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SOME BAYES ESTIMATION STRATEGIES IN EXPONENTIAL DISTRIBUTION UNDER A WEIGHTED LOSS FUNCTION

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Abstract. In Bayesian approach, prior knowledge is often vague and any elicited prior distribution is only an approximation to the true one. E-Bayes and robust Bayes approaches consider a class of prior distributions instead of a single prior. In this paper, we deal with Bayes, E-Bayes and robust Bayes estimation of the exponential scale-parameter under a weighted loss function. We conduct a simulation study for comparison of these estimators.

Key Words: Bayes estimation, Class of prior, E-Bayes estimation, Robust Bayesian estimation, Weighted loss function.
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1. Introduction

Bayesian estimation approach is used when practitioners suppose that a correct prior exists, but they are unable to apply the pure Bayesian assumption. Because they are not confident enough to specify it completely or when a problem must be solved by two or more decision-makers and they do not agree on the prior distribution to be used. Various solutions to this problem have been proposed. some proposed solutions are E-Bayes and robust Bayes methodology, which have been applied over the last three decades. E-Bayes and robust Bayes analysis is performed by using a class of prior distributions and then to calculate the

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range of Bayes estimation. This method was first introduced by [6]. The E-Bayes estimator of unknown parameter is obtain on the basis of different distributions of the hyperparameter(s). These distributions are used to investigate the influence of the different prior distributions for obtaining optimal estimator, see [8]. In robust Bayes approach, there are several methods for obtaining optimal estimator such as the conditional Γ-minimax rule ([4], [2]), the posterior regret -minimax rule ([10], [9]), the least sensitive rule ([1]) and the most stable rules ([5], [3]). In this paper, we consider the problem of estimation under the following weighted loss function

$$L(\theta,\delta) = \big(\frac{\delta}{\theta} - 1\big)^2, \tag{1.1}$$

where δ is an estimator of θ. This loss function is strictly convex function of $\Delta = \frac{\delta}{\theta}$ and has a unique minimum at $\Delta = 1$. It is useful for estimation of the scale parameter. In this paper, Bayes, E-Bayes and robust Bayes approaches have been used to obtain the estimators of the exponential scale-parameter under the loss function (1.1). In section 2, we state preliminary definitions and formulas of Bayes, E-Bayes and robust Bayes estimation of known parameter. In section 3, we find the Bayes estimator of the exponential scale-parameter under the loss function (1.1). E-Bayes and robust Bayes estimators are developed in sections 4 and 5, respectively. Finally, a comparison is made between the Bayes, E-Bayes and robust Bayes estimators using a simulation in section 6.

2. PRELIMINARIES

Let $\mathbf{X}^n = (X_1, ..., X_n)$ be independent and identically distributed (i.i.d.) random variables from a distribution p_θ indexed by a real unknown parameter θ. Also, let (χ, B, p) denoted the probability space generated by X, where $\chi \subset R^n$, B is the σ-field of χ, $p = \{p_\theta(x)|\theta \in \Theta\}$ and Θ is the space parameter. In estimation of θ, let $L(\theta,\delta)$ be the loss function (1.1). Then, the posterior risk of δ based on observations $\mathbf{x}^n = (x_1, ..., x_n)$ can be expressed as

$$\begin{aligned} \rho(\pi,\delta) &= E[L(\theta,\delta)|\mathbf{X}^n = \mathbf{x}^n] \\ &= \delta^2 E[\frac{1}{\theta^2}|\mathbf{x}^n] - 2E[\frac{1}{\theta}|\mathbf{x}^n] + 1. \end{aligned} \tag{2.1}$$

The Bayes estimate of θ based on observation $\mathbf{x}^n$ is any estimate $\delta^B(\mathbf{x}^n)$ that minimizes the posterior risk (2.1), which is given by

$$\delta^B(\mathbf{x}^n) = \frac{E[\frac{1}{\theta}|x]}{E[\frac{1}{\theta^2}|x]}. \tag{2.2}$$

Information on the appropriate prior is often inadequate to unambiguously specify a prior distribution. The problem of expressing uncertainty regarding prior information can be solved by using a class of prior distributions. E-Bayesian and Robust Bayesian inference deal with such a problem by constructing methods which are stable to such a lack of information.

E-Bayes estimator is the expectation of the Bayes estimator for the all hyperparameters and is defined as

$$\delta^{EB}(\mathbf{x}^n) = \int_{\beta \epsilon D} \delta^B(\mathbf{x}^n)\pi(\beta)d\beta, \tag{2.3}$$

where $\delta^B(\mathbf{x}^n)$ is the Bayes estimate, $\pi(\beta)$ is a prior density for the hyperparameter(s) β and D be the set of all possible values of hyperparameter(s) β.

In robust Bayes methods, our interest is to construct some optimal estimators, when the prior run over the class Γ. There are several methods for obtaining optimal estimator of unknown parameter which we recall them as follow:
Let $\overline{F}(\pi, \delta)$ be a posterior functional. The optimal decision δ satisfies

$$\sup_{\pi \in \Gamma} \overline{F}(\pi, \delta) = \inf_{\delta \in D} \sup_{\pi \in \Gamma} \overline{F}(\pi, \delta).$$

δ^C is the conditional Γ-minimax estimator, where $\overline{F}(\pi, \delta) = \rho(\pi, \delta)$, δ^S is the most stable estimator, where $\overline{F}(\pi, \delta) = \sup_{\pi \in \Gamma} \rho(\pi, \delta) - \inf_{\pi \in \Gamma} \rho(\pi, \delta)$, δ^{PR} is the posterior regret Γ-minimax estimator, where $\overline{F}(\pi, \delta) = \rho(\pi, \delta) - \rho(\pi, \delta^\pi)$ and δ^{LS} is the least sensitive estimator, where $\overline{F}(\pi, \delta) = \frac{\rho(\pi, \delta) - \rho(\pi, \delta^\pi)}{\rho(\pi, \delta^\pi)}$.

3. Bayesian estimation strategy.

Let $\mathbf{X}^n = (X_1, X_2, ..., X_n)$ be a sequence of iid random variables from exponential $EXP(\theta)$ distribution with probability density function (p.d.f.)

$$f(x|\theta) = \theta e^{-\theta x}, \ x > 0, \ \theta > 0, \tag{3.1}$$

where θ is unknown scale parameter. Assume that θ has a prior distribution $Gamma(\alpha, \beta^{-1})$, with p.d.f.

$$\pi_{\alpha,\beta}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^\alpha e^{-\beta\theta}, \ \alpha > 0, \ \beta > 0, \ \theta > 0. \tag{3.2}$$

If $\mathbf{X}^n = \mathbf{x}^n$, then the posterior distribution is $Gamma(n+\alpha, (T+\beta)^{-1})$ where $T = \sum_{i=1}^n x_i$ and the posterior risk of an estimator δ under the loss function (1.1) is equal to

$$\rho(\pi, \delta) = \frac{\delta^2(T+\beta)^2}{(n+\alpha-1)(n+\alpha-2)} - \frac{2\delta(T+\beta)}{n+\alpha-1} + 1.$$

Therefore, the Bayes estimator of θ under the loss function (1.1) is given by

$$\delta^B(x) = \frac{E[\frac{1}{\theta}|\mathbf{x}^n]}{E[\frac{1}{\theta^2}|\mathbf{x}^n]} = \frac{n+\alpha-2}{T+\beta}. \tag{3.3}$$

4. E-Bayesian estimation strategy

According to [6] the prior parameters should be selected to guarantee that $\pi(\beta)$ is a decreasing function of β. If we take the conjugate prior (3.2), hyperparameters α and β should be in the ranges $0 < \alpha < 1$ and $\beta > 0$, respectively, due to $\frac{d\pi(\alpha,\beta)}{d\theta} < 0$. When $\alpha = 1$, π is a decreasing function of θ. Accordingly, b should not too big while $\alpha = 1$. It is better to choose $\alpha = 1$ and $0 < \beta < c$. Then, the prior density π is given by

$$\pi(\theta \mid \beta) = \beta e^{-\beta\theta}. \tag{4.1}$$

Following [7] we consider the following three prior distributions for the hyperparameter β:

$$\begin{aligned} \pi_1(\beta) &= \frac{2(c-\beta)}{c^2}, \ \alpha = 1, \ 0 < \beta < c, \\ \pi_2(\beta) &= \frac{1}{c}, \ \alpha = 1, \ 0 < \beta < c, \\ \pi_3(\beta) &= \frac{2\beta}{c^2}, \ \alpha = 1, \ 0 < \beta < c. \end{aligned} \tag{4.2}$$

In the following theorem, we obtain the E-Bayes estimators of θ under the loss function (1.1) and prior distributions (4.2).

Theorem 3.1. *Let $X_1, X_2, \ldots$ be a sequence of random variables which are conditionally independent given θ, and suppose that X_i given θ has*

$EXP(\theta)$-distribution. Suppose that $\boldsymbol{X}^n = \boldsymbol{x}^n$. Then, the E-Bayes estimators of θ corresponding to the priors given in (4.2) under the loss function (1.1) are equal to

$$
\begin{aligned}
\delta^{EB_1}(\mathbf{x}^n) &= \frac{2(n-1)}{c^2}\{\ln(\frac{c+T}{T})^{c+T} - c\}, \\
\delta^{EB_2}(\mathbf{x}^n) &= \frac{n-1}{c}\ln(\frac{c+T}{T}), \\
\delta^{EB_3}(\mathbf{x}^n) &= \frac{2(n-1)}{c^2}\{c - \ln(\frac{c+T}{T})^{T}\}
\end{aligned}
\tag{4.3}
$$

Proof. For $\pi_1(\beta)$, the E-Bayes estimator under the function(1.1) is given by

$$
\begin{aligned}
\delta^{EB_1}(\mathbf{x}^n) = \int_{\beta\in D}\delta^{B}(\mathbf{x}^n)\pi_1(\beta)d\beta &= \int_0^c \frac{n-1}{T+\beta}\frac{2(c-\beta)}{c^2}d\beta \\
&= \frac{2(n-1)}{c^2}\{\ln(\frac{c+T}{T})^{c+T} - c\}.
\end{aligned}
$$

Similarly, the E-Bayes estimator of θ under priors $\pi_2(\beta)$ and $\pi_3(\beta)$ can be obtained.

5. Robust Bayesian estimation strategy

The Bayes estimation is used if precise information about the prior is given. On the other hand, the robust Bayes estimation is applied if no information about the prior is available. The robust Bayes approaches deals with partial prior information, using a class of prior distributions. Now suppose that the prior distribution is not exactly specified and consider Γ class of prior of θ as:

$$\Gamma = \{Gamma(\alpha, \beta^{-1})|\beta \in [\beta_1, \beta_2] \subset R^{+}, \alpha = \alpha_0 > 0\}.$$

where α_0, β_1 and β_2 are known. Note that the robust Bayes estimation becomes the usual Bayes when the class of prior distribution contains a single prior.

The following theorem gives the robust Bayes estimators of θ under the loss function (1.1).

Theorem 4.1. *If the class of priors is equal to Γ, then the most stable estimator, the conditional Γ-minimax estimator, the posterior regret Γ-minimax estimator and the least sensitive under the loss function (1.1)* *are equivalent and are of the form*

$$\delta^{RB} = \frac{2(n+\alpha_0-2)}{2T+\beta_1+\beta_2} \tag{5.1}$$

The estimator δ^{RB} *is the Bayes estimator under the prior* $Gamma(a, \beta^*)$ *and under the loss function (1.1), where* $\beta^* = \frac{\beta_1+\beta_2}{2}$.

Proof. First of all we consider the problem of constructing the conditional Γ-minimax estimator. We consider posterior risk of δ, i.e.,

$$\begin{aligned}\rho(\pi,\delta) &= \frac{\delta^2 b^2}{(a-1)(a-2)} - \frac{2\delta b}{(a-1)} + 1 \\ &= \rho(\pi_b,\delta).\end{aligned}$$

where $a = n+\alpha$ and $b = T+\beta$. The first and second derivatives of $\rho(\pi_b,\delta)$ are equal to

$$\frac{\partial\rho(\pi_b,\delta)}{\partial b} = \frac{-2\delta^2 b}{(a-1)(a-2)} - \frac{2\delta}{a-1},\ a>2,$$

and

$$\frac{\partial^2\rho(\pi_b,\delta)}{\partial b^2} = \frac{2\delta^2}{(a-1)(a-2)} > 0.\ a>2.$$

Thus $\rho(\pi_b,\delta)$ is a strictly convex function of b and has a minimum at

$$b_{min} = \frac{a-2}{\delta}.$$

The function $l(\delta) = \rho(\pi_{b_1},\delta) - \rho(\pi_{b_2},\delta)$ is a decreasing and continuous function of δ and $l(\delta) = 0$ if and only if

$$\delta = \delta^* = \frac{2(a-2)}{b_1+b_2}$$

Hence, we have

$$\sup_{b\in[b_1,b_2]} \rho(\pi_b,\delta) = \begin{cases} \rho(\pi_{b_1},\delta) & \delta \geq \delta^* \\ \rho(\pi_{b_2},\delta) & \delta \leq \delta^*. \end{cases}$$

Note that $\rho(\pi_b,\delta)$ is a strictly convex function of δ and has a minimum at $\delta^\pi = \frac{a-2}{b}$. Since $\delta^{b_1} = \frac{a-2}{b_1} < \delta^*$, Thus for $\delta \geq \delta^*$ we have $\inf_{\delta\geq\delta^*}\rho(\pi_b,\delta) = \rho(\pi_{b_1},\delta^*)$. Moreover $\delta^{b_2} = \frac{a-2}{b_2} > \delta^*$ and then for $\delta \leq \delta^*$, we obtain $\inf_{\delta\leq\delta^*}\rho(\pi_b,\delta) = \rho(\pi_{b_2},\delta^*)$. Finally, we get

$$\inf_{\delta\in R}\sup_{b\in[b_1,b_2]} \rho(\pi_b,\delta) = \rho(\pi_{b_1},\delta^*) = \rho(\pi_{b_2},\delta^*)$$

which implies that $\delta_\Gamma^C = \delta^* = \frac{2(a-2)}{b_1+b_2}$.

Calculation of the most stable estimator reduces to the problem of constructing δ_Γ^C. Since $\inf_{b\in[b_1,b_2]} \rho(\pi_b,\delta) = \rho(\pi_{b_{min}},\delta) = 1 - \frac{a-2}{a-1}$ is not a function of δ, then we obtin

$$\inf_{\delta\in R}[\sup_{b\in[b_1,b_2]} \rho(\pi_b,\delta) - \inf_{b\in[b_1,b_2]} \rho(\pi_b,\delta)] = \inf_{\delta\in R}\sup_{b\in[b_1,b_2]} \rho(\pi_b,\delta) - \{1 - \frac{a-2}{a-1}\}.$$

which implies that $\delta_\Gamma^S = \delta_\Gamma^C$.

To find the posterior regret Γ-minimax and least sensitive estimators, we consider the posterior risk of Bayes estimator δ^B as

$$\rho(\pi_b,\delta^B) = 1 - \frac{a-2}{a-1}.$$

Since the posterior risk of the Bayes estimator does not depend on parameter b, therefore, the estimator $\delta_\Gamma^{PR} = \delta_\Gamma^{LS}$ is equal to δ_Γ^C.

6. Simulation study

In this section, We perform a numerical comparison between the Bayes, E-Bayes and robust Bayes estimators. For this purpose, we generate sequences n of independent random samples from exponential distribution with true value of parameter $\theta = 4$. Let $\delta_i^k,\ k = 1,2,3,4,5$ stands for $\delta^B(\mathbf{x}^n)$ with $\alpha = 1$ and $\beta = 3$ given by (3.3), E-Bayes estimators $\delta^{EB_i}(\mathbf{x}^n),\ i = 1,2,3$ for $c = 3$ given by (4.3), and robust estimator $\delta^{RB}(\mathbf{x}^n)$ over the classes Γ with $\alpha = 1$ and $\beta \in [1,3]$ given by (5.1) in ith replication, respectively. Repeat these tasks 10^4 times and calculate the value of Estimated Risk (ER) using the following formula

$$ER(\delta^k) = \frac{1}{10^4}\sum_{i=1}^{N}(\frac{\delta_i^k}{\theta} - 1)^2,$$

The results are summarized in Table 1. It is seen from Table 1 that the performance of the Bayes and robust Bayes estimators are quite satisfactory in terms of estimated risks. As we observe from Table 1, the estimated risk of the Bayes estimators of θ are

$$ER(\delta^{EB_1}) < ER(\delta^{EB_2}) < ER(\delta^{EB_3}) < ER(\delta^{RB}) < ER(\delta^B).$$

Moreover, the estimated risk decreases as the sample size increases and even for small sample size, performance of the E-Bayes estimators are quite satisfactory than the robust Bayes estimator. Also, the robust Bayes estimator work better than the Bayes estimator.

Table 1. Risks of the Bayes, E-Bayes and robust Bayes estimators for $\theta = 4$

n	δ^{B}	δ^{EB_1}	δ^{EB_2}	δ^{EB_3}	δ^{RB}
5	0.57926	0.21352	0.32825	0.44009	0.47062
10	0.33897	0.08273	0.16146	0.22712	0.24224
20	0.15558	0.02773	0.07312	0.10061	0.10487
30	0.08803	0.01348	0.04481	0.05952	0.06120
40	0.05704	0.00826	0.03246	0.04161	0.04242
50	0.04006	0.00566	0.02483	0.03116	0.03162

7. Conclusions

In this paper, we have studied the Bayes, E-Bayes and robust Bayes estimators of parameter exponential distribution (3.1) under the loss function (1.1). The Bayes estimator is obtained by choosing an explicit prior distribution over the interesting parameter. Usually in practical situations, there is a debate in choosing a unique prior or in case there are no unique prior information. In such cases, E-Bayes or robust Bayes can be employed to handle the uncertainty in specifying the prior distribution by considering a class of priors instead of a single prior. We first to obtain and to compare the Bayes, E-Bayes and robust Bayes estimators. Reviwing the simulation study, we find that E-Bayes estimator δ^{EB_1} better than the robust Bayes and Bayes estimators. Also, observed that by increasing n ,the almost performances of all estimators improves in terms of ER values, as illustrated in Table 1.

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