \vee f_B)(0,2).$$

Hence, $A \vee B$ is not a soft int-AG-group over U.

Definition 3.11. Let $A, B \in S_{\cap AG}(U)$ on AG-groups G_1 and G_2 respectively. Then, **the product of** A **and** B is denoted by $A \times B$ and is defined by

$$\begin{aligned} A \times B &= \Big\{\{(g,g'),\, (f_{A\times B})\,(g,g')\} :\ \forall\ (g,g') \in G_1 \times G_2\Big\} \\ &= \Big\{\{(g,g'),\, \big(f_A(g) \times f_B(g')\big)\,\} :\ \forall\ (g,g') \in G_1 \times G_2\Big\}. \end{aligned}$$

Example 3.12. Let $U = \{0,-1,1\}$ is the universal set, and $G_1 = \{a,b,c,d\}$ and $G_2 = \{x,y,z\}$ are AG-groups of order 4 and 3 defined in the following tables (i) and (ii) respectively:

.	a	b	c	d
a	d	a	b	c
b	c	d	a	b
c	b	c	d	a
d	a	b	c	d

(i)

.	x	y	z
x	x	y	z
y	z	x	y
z	y	z	x

(ii)

Let $A, B \in S_{\cap AG}(U)$ on AG-groups G_1 and G_2 respectively defined by:

$$f_A(a) = \{0\} = f_A(c),\ f_A(b) = \{0,-1\},\ f_A(d) = \{0,-1,1\},$$

and

$$f_B(x) = \{0,-1,1\}\,,\ f_B(y) = \{-1\} = f_B(z).$$

Then,

$$\begin{aligned} A \times B &= \Big\{\{(g,g'),\, \big(f_A(g) \times f_B(g')\big)\,\} :\ \forall\ (g,g') \in G_1 \times G_2\Big\}, \\ &= \Big\{\ \{(a,x)\,,((0,0)\,,(0,-1)\,,(0,1))\}\,,\{(a,y)\,,(0,-1)\}\,,\{(a,z)\,,(0,-1)\}\,, \\ &\quad \{(b,x)\,,((0,0)\,,(0,-1)\,,(0,1)\,,(-1,0)\,,(-1,-1)\,,(-1,1))\}\,, \\ &\quad \{(b,y)\,,((0,-1)\,,(-1,-1))\}\,,\{(b,z)\,,((0,-1)\,,(-1,-1))\}\,, \\ &\quad \{(c,x)\,,((0,0)\,,(0,-1)\,,(0,1))\}\,,\{(c,y)\,,(0,-1)\}\,,\{(c,z)\,,(0,-1)\}\,, \\ &\quad \{(d,x)\,,((0,0)\,,(0,-1)\,,(0,1)\,,(-1,0)\,,(-1,-1)\,,(-1,1)\,,(1,0)\,, \\ &\quad (1,-1)\,,(-1,1))\},\{(d,y)\,,((0,-1)\,,(-1,-1)\,,(1,-1))\}\,, \\ &\quad \{(d,z)\,,((0,-1)\,,(-1,-1)\,,(1,-1))\}\ \Big\}. \end{aligned}$$

Theorem 3.13. *Let $A, B \in S_{\cap AG}(U)$ on G_1 and G_2 respectively. Then $A \times B \in S_{\cap AG}(U \times U)$.*

Proof. For any $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$,

$$\begin{aligned}
(f_{A\times B})\left((x_1,y_1),(x_2,y_2)^{-1}\right) &= (f_{A\times B})\left((x_1,y_1),(x_2,y_2)^{-1}\right) \\
&= (f_{A\times B})\left((x_1,y_1),(x_2^{-1},y_2^{-1})\right) \\
&= (f_{A\times B})\left((x_1x_2^{-1},y_1y_2^{-1}\right) \\
&= f_A(x_1x_2^{-1}) \times f_B(y_1y_2^{-1}) \qquad \text{(by Definition 3.11)} \\
&\supseteq (f_A(x_1)\cap f_A(x_2)) \times (f_B(y_1)\cap f_B(y_2)) \\
&= (f_A(x_1)\times f_B(y_1)) \cap (f_A(x_2)\times f_B(y_2)) \\
&= (f_{A\times B})(x_1,y_1) \cap (f_{A\times B})(x_2,y_2).
\end{aligned}$$

Hence, $A \times B \in S_{\cap AG}(U \times U)$. □

Theorem 3.14. *Let $A, B \in S_{\cap AG}(U)$, then $A\tilde{\cap}B \in S_{\cap AG}(U)$.*

Proof. Since $A, B \in S_{\cap AG}(U)$. Therefore, $A\tilde{\cap}B \neq \Phi$. For any $x, y \in A\tilde{\cap}B$, we have

$$\begin{aligned}
(f_A\tilde{\cap}f_B)(xy^{-1}) &= f_A(xy^{-1}) \cap f_B(xy^{-1}) \qquad \text{(by Definition 2.2-(v))} \\
&\supseteq (f_A(x)\cap f_A(y)) \cap (f_B(x)\cap f_B(y)) \\
&= (f_A(x)\cap f_B(x)) \cap (f_A(y)\cap f_B(y)) \\
&= (f_A\tilde{\cap}f_B)(x) \cap (f_A\tilde{\cap}f_B)(y).
\end{aligned}$$

Hence, $A\tilde{\cap}B \in S_{\cap AG}(U)$. □

Note that $A\tilde{\cup}B$ may not be a soft int-AG-group over U as given by the following counter example.

Example 3.15. Let $G = \{0, 1, 2, 3, 4, 5\}$ be an AG-group of order 6 defined in the following table:

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	2	0	1	5	3	4
2	1	2	0	4	5	3
3	3	4	5	0	1	2
4	5	3	4	2	0	1
5	4	5	3	1	2	0

Let A and B are any two soft int-AG-groups over $U = \mathbb{Z}$ defined as follow:

$f_A(0) = \mathbb{Z}$	$f_B(0) = \mathbb{Z}$
$f_A(1) = \{0,1,4\} = f_A(5)$	$f_B(1) = \{6,7\} = f_B(5)$
$f_A(2) = \{0,1,4,11\} = f_A(4)$	$f_B(2) = \{6,7,10,13\} = f_B(4)$
$f_A(3) = \{0,1,4,12,13\}$	$f_B(3) = \{6,7,8,9\}$

It is clear that

$$(f_A\tilde{\cup}f_B)\left(2\cdot 4^{-1}\right) = (f_A\tilde{\cup}f_B)(5) = f_A(5)\cup f_B(5) = \{0,1,4,6,7\}, \tag{3.5}$$

and

$$\begin{aligned} ((f_A\tilde{\cup}f_B)(2)) \cap ((f_A\tilde{\cup}f_B)(4)) &= (f_A(2)\cup f_B(2)) \cap (f_A(4)\cup f_B(4)) \\ &= \{0,1,4,6,7,10,11,13\}, \end{aligned}$$

this implies that

$$((f_A\tilde{\cup}f_B)(2)) \cap ((f_A\tilde{\cup}f_B)(4)) = \{0,1,4,6,7,10,11,13\}. \tag{3.6}$$

From Equations (3.5) and (3.6) it is clear that

$$(f_A\tilde{\cup}f_B)\left(2\cdot 4^{-1}\right) \supsetneq ((f_A\tilde{\cup}f_B)(2)) \cap ((f_A\tilde{\cup}f_B)(4)).$$

Hence the union of two soft int-AG-groups may not be a soft int-AG-group.

Definition 3.16. Let H be an AG-subgroup of an AG-group G, $A \in S_{\cap AG}(U)$ on G, and $\Phi \neq B \in S(U)$ on H. If $B \in S_{\cap AG}(U)$ on H, then B is called a **soft int-AG-subgroup** of A over U, on H and denoted by $B\tilde{\leq}A$.

Example 3.17. Let $U = \{u_1, u_2, \ldots, u_{10}\}$ be the universal set and G be any AG-group of order 9 defined in the following table,

·	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	2	0	1	4	5	3	7	8	6
2	1	2	0	5	3	4	8	6	7
3	7	6	8	0	2	1	5	3	4
4	6	8	7	1	0	2	4	5	3
5	8	7	6	2	1	0	3	4	5
6	4	3	5	8	6	7	0	2	1
7	3	5	4	7	8	6	1	0	2
8	5	4	3	6	7	8	2	1	0

Define a soft int-AG-group A as follows:

$$\begin{aligned} f_A(0) &= U, \\ f_A(3) &= \{u_1, u_2, u_3, u_4\} = f_A(7), \\ f_A(1) &= \{u_1, u_2\} = f_A(2) = f_A(4) = f_A(5) = f_A(6) = f_A(8). \end{aligned}$$

Let $H_1 = \{0, 3, 7\}$ and $H_2 = \{0, 1, 2\}$ be two AG-subgroups of G. Define soft int-AG-groups B and C

$$B = \{(0, U), (3, \{u_2, u_4\}), (7, \{u_2, u_4\})\},$$

and

$$C = \{(0, U), (1, \{u_1, u_2\}), (2, \{u_1, u_2\})\}.$$

As $B, C \tilde{\subseteq} A$. Therefore, B, $C \tilde{\leq} A$.

Theorem 3.18. *Let B, $C \tilde{\leq} A$. Then, $B \tilde{\cap} C \tilde{\leq} A$.*

Proof. By Çağman et al. ([7], Theorem 7), we have the proof for two soft int-groups. Let us prove for two soft int-AG-groups. Since B, $C \tilde{\leq} A$, $B \tilde{\cap} C \neq \Phi$. Let x, $y \in B \tilde{\cap} C$. Then

$$\begin{aligned} (f_B \tilde{\cap} f_C)(xy^{-1}) &= \left((f_{B\tilde{\cap}C})(xy^{-1})\right) \\ &= f_B(xy^{-1}) \cap f_C(xy^{-1}) \\ &\supseteq (f_B(x) \cap f_B(y)) \cap (f_C(x) \cap f_C(y)) \\ &= (f_B(x) \cap f_C(x)) \cap (f_B(y) \cap f_C(y)) \\ &= f_{B\tilde{\cap}C}(x) \cap f_{B\tilde{\cap}C}(y). \end{aligned}$$

Thus, $xy^{-1} \in B \tilde{\cap} C$. Hence, $B \tilde{\cap} C \leq^{\tilde{}} A$. □

Theorem 3.19. *Let $\{B_i : i \in I\} \tilde{\leq} A$ for all $i \in I$. Then $\cup_{i \in I} B_i \tilde{\leq} A$.*

Proof. By Kaygısız ([10], Theorem 3.4), we have the proof for soft int-groups. We can prove by similar way for soft int-AG-groups. Since $\{B_i : i \in I\} \tilde{\leq} A$ for all $i \in I$, $\cap_{i \in I} B_i \neq \Phi$. Let x, $y \in \cap_{i \in I} B_i$. Then,

$$\begin{aligned} (\tilde{\cap}_{i \in I} f_{B_i})(xy^{-1}) &= \left(\left(f_{\tilde{\cap}_{i \in I} B_i}\right)(xy^{-1})\right) \\ &= \cap_{i \in I}\left(f_{B_i}(xy^{-1}) \ : \ i \in I\right) \\ &\supseteq \tilde{\cap} i \in I\left((f_{B_i}(x) \cap f_{B_i}(y)) \ : \ i \in I\right) \\ &= \left(\cap_{i \in I}\left(f_{B_i}(x) \ : \ i \in I\right)\right) \cap \left(\cap_{i \in I}\left(f_{B_i}(y) \ : \ i \in I\right)\right) \\ &= \left(\left(f_{\tilde{\cap} i \in I B_i}\right)(x)\right) \cap \left(\left(f_{\tilde{\cap}_{i \in I} B_i}\right)(y)\right). \end{aligned}$$

Thus, $xy^{-1} \in \tilde{\cap}_{i \in I} B_i$. Hence, $\tilde{\cap}_{i \in I} B_i \tilde{\leq} A$. □

Note that if B, $C\tilde{\leq}A$. Then it is not necessary that, $B\tilde{\cup}C\tilde{\leq}A$ in general.

Example 3.20. From, Example 3.17, it can be easily shown that:

$$(f_B\tilde{\cup}f_C)\left(3\cdot 2^{-1}\right) = \left(f_{B\tilde{\cup}C}\right)(3\cdot 2) = \left(f_{B\tilde{\cup}C}\right)(8) = f_B(8)\cup f_C(8) = \emptyset, \tag{3.7}$$

and

$$\begin{aligned}((f_B\tilde{\cup}f_C)(3))\cap((f_B\tilde{\cup}f_C)(2)) &= (f_B(3)\cup f_C(3))\cap(f_B(2)\cup f_C(2)) \\ &= \{u_2, u_4\}\cap\{u_1, u_2\},\end{aligned}$$

this implies that

$$((f_B\tilde{\cup}f_C)(3))\cap((f_B\tilde{\cup}f_C)(2)) = \{u_2\}. \tag{3.8}$$

By Equations (3.7) and (3.8), we get

$$(f_B\tilde{\cup}f_C)\left(3\cdot 2^{-1}\right) \supsetneq ((f_B\tilde{\cup}f_C)(3))\cap((f_B\tilde{\cup}f_C)(2)).$$

Hence, $B\tilde{\cup}C\tilde{\nleq}A$.

4. Conjugate Soft int-AG-groups

Definition 4.1. Let $A\in S_{\cap AG}(U)$ and $u\in G$. Then A_u is called **conjugate soft int-AG-group** (with respect to u) denoted by $A_u\sim_c A$, and is given as

$$f_{A_u}(x) \quad = \quad f_A\left((ux)u^{-1}\right), \text{ for all } x\in G.$$

Remark 4.2. It is noted that a conjugate soft int-AG-group may or may not be a soft-int-AG-group.

Example 4.3. Consider an AG-group G of order 6 defined in Example 3.15. Let $A\in S_{\cap AG}(\mathbb{Z})$, defined as follows:

$$\begin{aligned}f_A(0) &= \mathbb{Z}, \\ f_A(2) &= \{1,3,\ldots,9\} = f_A(4), \\ f_A(1) &= \{3,6,9\} = f_A(3) = f_A(5).\end{aligned}$$

The conjugates soft int-AG-group A is given by:

$$\begin{aligned}
f_{A_0}(0) = f_{A_3}(0) &= f_A(0) = && \mathbb{Z}, \\
f_{A_0}(1) = f_{A_3}(1) &= f_A(5) = && \{3, 6, 9\}, \\
f_{A_0}(2) = f_{A_3}(2) &= f_A(4) = && \{1, 3, \ldots, 9\}, \\
f_{A_0}(3) = f_{A_3}(3) &= f_A(3) = && \{3, 6, 9\}, \\
f_{A_0}(4) = f_{A_3}(4) &= f_A(2) = && \{1, 3, \ldots, 9\}, \\
f_{A_0}(5) = f_{A_3}(5) &= f_A(1) = && \{3, 6, 9\}.
\end{aligned}$$

$$\begin{aligned}
f_{A_1}(0) = f_{A_4}(0) &= f_A(2) = && \{1, 3, \ldots, 9\}, \\
f_{A_1}(1) = f_{A_4}(1) &= f_A(1) = && \{3, 6, 9\}, \\
f_{A_1}(2) = f_{A_4}(2) &= f_A(0) = && \mathbb{Z}, \\
f_{A_1}(3) = f_{A_4}(3) &= f_A(5) = && \{3, 6, 9\}, \\
f_{A_1}(4) = f_{A_4}(4) &= f_A(4) = && \{1, 3, \ldots, 9\}, \\
f_{A_1}(5) = f_{A_4}(5) &= f_A(3) = && \{3, 6, 9\}.
\end{aligned}$$

$$\begin{aligned}
f_{A_2}(0) = f_{A_5}(0) &= f_A(4) = && \{1, 3, \ldots, 9\}, \\
f_{A_2}(1) = f_{A_5}(1) &= f_A(3) = && \{3, 6, 9\}, \\
f_{A_2}(2) = f_{A_5}(2) &= f_A(2) = && \{1, 3, \ldots, 9\}, \\
f_{A_2}(3) = f_{A_5}(3) &= f_A(1) = && \{3, 6, 9\}, \\
f_{A_2}(4) = f_{A_5}(4) &= f_A(0) = && \mathbb{Z}, \\
f_{A_2}(5) = f_{A_5}(5) &= f_A(5) = && \{3, 6, 9\}.
\end{aligned}$$

A_1 and A_2 are conjugate soft int-AG-groups but are not soft int-AG-groups over $\mathbb{Z}$, as

$$f_{A_1}(2 \cdot 2) = f_{A_1}(0) = \{1, 3, \ldots, 9\} \supsetneq f_{A_1}(2) \cap f_{A_1}(2) = \mathbb{Z},$$

and

$$f_{A_2}(4 \cdot 4) = f_{A_2}(0) = \{1, 3, \ldots 9\} \supsetneq f_{A_2}(4) \cap f_{A_2}(4) = \mathbb{Z}.$$

Definition 4.4. Let $A \in S_{\cap AG}(U)$. Then A is called a **normal soft int-AG-group** over U if

$$f_A\left((xy)x^{-1}\right) = f_A(y) \;\forall\; x, y \in G.$$

Or in other words A is a normal soft int-AG-group over U, if A is self conjugate soft int-AG-group.

The set of all normal soft int-AG-groups over U is represented by $NS_{\cap AG}(U)$.

Example 4.5. Let G be an AG-group of order 6 defined as in Example 3.15. Let $A \in S_{\cap AG}(\mathbb{Z})$, defined by

$$\begin{aligned} f_A(0) &= \mathbb{Z} = f_A(2) = f_A(4), \\ f_A(1) &= \{2,4,6,8,10\} = f_A(3) = f_A(5). \end{aligned}$$

The conjugates soft int-AG-groups of A, are given by:

$$\begin{aligned} f_{A_0}(0) = f_{A_3}(0) &= f_A(0) = \mathbb{Z}, \\ f_{A_0}(1) = f_{A_3}(1) &= f_A(5) = \{2,4,6,8,10\}, \\ f_{A_0}(2) = f_{A_3}(2) &= f_A(4) = \mathbb{Z}, \\ f_{A_0}(3) = f_{A_3}(3) &= f_A(3) = \{2,4,6,8,10\}, \\ f_{A_0}(4) = f_{A_3}(4) &= f_A(2) = \mathbb{Z}, \\ f_{A_0}(5) = f_{A_3}(5) &= f_A(1) = \{2,4,6,8,10\}. \end{aligned}$$

$$\begin{aligned} f_{A_1}(0) = f_{A_4}(0) &= f_A(2) = \mathbb{Z}, \\ f_{A_1}(1) = f_{A_4}(1) &= f_A(1) = \{2,4,6,8,10\}, \\ f_{A_1}(2) = f_{A_4}(2) &= f_A(0) = \mathbb{Z}, \\ f_{A_1}(3) = f_{A_4}(3) &= f_A(5) = \{2,4,6,8,10\}, \\ f_{A_1}(4) = f_{A_4}(4) &= f_A(4) = \mathbb{Z}, \\ f_{A_1}(5) = f_{A_4}(5) &= f_A(3) = \{2,4,6,8,10\}. \end{aligned}$$

$$\begin{aligned} f_{A_2}(0) = f_{A_5}(0) &= f_A(4) = \mathbb{Z}, \\ f_{A_2}(1) = f_{A_5}(1) &= f_A(3) = \{2,4,6,8,10\}, \\ f_{A_2}(2) = f_{A_5}(2) &= f_A(2) = \mathbb{Z}, \\ f_{A_2}(3) = f_{A_5}(3) &= f_A(1) = \{2,4,6,8,10\}, \\ f_{A_2}(4) = f_{A_5}(4) &= f_A(0) = \mathbb{Z}, \\ f_{A_2}(5) = f_{A_5}(5) &= f_A(5) = \{2,4,6,8,10\}. \end{aligned}$$

Hence, $A \in NS_{\cap AG}(\mathbb{Z})$, as A is self conjugate soft int-AG-group.

Theorem 4.6. *Let $A \in S_{\cap AG}(U)$. Then the following assertions are equivalent for all $x, y \in G$,*

(1) $f_A\left((xy)x^{-1}\right) = f_A(y)$,
(2) $f_A\left((xy)x^{-1}\right) \supseteq f_A(y)$,
(3) $f_A\left((xy)x^{-1}\right) \subseteq f_A(y)$.

Proof. $(i) \Rightarrow (ii)$: Obvious.

$(ii) \Rightarrow (iii)$: Assume that (ii) holds. Consider,

$$\begin{aligned}
f_A\left((xy)x^{-1}\right) &\subseteq f_A\left(\left(x^{-1}\left((xy)x^{-1}\right)\right)\left((x^{-1})^{-1}\right)\right) \\
&= f_A\left(\left(x^{-1}\left((xy)\,x^{-1}\right)\right)x\right) \\
&= f_A\left(\left(x\left((xy)\,x^{-1}\right)\right)x^{-1}\right) && \text{(by the left invertive law)} \\
&= f_A\left(\left((xy)\left(xx^{-1}\right)\right)x^{-1}\right) && \text{(by Lemma 2.5-(ii))} \\
&= f_A\left(\left((xy)\,e\right)x^{-1}\right) \\
&= f_A\left(\left((ey)\,x\right)x^{-1}\right) && \text{(by the left invertive law)} \\
&= f_A\left((yx)\,x^{-1}\right) \\
&= f_A\left(\left(x^{-1}x\right)y\right) && \text{(by the left invertive law)} \\
&= f_A\left(ey\right) = f_A(y) \\
\Rightarrow f_A\left((xy)x^{-1}\right) &\subseteq f_A(y)\ \forall\ x,\ y \in G.
\end{aligned}$$

$(iii) \Rightarrow (i)$: Assume that (iii) holds. Consider,

$$\begin{aligned}
f_A\left((xy)\,x^{-1}\right) &\supseteq f_A\left(\left(x^{-1}\left((xy)\,x^{-1}\right)\right)\left(\left(x^{-1}\right)^{-1}\right)\right) \\
&= f_A(y),\ \text{as in the proof}\ (ii) \Rightarrow (iii) \\
\Rightarrow f_A\left((xy)\,x^{-1}\right) &\supseteq f_A(y)\ \forall\ x, y \in G.
\end{aligned}$$

Consequently, $f_A\left((xy)x^{-1}\right) \subseteq f_A(y) \subseteq f_A\left((xy)\,x^{-1}\right)$. Hence, $f_A\left((xy)\,x^{-1}\right) = f_A(y)$. □

Theorem 4.7. *Let $A \in S_{\cap AG}(U)$. Then $A \in NS_{\cap AG}(U)$ if and only if $f_A([x,\ y]) \supseteq f_A(x)\ \forall\ x,\ y \in G$, where $[x, y] = xy \cdot y^{-1}x^{-1}$ is a commutator of x and y in AG-group G.*

Proof. Let $A \in NS_{\cap AG}(U)$. Then,

$$\begin{aligned}
f_A([x,\ y]) &= f_A\left((xy)\left(y^{-1}x^{-1}\right)\right) && \text{(by Definition of Commutator's in } G\text{)} \\
&= f_A\left(\left(y^{-1}x^{-1}\right)(xy)\right) && \text{(by Lemma 3.5)} \\
&= f_A\left((yx)\left(x^{-1}y^{-1}\right)\right) && \text{(by Lemma 2.5-(iv))} \\
&= f_A\left(x^{-1}\left((yx)\,y^{-1}\right)\right) && \text{(by Lemma 2.5-(ii))} \\
&\supseteq f_A\left(x^{-1}\right) \cap f_A\left((yx)\,y^{-1}\right) \\
&= f_A(x) \cap f_A(x) && \text{(as } A \in NS_{\cap AG}(U)\text{)} \\
&= f_A(x).
\end{aligned}$$

Hence, $f_A([x, y]) \supseteq f_A(x)\ \forall\ x,\ y \in G$.

Conversely, assume that $f_A([x, y]) \supseteq f_A(x)\ \forall\ x, y \in G$. Then, for any $z \in G$,

$$\begin{aligned} f_A\left((xz)\,x^{-1}\right) &= f_A\left(e\left((xz)\,x^{-1}\right)\right) \\ &= f_A\left(\left(zz^{-1}\right)\left((xz)\,x^{-1}\right)\right) \end{aligned}$$

$$\begin{aligned} &= f_A\left(\left(\left((xz)\,x^{-1}\right)z^{-1}\right)z\right) && \text{(by the left invertive law)} \\ &= f_A\left(\left(\left(z^{-1}x^{-1}\right)(xz)\right)z\right) && \text{(by the left invertive law)} \\ &= f_A\left(\left((zx)\left(x^{-1}z^{-1}\right)\right)z\right) && \text{(by Lemma 2.5-(iv))} \\ &= f_A\left([z, x]z\right) \\ &\supseteq f_A([z, x]) \cap f_A(z) \\ &\supseteq f_A(z) \cap f_A(z) = f_A(z). \end{aligned}$$

This implies that $f_A\left((xz)\,x^{-1}\right) \supseteq f_A(z)\ \forall\ x \in G$. Now by Theorem 4.6, we have $f_A\left((xz)\,x^{-1}\right) = f_A(z)\ \forall\ x \in G$. Hence, $A \in NS_{\cap AG}(U)$. □

Proposition 4.8. *Let $A \in S_{\cap AG}(U)$. Then $f_A([x, y]) = f_A(e)\ \forall\ x, y \in G$ if and only if $A \in NS_{\cap AG}(U)$.*

Proof. Let $A \in NS_{\cap AG}(U)$. Then,

$$\begin{aligned} f_A\left((yx)\,y^{-1}\right) &= f_A(x)\ \forall\ x, y \in G \\ \Leftrightarrow\quad f_A\left(e\left((yx)\,y^{-1}\right)\right) &= f_A(x) \\ \Leftrightarrow f_A\left(\left(xx^{-1}\right)\left((yx)\,y^{-1}\right)\right) &= f_A(x) \\ \Leftrightarrow f_A\left(\left(\left((yx)\,y^{-1}\right)x^{-1}\right)x\right) &= f_A(x) && \text{(by the left invertive law)} \\ \Leftrightarrow f_A\left(\left(\left(x^{-1}y^{-1}\right)(yx)\right)x\right) &= f_A(x) && \text{(by the left invertive law)} \\ \Leftrightarrow f_A\left(\left((xy)\left(y^{-1}x^{-1}\right)\right)x\right) &= f_A(x) && \text{(by Lemma 2.5-(iv))} \\ \Leftrightarrow f_A\left(([x, y])\,x\right) &= f_A(x) \\ \Leftrightarrow f_A([x, y]) &= f_A(e). && \text{(by Lemma 3.7)} \end{aligned}$$

Hence, $A \in NS_{\cap AG}(G)$ if and only if $\mu_A([x, y]) = \mu_A(e)\ \forall\ x, y \in G$. □

5. α-inclusion of Soft Int-AG-groups

Definition 5.1. Let $A \in S_{\cap AG}(U)$. Then, e-set of A is denoted by $A_{\tilde{e}}$ and defined by

$$A_{\tilde{e}} = \{x \in G : f_A(x) = f_A(e)\}.$$

Example 5.2. In Example 3.3, $A_{\tilde{e}} = \{d\}$.

Theorem 5.3. *Let $A \in S_{\cap AG}(U)$. Then, $A_{\tilde{e}}$ is an AG-subgroup of G.*

Proof. By definition of $A_{\tilde{e}}$, it is obvious that $A_{\tilde{e}} \neq \emptyset$. Let $x, y \in A_{\tilde{e}}$. Then, $f_A(x) = f_A(e) = f_A(y)$. Consider,

$$\begin{aligned} f_A(xy^{-1}) &\supseteq f_A(x) \cap f_A(y) \\ &= f_A(e) \cap f_A(e) \\ &= f_A(e), \end{aligned}$$

also by Theorem 3.4, $f_A(e) \supseteq f_A(xy^{-1})\ \forall\ x, y \in G$. Consequently, $f_A(xy^{-1}) = f_A(e)$. This implies that $xy^{-1} \in A_{\tilde{e}}$. Hence $A_{\tilde{e}}$ is an AG-subgroup of G. □

Definition 5.4. Let $A \in S_{\cap AG}(U)$ and $\alpha \in P(U)$. Then α-inclusion of A, is denoted by $A_{\tilde{\alpha}}$, and defined by

$$A_{\tilde{\alpha}} = \{x \in G : f_A(x) \supseteq \alpha\},$$

while the set

$$A_{\tilde{\alpha}^+} = \{x \in G : f_A(x) \supset \alpha\},$$

is called the strong α-inclusion of A.

Note that if $\alpha = \emptyset$. Then $A_{\tilde{\alpha}} = \{x \in G : f_A(x) \neq \emptyset\}$, and is called support of A, and denoted by supp(A).

Example 5.5. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the universal set and $G = \{0, 1, 2, 3, 4, 5\}$ be an AG-group of order 6 defined as in Example 3.15. If we define soft int-AG-group A over U by:

$$\begin{aligned} f_A(0) &= U, \\ f_A(1) &= \{u_2, u_3, u_4, u_5, u_6\} = f_A(5), \\ f_A(2) &= \{u_1, u_2, u_3, u_4, u_5, u_6\} = f_A(4), \\ f_A(3) &= \{u_2, u_3, u_4, u_5, u_6, u_7\}. \end{aligned}$$

Let $\alpha = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, then $A_{\tilde{\alpha}} = \{0, 2, 4\}$ and $A_{\tilde{\alpha}^+} = \{0\}$.

Corollary 5.6, Theorem 5.7, 5.9 and 5.10 are available for both soft int-group and soft int-AG-groups. Therefore, proofs of them is similar in [7] and [10].

Corollary 5.6. *Let $B, C \tilde{\leq} A$. Then, the following assertions hold;*

(1) *$B \tilde{\subseteq} C$, $\alpha \in P(U)$. Then $B_{\tilde{\alpha}} \subseteq C_{\tilde{\alpha}}$,*
(2) *Let $\alpha_1 \subseteq \alpha_2$, $\alpha_1, \alpha_2 \in P(U)$. Then $B_{\tilde{\alpha}_2} \subseteq B_{\tilde{\alpha}_1}$,*
(3) *$B \tilde{=} C \Leftrightarrow B_{\tilde{\alpha}} = C_{\tilde{\alpha}}$, for all $\alpha \in P(U)$.*

Proof. Let $B, C \tilde{\leq} A$.

(1) Let $x \in B_{\tilde{\alpha}}$, then, $f_B(x) \supseteq \alpha$. Since $f_B \tilde{\subseteq} f_C$, $\alpha \in P(U)$. This implies that $\alpha \subseteq f_B \tilde{\subseteq} f_C \Rightarrow f_C(x) \supseteq \alpha \Rightarrow x \in C_{\tilde{\alpha}}$. Hence $B_{\tilde{\alpha}} \subseteq C_{\tilde{\alpha}}$.
(2) Let $\alpha_1 \subseteq \alpha_2$, $\alpha_1, \alpha_2 \in P(U)$, and $x \in B_{\tilde{\alpha}_2}$. Then $f_B(x) \supseteq \alpha_2$. Since, $\alpha_1 \subseteq \alpha_2$ implies that $f_B(x) \supseteq \alpha_1 \Rightarrow x \in B_{\tilde{\alpha}_1}$. Therefore, $B_{\tilde{\alpha}_2} \subseteq B_{\tilde{\alpha}_1}$.
(3) The proof is straight forward. □

Theorem 5.7. *Let $B, C \tilde{\leq} A$ and $\alpha \in P(U)$. Then,*

(1) $B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} \subseteq (B \tilde{\cup} C)_{\tilde{\alpha}}$,
(2) $B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} = (B \tilde{\cap} C)_{\tilde{\alpha}}$.

Proof. Let $B, C \tilde{\leq} A$, and G be the corresponding AG-group then:

(1) For any $x \in G$, let

$$\begin{aligned} x \in B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} &\Rightarrow x \in B_{\tilde{\alpha}} \text{ or } x \in C_{\tilde{\alpha}} \\ &\Rightarrow f_B(x) \supseteq \alpha \text{ or } f_C(x) \supseteq \alpha, \\ &\Rightarrow f_B(x) \cup f_C(x) \supseteq \alpha, \\ &\Rightarrow f_{B \tilde{\cup} C}(x) \supseteq \alpha, \\ &\Rightarrow x \in (B \tilde{\cup} C)_{\tilde{\alpha}}. \end{aligned}$$

Therefore, $B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} \subseteq (B \tilde{\cup} C)_{\tilde{\alpha}}$.
(2) Again, for any $x \in G$, let $x \in B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}}$. Then,

$$\begin{aligned} x \in B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} &\Leftrightarrow x \in B_{\tilde{\alpha}} \text{ and } x \in C_{\tilde{\alpha}} \\ &\Leftrightarrow f_B(x) \supseteq \alpha \text{ and } f_C(x) \supseteq \alpha, \\ &\Leftrightarrow f_B(x) \cap f_C(x) \supseteq \alpha, \\ &\Leftrightarrow f_{B \tilde{\cap} C}(x) \supseteq \alpha, \\ &\Leftrightarrow x \in (B \tilde{\cap} C)_{\tilde{\alpha}}. \end{aligned}$$

Hence, $B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} = (B \tilde{\cap} C)_{\tilde{\alpha}}$. □

Example 5.8. Let $U = \{u_1, u_2, \ldots, u_{10}\}$ be the universal set and G be any AG-group of order 8, defined by:

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	5	4	7	6	1	0	3	2
5	4	5	6	7	0	1	2	3
6	7	6	5	4	3	2	1	0
7	6	7	4	5	2	3	0	1

Define a soft int-AG-group A as follows:

$$\begin{aligned} f_A(0) &= U, \\ f_A(1) &= \{u_1, u_2, \ldots, u_6\}, \\ f_A(2) &= \{u_1, u_2, u_3, u_4\} = f_A(3), \\ f_A(4) &= \{u_1, u_2\} = f_A(5) = f_A(6) = f_A(7). \end{aligned}$$

Let $H_1 = \{0, 1, 2, 3\}$ and $H_2 = \{0, 1\}$ be any AG-subgroups of G. Define soft int-AG-groups B and C on H_1 and H_2 respectively as follows:

$$B = \{(0, U), (1, \{u_1, u_2\}), (2, \{u_1\}), (3, \{u_1\})\},$$

and

$$C = \{(0, U), (1, \{u_3\})\}.$$

As $B, C \tilde{\subseteq} A$. Therefore, B, $C \tilde{\leq} A$.
Now, let $\alpha = \{u_1, u_2, u_3\}$, then $B_{\tilde{\alpha}} = \{0\}$ and $C_{\tilde{\alpha}} = \{0\}$. Therefore, $B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} = \{0\}$ and $B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} = \{0\}$.
Also,

$$B \tilde{\cup} C = \{(0, U), (1, \{u_1, u_2, u_3\}), (2, \{u_1\}), (3, \{u_1\})\},$$

and,

$$B \tilde{\cap} C = \{(0, U)\},$$

$$(B \tilde{\cup} C)_{\tilde{\alpha}} = \{0, 1\} \text{ and } (B \tilde{\cap} C)_{\tilde{\alpha}} = \{0\}.$$

Hence, $B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} \subseteq (B \tilde{\cup} C)_{\tilde{\alpha}}$. While, $B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} = (B \tilde{\cap} C)_{\tilde{\alpha}}$.

Theorem 5.9. *Let $\{B_i : i \in I\}$ be the family of soft int-AG-subgroups of A over U. Then, for any $\alpha \in P(U)$,*

(1) $\bigcup_{i \in I} (B_{i\tilde{\alpha}}) \subseteq (\tilde{\cup}_{i \in I} B_i)_{\tilde{\alpha}}$,
(2) $\cap_{i \in I} (B_{i\tilde{\alpha}}) = (\tilde{\cap}_{i \in I} B_i)_{\tilde{\alpha}}$.

Proof. The proof is straight forward. □

Theorem 5.10. *Let $A \in S_{\cap AG}(U)$ and $\{\alpha_i : i \in I\}$ be a family of non-empty subsets of $P(U)$. If $\beta = \cap\{\alpha_i : i \in I\}$, and $\gamma = \cup\{\alpha_i : i \in I\}$. Then the following assertions hold,*

(1) $\cup_{i \in I} A_{\tilde{\alpha}_i} \subseteq A_\beta$,
(2) $\cap_{i \in I} A_{\tilde{\alpha}_i} = A_\gamma$.

Proof. The proof is clear from Definition 5.4. □

Theorem 5.11. *Let G be an AG-group and $\alpha \in P(U)$. Then $A \in S_{\cap AG}(U)$ if and only if $A_{\tilde{\alpha}}$ is a subgroup of G, where $A_{\tilde{\alpha}} \neq \emptyset$.*

Proof. Let $A \in S_{\cap AG}(U)$ and $A_{\tilde{\alpha}} \neq \emptyset$. Suppose that $x, y \in A_{\tilde{\alpha}}$, then $f_A(x) \supseteq \alpha$ and $f_A(y) \supseteq \alpha$. Therefore,

$$f_A(xy^{-1}) \quad \supseteq \quad f_A(x) \cap f_A(y) \supseteq \alpha.$$

By definition of α-inclusion, $xy^{-1} \in A_{\tilde{\alpha}}$. Hence, $A_{\tilde{\alpha}}$ is a subgroup of G.

Conversely, suppose that $A_{\tilde{\alpha}}$ is a subgroup of G for any $A_{\tilde{\alpha}} \neq \emptyset$. Let $x, y \in G$ such that $f_A(x) = \beta$ and $f_A(y) = \gamma$ and let $\delta = \beta \cap \gamma$. Then $x, y \in A_{\tilde{\delta}}$ and $A_{\tilde{\delta}} \leq G$ by hypothesis. So $xy^{-1} \in A_{\tilde{\delta}}$. Therefore, $f_A(xy^{-1}) \supseteq \delta = \beta \cap \gamma = f_A(x) \cap f_A(y)$. Hence, $A \in S_{\cap AG}(U)$. □

Theorem 5.12. *Let $A \in NS_{\cap AG}(U)$. Then, $A_{\tilde{e}}$ is a normal AG-subgroup of G.*

Proof. By Theorem 5.3, $A_{\tilde{e}} \leq G$. Let $x \in A_{\tilde{e}}$ and $g \in G$. Then, by Definition 4.4, we get

$$f_A(gx \cdot g^{-1}) \quad = \quad f_A(x) = f_A(e) \Rightarrow gx \cdot g^{-1} \in A_{\tilde{e}}.$$

Hence, $A_{\tilde{e}}$ is a normal AG-subgroup of G. □

Lemma 5.13. *Let G be an AG-group and $A \in S(U)$. Then, $A \in S_{\cap AG}(U)$ if and only if $A_{\tilde{\alpha}}$ is a AG-subgroup of G $\forall \alpha \in Im(A) \cup \{\beta \in P(U) : \beta \subseteq f_A(e)\}$.*

Proof. Let $A \in S_{\cap AG}(U)$ and $\alpha \in Im(G)$. As $f_A(e) \supseteq f_A(x)$ for all $x \in G, e \in A_{\tilde{\alpha}}$. Therefore $A_{\tilde{\alpha}} \neq \emptyset$. For $x, y \in A_{\tilde{\alpha}}$ $f_A(x) \supseteq \alpha$ and $f_A(y) \supseteq \alpha$. Since A is soft int-AG-group, $f_A(xy^{-1}) \supseteq f_A(x) \cap f_A(y) \supseteq \alpha \cap \alpha = \alpha$. Thus $xy^{-1} \in A_{\tilde{\alpha}}$. Similarly, if $\alpha \subseteq f_A(e)$, then it can be shown that $A_{\tilde{\alpha}}$ is an AG-subgroup of G. Conversely, let $A_{\tilde{\alpha}}$ be an AG-subgroup of G for all $\alpha \in Im(A) \cup \{\beta \in P(U) : \beta \subseteq f_A(e)\}$. Then for all $\alpha \in Im(A)$ we must have $e \in A_{\tilde{\alpha}}$ and so $f_A(e) \supseteq \alpha$. Suppose

$x, y \in G$ and $f_A(x) = \alpha$, $f_A(y) = \beta$. Let $\gamma = \alpha \cap \beta$. Then $x, y \in A_{\tilde{\gamma}}$ and $\gamma \subseteq f_A(e)$. By hypothesis, $A_{\tilde{\gamma}}$ is an AG-subgroup of G and so $xy^{-1} \in A_{\tilde{\gamma}}$. Thus $f_A(xy^{-1}) \supseteq \gamma = \alpha \cap \beta = f_A(x) \cap f_A(y)$. Hence A is soft int-AG-group of G.

□

Theorem 5.14. *Let G be an AG-group and $A \in S(U)$. Then, $A \in NS_{\cap AG}(U)$ if and only if $A_{\tilde{\alpha}}$ is a normal AG-subgroup of G $\forall \alpha \in Im(A) \cup \{\beta \in P(U) : \beta \subseteq f_A(e)\}$.*

Proof. Suppose that $A \in NS_{\cap AG}(U)$ and $\alpha \in Im(A) \cup \{\beta \in P(U) : \beta \subseteq f_A(e)\}$. Since A is a soft int-AG-group, $A_{\tilde{\alpha}}$ is a subgroup of G by Theorem 5.11. If $x \in G$ and $y \in A_{\tilde{\alpha}}$, from Definition 4.4, we know that $f_A(xy \cdot x^{-1}) = f_A(y) \supseteq \alpha$. Hence $xy \cdot y^{-1} \in A_{\tilde{\alpha}}$. Thus $A_{\tilde{\alpha}}$ is a normal AG-subgroup of G. Conversely, suppose that $A_{\tilde{\alpha}}$ is a normal AG-subgroup of G for all $\alpha \in Im(A) \cup \{\beta \in P(U) : \beta \subseteq f_A(e)\}$. From Lemma 5.13, $A \in S_{\cap AG}(U)$. Assume that $x, y \in G$ and $\alpha = f_A(y)$. Then $y \in A_{\tilde{\alpha}}$ and so $xy \cdot x^{-1} \in A_{\tilde{\alpha}}$. Thus $f_A(xy \cdot x^{-1}) \supseteq \alpha = f_A(y)$. This shows that A satisfies condition (iii) of Theorem 4.6. Consequently, from Theorem 4.6 $A \in NS_{\cap AG}(U)$. □

6. Conclusion

In this paper one of the most interesting concepts of soft set theory "Soft Intersection group" is extended to soft intersection AG-group. The notion of conjugates soft int-AG-group, normal soft int-AG-group, e-set and α-inclusion of soft int-AG-groups are presented and investigated. In future, these concepts can further be generalized to bipolar soft intersection AG-groups and soft intersection AG-rings. Moreover the study of homomorphism theorems in soft intersection AG-group may also be a nice work in this area.

Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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References

[1] M. I. Ali, F. Feng, X. Lui. W. K. Min and M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57** (2009), 1547-1553.

[2] H. Aktaş & N. Çağman, *Soft sets and soft groups,* Information Sciences,**177(13)** (2007), 2726-2735.

[3] A. Aygünoğlu, H. Aygün, *Introduction to fuzzy soft groups*, Computer and Mathematics with Applications, **58** (2009), 1279-1286.

[4] I. Ahmad, Amanullah, M. Shah, *Fuzzy AG-subgroups*, Life Science Journal, **9(4)** (2012), 3931-3936.

[5] Amanullah, Imtiaz Ahmad, and Faruk Karaaslan, *Cubic Abel-grassmann's subgroups*, Journal of Computational and Theoretical Nanoscience, **13(1)** (2016), 628-635,.

[6] Amanullah, I. Ahmad, M. Shah, *On the equal-height elements of fuzzy AG-subgroups*, Life Science Journal, **10(4)** (2013), 3143-3146, .

[7] N. Çağman, F. Çıtak and H. Aktaş, *Soft int-group and its applications to group theory*, Neural Computing and Applications, **21(1)** (2012), 151-158.

[8] N. Çağman, S. Enginoğlu, *Soft set theory and uni-int decision making*, European Journal of Operation Research, **207** (2010), 848-855.

[9] F. Feng, M.I. Ali, M. Shabir, *Soft relations applied to semigroups*, Filomat, **27(7)** (2013), 1183-1196.

[10] K. Kaygısız, *On soft int-groups*, Annals of Fuzzy Mathematics and Informatics, **4(2)** (2012), 365-375.

[11] F. Karaaslan, I. Ahmad, Amanullah, *Bipolar soft groups*, Journal of Intelligent and Fuzzy system, **31(1)** (2016), 651-662.

[12] X. Ma, J. Zhan, B. Davvaz, *Applications of soft intersection sets to hemirings via Si-h-Bi-ideals and Si-h-Quasi-ideals*, Filomat, **30(8)** (2016), 2295-2313.

[13] P. K. Maji, A.R. Roy and R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications, **44** (2002), 1077-1083.

[14] P. K. Maji, R. Biswas and A.R. Roy, *Soft set theory*, Computer and Mathematics with Application, **45** (2003), 555-562.

[15] M. S. Kamran, *Conditions for LA-semigroups to resemble associative structures, PhD thesis*, Quaid-i-Azam University, Islamabad 1993.

[16] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, **37** (1999), 19-31.

[17] A. Rosenfeld, *Fuzzy group*, Journal of Mathematical Analysis and Applications, **35** (1971), 512-517.

[18] A. S. Sezer, *A new approach to LA-semigroup theory via the soft sets*, Journal of Intelligent and Fuzzy Systems, **26** (2014), 2483-2495.

[19] A. Sezgin, N. Çağman, A.O. Atagün, M.I. Ali, E. Türkmen, *Soft Intersection Semigroups, Ideals and Bi-Ideals;a New Application on Semigroup Theory I*, Filomat **29(5)** (2015), 917-946.

[20] T. Shah, I. Rehman and A. Razzaq, *Soft ordered Abel-Grassman's groupoid (AG-groupoid)*, International Journal of the Physical Sciences, **6(25)** (2011), 6118-6126.

[21] M. Shah and A. Ali, *Some structural properties of AG-groups*, International Mathematical Forum, **6(34)** (2011), 1661-1667.

[22] M. Shah, C. Gretton, V. Sorge, *Enumerating AG-groups with a study of smarandache AG-groups*, International Mathematical Forum, **6(62)** (2011), 3079-3086.

[23] N. Sultana, N. Rani, M.I. Ali, A. Hussain, *Soft translations and soft extensions of BCI/BCK-algebras*, Applied Soft Computing, **36** (2015), 499-505.

[24] H. Zhang, J. Zhan, *Rough soft lattice implication algebras and corresponding decision making methods* , International Journal of Machine Learning and Cybernetics, **8(4)** (2016),1301-1308.

[25] J. Zhan, K. Zhu, *A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making,* Soft Computing, **21(8)** (2017), 1923-1936.

[26] J. Zhan, X. Zhou, D.Xiang, *Rough soft n-ary semigroups based on a novel congruence relation and corresponding decision making*, Journal of Intelligent and Fuzzy Systems, **33(2)** (2017), 693-703.

[27] K. Zhu, J. Zhan, *A study on soft Z-congruence relations over hemirings*, Journal of Intelligent and Fuzzy Systems, **30(1)** (2015), 467-474.

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