Journal of Hyperstructures 6 *(Spec. 13th AHA)* (*2017*), *27-37.*
ISSN: 2322-1666 print/2251-8436 online

SOME ORDERED HYPERSEMIGROUPS WHICH ENTER THEIR PROPERTIES INTO THEIR σ-CLASSES

NIOVI KEHAYOPULU

Abstract. An important problem in the theory of ordered hypersemigroups is to describe the ordered hypersemigroups which enter their properties into their σ-classes. In this respect, we prove the following: If H is a regular, left (right) regular, completely regular, intra-regular, left (right) quasi-regular, semisimple, k-regular, archimedean, weakly commutative, left (right) simple, simple, left (right) strongly simple ordered semigroup and σ a complete semilattice congruence on H then, for each $a \in H$, the σ-class $(a)_\sigma$ of H is, respectively, so.

Key Words: Ordered hypersemigroup, regular, left regular, intra-regular, left quasi-regular, semisimple, k-regular, archimedean, weakly commutative, left simple, simple, left strongly simple, complete semilattice congruence.

2010 Mathematics Subject Classification: Primary: 20N20; Secondary: 06F05, 20M10.

1. Introduction

The concept of the hypergroup introduced by the French Mathematician F. Marty at the 8th Congress of Scandinavian Mathematicians in 1933 is as follows: An hypergroup is a nonempty set H endowed with a multiplication xy such that (i) $xy \subseteq H$; (ii) $x(yz) = (xy)z$; (iii) $xH = Hx = H$ for every x, y, z in H (cf. [34]). Hundreds of papers appeared on hyperstructures since Marty introduced this concept, and

Received: 17 October 2017. Communicated by Ali Taghavi;
∗Address correspondence to N. Kehayopulu; E-mail: nkehayop@math.uoa.gr.

in the recent years, many groups in the world investigate the hypersemigroups in research programs using the definition given by Marty. Being impossible to give a complete information regarding the bibliography, we will refer only some recent books and articles such as the [1–3, 5–25, 27–36, 38–42]. This is the Theorem 3.3.2 in [33]: Let $\mathcal{K}$ be one of the following classes of semigroups: quasi left regular, quasi right regular, quasi-regular, quasi left π-regular, quasi right π-regular, quasi π-regular, left quasi-regular, right quasi-regular, completely quasi-regular, left quasi-π-regular, right quasi-π-regular, completely quasi-π-regular. Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$. Then S is a semigroup from class $\mathcal{K}$ if and only if S_α is in class $\mathcal{K}$, for every $\alpha \in Y$.

In the present paper we study some ordered hypersemigroups H which inter their properties into their σ-classes, where σ is a complete semilattice congruence on H. We prove that an ordered hypersemigroup H is regular, left (right) regular, intra-regular, left (resp. right) quasi-regular, left (right) simple, simple, archimedean, weakly commutative, k-regular if and only if, for each $a \in H$ the σ-classes $(a)_\sigma$ of H are, respectively, so. The corresponding results for hypersemigroups (without order) can be also obtained as application of the results of this paper, and this is because every hypersemigroup endowed with the equality relation is an ordered hypersemigroup.

According to Clifford-Preston, a relation σ on a groupoid $(S, \cdot, \le)$ is called right (resp. left) compatible (or regular or homogenous) if $a\rho b$ $(a, b \in S)$ implies $ac\rho bc$ (resp. $ca\rho cb$) for every $c \in S$. Although they could say "regular" the relation that is both right and left compatible, they call it "congruence". Petrich keeps the same definition of congruences, he goes a step further adding two additional properties and defines the "semilattice congruences". So, according to Petrich a semilattice congruence on a groupoid S is an equivalence relation σ on S such that $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$, $(ca, cb) \in \sigma$, $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for all $a, b, c \in S$. A semilattice congruence on an ordered groupoid $(S, \cdot, \le)$ is called complete [26] if $a \le b$ implies $(a, ab) \in \sigma$. These concepts can be naturally transferred to ordered hypersemigroups as follows: An equivalence relation σ on an hypersemigroup H is called *congruence* if $(a, b) \in \sigma$ implies $(a \circ c, b \circ c) \in \sigma$ and $(c \circ a, c \circ b) \in \sigma$ for every $c \in H$. A congruence σ on H is called *semilattice congruence* if $(a \circ a, a) \in \sigma$ and $(a \circ b, b \circ a) \in \sigma$ for every $a, b \in H$. A semilattice congruence σ on H is called *complete* if $a \le b$ implies $(a, a \circ b) \in \sigma$.

The concepts of regular, left (right) regular or intra-regular ordered hypersemigroups are well known but, for the sake of completeness, we will give these definitions below. When we write $(A]$ we mean the set of the elements x of H such that $x \le t$ for some $t \in H$. If we have an ordered hypersemigroup $(H, \circ, \le)$ and write the "$\le$" between subsets of H, that is if we write $A \le B$ (where $A, B \subseteq H$), then this means that for every $a \in A$ there exists $b \in B$ such that $a \le b$.

2. MAIN RESULTS

An ordered hypersemigroup $(H, \circ, \le)$ is called *regular* if for every $a \in H$ there exists $x \in H$ such that $a \le a \circ x \circ a$. This is equivalent to saying that $a \in (a \circ H \circ a]$ for every $a \in H$ or $A \subseteq (A \circ H \circ A]$ for every nonempty subset A of H.

Theorem 1. *Let H be a regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a regular subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b \in (a)_\sigma$. Then there exists $y \in (a)_\sigma$ such that $b \le b \circ y \circ b$. In fact: Since $b \in H$ and H is regular, we have $b \le b \circ x \circ b$ for some $x \in H$. Then $b \le b \circ x \circ (b \circ x \circ b) = b \circ (x \circ b \circ x) \circ b$. On the other hand, $x \circ b \circ x \subseteq (a)_\sigma$. Indeed: Since $b \le b \circ x \circ b$ and σ is a complete semilattice congruence on H, we have $(b, b^2 \circ x \circ b) \in \sigma$. Since σ is a semilattice congruence on H, we have $(x \circ b, b \circ x) \in \sigma$, $(b^2 \circ x \circ b, b^3 \circ x) \in \sigma$, $(b^3, b) \in \sigma$, $(b^3 \circ x, b \circ x) \in \sigma$, so $(b^2 \circ x \circ b, b \circ x) \in \sigma$. Thus we have $(b, b \circ x) \in \sigma$, $(b, x \circ b) \in \sigma$ and $(b \circ x, x \circ b \circ x) \in \sigma$. Since $(b, b^2 \circ x \circ b) \in \sigma$, $(b^2 \circ x \circ b, x \circ b) \in \sigma$ and $(b \circ x, x \circ b \circ x) \in \sigma$, we have $(b, x \circ b \circ x) \in \sigma$, and then $x \circ b \circ x \subseteq (b)_\sigma = (a)_\sigma$. $\square$

An ordered hypersemigroup $(H, \circ, \le)$ is called *left regular* if for every $a \in H$ there exists $x \in H$ such that $a \le x \circ a^2$. Equivalently, if $a \in (H \circ a^2]$ for every $a \in H$ or $A \subseteq (H \circ A^2]$ for every nonempty subset A of H. It is called *right regular* if for every $a \in H$ there exists $x \in H$ such that $a \le a^2 \circ x$. This is equivalent to saying that $a \in (a^2 \circ H]$ for every $a \in H$ or $A \subseteq (A^2 \circ H]$ for every nonempty subset A of H.

Theorem 2. *Let H be a left (resp. right) regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a left (resp. right) regular subsemigroup of H for every $a \in H$.*

Proof. Let H be left regular, $a \in H$ and $b \in (a)_\sigma$. Then there exists $z \in (a)_\sigma$ such that $b \le z \circ b^2$. In fact: Since $b \in H$ and H is left regular, we have $b \le x \circ b^2$ for some $x \in H$. Then we have $b \le x \circ b \circ (x \circ b^2) =$

$(x \circ b \circ x) \circ b^2$. On the other hand, $x \circ b \circ x \subseteq (a)_\sigma$. Indeed: Since σ is complete, we have $(b, b \circ x \circ b^2) \in \sigma$. Since σ is a semilattice congruence, we have $(b \circ x \circ b^2, x \circ b \circ x) \in \sigma$. Thus we have $(b, x \circ b \circ x) \in \sigma$ and so $x \circ b \circ x \subseteq (b)_\sigma = (a)_\sigma$. If H is right regular, the proof is similar. $\square$

An ordered hypersemigroup H is called *completely regular* if it is at the same time regular, left regular and right regular. This is equivalent to saying that for every $a \in H$ there exists $x \in H$ such that $a \leq a^2 \circ x \circ a^2$. That is, if $a \in (a^2 \circ S \circ a^2]$ for every $a \in H$ or if $A \subseteq (A^2 \circ H \circ A^2]$ for every nonempty subset A of H.

The following theorem is a consequence of Theorems 1 and 2. An independent proof is the following.

Theorem 3. *Let H be a completely regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a completely regular subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b \in (a)_\sigma$. Then there exists $y \in (a)_\sigma$ such that $b \leq b^2 \circ z \circ b^2$. In fact: Since $b \in H$ and H is completely regular, we have $b \leq b^2 \circ x \circ b^2$ for some $x \in H$. Then we have

$$\begin{aligned} b &\leq b \circ (b^2 \circ x \circ b^2) \circ x \circ (b^2 \circ x \circ b^2) \circ b \\ &= b^2 \circ (b \circ x \circ b^2 \circ x \circ b^2 \circ x \circ b) \circ b^2. \end{aligned}$$

On the other hand, $b \circ x \circ b^2 \circ x \circ b^2 \circ x \circ b \subseteq (a)_\sigma$. In fact: Since $b \leq b^2 \circ x \circ b^2$ and σ is complete, we have $(b, b^3 \circ x \circ b^2) \in \sigma$. Since σ is a semilattice congruence, we have $(b^3 \circ x \circ b^2, b \circ x \circ b^2 \circ x \circ b^2 \circ x \circ b) \in \sigma$. Thus we have $(b, b \circ x \circ b^2 \circ x \circ b^2 \circ x \circ b) \in \sigma$ and so $b \circ x \circ b^2 \circ x \circ b^2 \circ x \circ b \subseteq (b)_\sigma = (a)_\sigma$. $\square$

An ordered hypersemigroup $(H, \circ, \leq)$ is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ such that $a \leq x \circ a^2 \circ y$. This is equivalent to saying that $a \in (H \circ a^2 \circ H]$ for every $a \in H$ or $A \subseteq (H \circ A^2 \circ H]$ for every nonempty subset A of H.

Theorem 4. *Let H be an intra-regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is an intra-regular subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b \in (a)_\sigma$. Then there exist $z, w \in (a)_\sigma$ such that $b \leq z \circ b^2 \circ w$. In fact: Since $b \in H$ and H is intra-regular, we have

$b \le x \circ b^2 \circ y$ for some $x, y \in H$. Then we have

$$\begin{aligned} b &\le x \circ (x \circ b^2 \circ y) \circ (x \circ b^2 \circ y) \circ y \\ &\le x^2 \circ b^2 \circ y \circ x \circ (x \circ b^2 \circ y) \circ (x \circ b^2 \circ y) \circ y^2 \\ &= (x^2 \circ b^2 \circ y \circ x^2) \circ b^2 \circ (y \circ x \circ b^2 \circ y^3). \end{aligned}$$

On the other hand, $x^2 \circ b^2 \circ y \circ x^2 \subseteq (a)_\sigma$ and $y \circ x \circ b^2 \circ y^3 \subseteq (a)_\sigma$. In fact: Since σ is complete, we have $(b, b \circ x \circ b^2 \circ y) \in \sigma$. Since σ is a semilattice congruence, we have $(b \circ x \circ b^2 \circ y, x^2 \circ b^2 \circ y \circ x^2) \in \sigma$. Then $(b, x^2 \circ b^2 \circ y \circ x^2) \in \sigma$ and so $x^2 \circ b^2 \circ y \circ x^2 \subseteq (b)_\sigma = (a)_\sigma$. In a similar way we prove that $y \circ x \circ b^2 \circ y^3 \subseteq (a)_\sigma$. □

An ordered hypersemigroup H is called *left quasi-regular* if for every $a \in H$ there exists $x, y \in H$ such that $a \le x \circ a \circ y \circ a$. This is equivalent to saying $a \in (H \circ a \circ H \circ a]$ for every $a \in H$ or $A \subseteq (H \circ A \circ H \circ A]$ for every nonempty subset A of H. It is called *right quasi-regular* if for every $a \in H$ there exists $x, y \in H$ such that $a \le a \circ x \circ a \circ y$. That is, if $a \in (a \circ H \circ a \circ H]$ for every $a \in H$ or $A \subseteq (A \circ H \circ A \circ H]$ for every nonempty subset A of H.

Theorem 5. *Let H be a left (resp. right) quasi-regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a left (resp. right) regular subsemigroup of H for every $a \in H$.*

Proof. Let H be a left quasi-regular and $b \in (a)_\sigma$. Then there exist $u, v \in (a)_\sigma$ such that $b \le u \circ b \circ v \circ b$. In fact: Since $b \in H$ and H is left quasi-regular, we have $b \le s \circ b \circ t \circ b$ for some $s, t \in H$. Then we have

$$\begin{aligned} b &\le s \circ b \circ t \circ (s \circ b \circ t \circ b) \le s \circ b \circ t \circ s \circ b \circ t \circ (s \circ b \circ t \circ b) \\ &= (s \circ b \circ t \circ s) \circ b \circ (t \circ s \circ b \circ t) \circ b. \end{aligned}$$

Moreover we have $s \circ b \circ t \circ s \subseteq (a)_\sigma$ and $t \circ s \circ b \circ t \subseteq (a)_\sigma$. In fact, since $b \le s \circ b \circ t \circ b$ and σ is a complete semilattice congruence on H, we have $(b, b \circ s \circ b \circ t \circ b) \in \sigma$, then $(b, s \circ b \circ t \circ b) \in \sigma$. Since $(a, b) \in \sigma$, we have $(a, s \circ b \circ t \circ b) \in \sigma$. Since $(t \circ b, b \circ t) \in \sigma$, we have $(s \circ b \circ t \circ b, s \circ b^2 \circ t) \in \sigma$, then $(a, s \circ b \circ t) \in \sigma$, $(a, s \circ b \circ t \circ s) \in \sigma$, and $s \circ b \circ t \circ s \subseteq (a)_\sigma$. Moreover, since $(a, s \circ b \circ t) \in \sigma$, we have $(a, s \circ b \circ t^2) \in \sigma$, $(a, t \circ s \circ b \circ t) \in \sigma$, and $t \circ s \circ b \circ t \subseteq (a)_\sigma$. □

An ordered hypersemigroup H is called *semisimple* if for every $a \in H$ there exist $x, y, z \in H$ such that $a \le x \circ a \circ y \circ a \circ z$. Equivalently, if $a \in (H \circ a \circ H \circ a \circ H]$ for every $a \in H$ or $A \subseteq (H \circ A \circ H \circ A \circ H]$ for every nonempty subset A of H. In a similar way as in Theorem 5, we can prove the following theorem.

Theorem 6. *If H is a semisimple ordered hypersemigroup and σ a complete semilattice congruence on H, then the σ-class $(a)_\sigma$ is a semisimple subsemigroup of H for every $a \in H$.*

In the following we denote by N the set of natural numbers $\{1, 2, ..., n\}$.

An ordered hypersemigroup H is called *k-regular* ($k \in N$) if, for every $a \in H$, the k-power of a is regular. That is, for every $a \in H$ there exists $x \in H$ such that $a^k \le a^k \circ x \circ a^k$. In other words, $a^k \in (a^k \circ H \circ a^k]$ for every $a \in H$ or $A^k \subseteq (A^k \circ H \circ A^k]$ for every nonempty subset A of H.

Theorem 7. *Let H be a k-regular ordered hypersemigroup and σ a complete semilattice congruence on H. Then $(a)_\sigma$ is a k-regular subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b \in (a)_\sigma$. Then there exists $z \in (a)_\sigma$ such that $b \le b^k \circ z \circ b^k$. In fact: Since H is k-regular, there exists $x \in H$ such that $b^k \le b^k \circ x \circ b^k$. Then we have

$$b^k \le b^k \circ x \circ (b^k \circ x \circ b^k) = b^k \circ (x \circ b^k \circ x) \circ b^k.$$

Since σ is complete, we have $(b^k, b^k \circ b^k \circ x \circ b^k) \in \sigma$. Since σ is a semilattice congruence, $(b^k, b^k \circ b^k \circ x \circ b^k) \in \sigma$ implies $(b, x \circ b^k \circ x) \in \sigma$. Thus we have $x \circ b^k \circ x \subseteq (b)_\sigma = (a)_\sigma$. $\square$

An ordered hypersemigroup H is called *archimedean* if for every $a, b \in H$ there exists $n \in N$ such that $a^n \in (H \circ b \circ H]$.

Theorem 8. *Let H be an archimedean ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is an archimedean subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b, c \in (a)_\sigma$. Then there exist $m \in N$ and $x, y \in (a)_\sigma$ such that $b^m \le x \circ c \circ y$. In fact: Since $b, c \in H$ and H is archimedean, there exist $n \in N$ and $s, t \in H$ such that $b^n \le s \circ c \circ t$. Since σ is complete, we have $(b^n, b^n \circ s \circ c \circ t) \in \sigma$. Then, since σ is a semilattice congruence, we have $(b, b \circ s \circ c \circ t) \in \sigma$. Moreover we have

$$\begin{aligned} b^{3n+2} &= b \circ b^n \circ b^n \circ b^n \circ b \le b \circ (s \circ c \circ t) \circ (s \circ c \circ t) \circ (s \circ c \circ t) \circ b \\ &= (b \circ s \circ c \circ t \circ s) \circ c \circ (t \circ s \circ c \circ t \circ b). \end{aligned}$$

Since $(b, b \circ s \circ c \circ t) \in \sigma$ and σ is a semilattice congruence, we have $(b, b \circ s \circ c \circ t \circ s) \in \sigma$ and $(b, t \circ s \circ c \circ t \circ b) \in \sigma$, thus we have $b \circ s \circ c \circ t \circ s \subseteq (b)_\sigma = (a)_\sigma$ and so $t \circ s \circ c \circ t \circ s \subseteq (b)_\sigma = (a)_\sigma$. $\square$

An ordered hypersemigroup H is called *weakly commutative* if for every $a, b \in H$ there exists $n \in N$ such that $(a \circ b)^n \in (b \circ S \circ a]$.

Theorem 9. *Let H be a weakly commutative ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a weakly commutative subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $x, y \in (a)_\sigma$. Then there exist $m \in N$ and $s \in (a)_\sigma$ such that $(x \circ y)^m \le y \circ s \circ x$. In fact: Since $x, y \in H$ and H is weakly commutative, there exist $n \in N$ and $t \in H$ such that $(x \circ y)^n \le y \circ t \circ x$. Since σ is complete, we have $\Big((x \circ y)^n, (x \circ y)^n \circ y \circ t \circ x\Big) \in \sigma$. Since σ is a semilattice congruence, we get $(x \circ y, t \circ x \circ y \circ t) \in \sigma$. Moreover we have

$$\begin{aligned} (x \circ y)^{2n} &= (x \circ y)^n \circ (x \circ y)^n \le (y \circ t \circ x) \circ (y \circ t \circ x) \\ &= y \circ (t \circ x \circ y \circ t) \circ x. \end{aligned}$$

On the other hand, $t \circ x \circ y \circ t \subseteq (a)_\sigma$. Indeed: Since $x, y \in (a)_\sigma$ and $(a)_\sigma$ is a subsemigroup of H, we have $x \circ y \subseteq (a)_\sigma$, so $(a, x \circ y) \in \sigma$. Since $(a, x \circ y) \in \sigma$ and $(x \circ y, t \circ x \circ y \circ t) \in \sigma$, we have $(a, t \circ x \circ y \circ t) \in \sigma$ and so $t \circ x \circ y \circ t \subseteq (a)_\sigma$. $\square$

An ordered hypersemigroup H is called *left* (resp. *right*) *simple* if H is the only left (resp. right) ideal of H, that is, if for every left (resp. right) ideal T of H, we have $T = H$. One can easily prove that H is left (resp. right) simple if and only if $H = (H \circ a]$ (resp. $H = (a \circ H]$) for every $a \in H$.

Theorem 10. *Let H be a left (resp. right) simple ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a left (resp. right) simple subsemigroup of H for every $a \in H$.*

Proof. Let H be left simple, $a \in H$ and $b, c \in (a)_\sigma$. Then there exists $y \in (a)_\sigma$ such that $c \le y \circ b$. In fact: Since $b, c \in H$ and H is left simple, there exists $x \in H$ such that $c \le x \circ b$. Since $c \circ b, \{b\} \subseteq H$ and H is left simple, there exists $z \in H$ such that $b \le z \circ (c \circ b)$. Then we have

$$c \le x \circ (z \circ c \circ b) = (x \circ z \circ c) \circ b.$$

Moreover, $x \circ z \circ c \subseteq (a)_\sigma$. Indeed: Since σ is complete, we have $(c, c \circ x \circ z \circ c \circ b) \in \sigma$. Since σ is a semilattice congruence $(c, c \circ x \circ z \circ c \circ b) \in \sigma$ implies $(c, x \circ z \circ c \circ b) \in \sigma$. Since $b, c \in (a)_\sigma$ and $(a)_\sigma$ is a subsemigroup of H, we have $c \circ b \subseteq (a)_\sigma$ and so $(c \circ b, a) \in \sigma$. Since $(c \circ b, a) \in \sigma$ and $(a, c) \in \sigma$, we have $(c \circ b, c) \in \sigma$ then, since σ is a semilattice congruence, we have $(x \circ z \circ c \circ b, x \circ z \circ c) \in \sigma$. Since $(c, x \circ z \circ c \circ b) \in \sigma$ and $(x \circ z \circ c \circ b, x \circ z \circ c) \in \sigma$, we have $(c, x \circ z \circ c) \in \sigma$. Then $x \circ z \circ c \subseteq (c)_\sigma = (a)_\sigma$. The proof of the right analogue is similar. $\square$

An ordered hypersemigroup H is called *simple* if H is the only ideal of H. An ordered hypersemigroup H is simple if and only if $H = (H \circ a \circ H]$ for every $a \in H$. That is, for every $a, b \in H$ there exist $x, y \in H$ such that $b \le x \circ a \circ y$.

Theorem 11. *Let H be a simple ordered hypersemigroup and σ a complete semilattice congruence on H. Then the σ-class $(a)_\sigma$ is a simple subsemigroup of H for every $a \in H$.*

Proof. Let $a \in H$ and $b, c \in (a)_\sigma$. Then there exist $z, w \in (a)_\sigma$ such that $c \le z \circ b \circ w$. In fact: Since $b, c \in H$ and H is simple, there exist $x, y \in H$ such that $c \le x \circ b \circ y$. Since $c \circ b \circ c, \{b\} \subseteq H$ and H is simple, there exist $s, t \in H$ such that $b \le s \circ (c \circ b \circ c) \circ t$. Then we have

$$c \le x \circ (s \circ c \circ b \circ c \circ t) \circ y = (x \circ s \circ c) \circ b \circ (c \circ t \circ y).$$

Moreover $x \circ s \circ c, c \circ t \circ y \subseteq (a)_\sigma$. Indeed: Since $c \le x \circ b \circ y$, we have $x \circ s \circ c \le x \circ s \circ x \circ b \circ y$. Since σ is complete, we have

$$(x \circ s \circ c, x \circ s \circ c \circ x \circ s \circ x \circ b \circ y) \in \sigma.$$

Since σ is a semilattice congruence, $(x \circ s \circ c, x \circ s \circ c \circ x \circ s \circ x \circ b \circ y) \in \sigma$ implies $(x \circ s \circ c, c \circ x \circ s \circ b \circ y) \in \sigma$. In a similar way, from $b \le s \circ c \circ b \circ c \circ t$, we have $s \circ b \le s^2 \circ c \circ b \circ c \circ t$, $(s \circ b, s \circ b \circ s^2 \circ c \circ b \circ c \circ t) \in \sigma$, $(s \circ b \circ s^2 \circ c \circ b \circ c \circ t, b \circ s \circ c \circ b \circ c \circ t) \in \sigma$ and $(s \circ b, b \circ s \circ c \circ b \circ t \circ c \circ t) \in \sigma$. Form $b \le s \circ c \circ b \circ c \circ t$, we get $(b, b \circ s \circ c \circ b \circ c \circ t) \in \sigma$, then $(b, s \circ b) \in \sigma$, and $(c \circ x \circ b \circ y, c \circ x \circ s \circ b \circ y) \in \sigma$. From $c \le x \circ b \circ y$, we have $(c, c \circ x \circ b \circ y) \in \sigma$, then $(c, c \circ x \circ s \circ b \circ y) \in \sigma$. Since $(c, c \circ x \circ s \circ b \circ y) \in \sigma$ and $(x \circ s \circ c, c \circ x \circ s \circ b \circ y) \in \sigma$, we have $(c, x \circ s \circ c) \in \sigma$, then $x \circ s \circ x \subseteq (c)_\sigma = (a)_\sigma$. In a similar way we prove that $c \circ t \circ y \subseteq (a)_\sigma$. $\square$

An ordered hypersemigroup is called *left* (resp. *right*) *strongly simple* if it is both simple and left (resp. right) quasi-regular.

By Theorems 5 and 11 we have the following theorem.

Theorem 12. *If H is a left (resp. right) strongly simple ordered hypersemigroup and σ a complete semilattice congruence on H, then the σ-class $(a)_\sigma$ is a left (resp. right) simple subsemigroup of H for every $a \in H$.*

An ordered hypersemigroup H is called *intra-k-regular*, *left* (*right*) *k-regular*, *left* (*right*) *quasi-k-regular* or k-semisimple if for every element $a \in H$, the k-power of a is intra-regular, left (right) regular, left (right) quasi-regular or semisimple. The following should also be true: If H is an intra-k-regular, left or right k-regular, left or right quasi-k-regular or k-semisimple ordered hypersemigroup and σ a complete semilattice congruence of H then, for any $a \in H$, the σ-class of H containing a is respectively so, we leave the proof to the reader.

Remark. The converse statements of the theorems above obviously hold: If H is an ordered hypersemigroup, σ a semilattice congruence on H and $(a)_\sigma$ is a regular subsemigroup of H for every $a \in H$, then H is regular. The same holds if we replace the word regular by intra-regular, left (right) regular, completely regular, left (right) quasi-regular, k-regular, archimedean, weakly commutative, left (right) simple or simple, respectively.

Problem. Write a problem which, for a finite hypersemigroup H given by a table of multiplication and an order finds the semilattice congruences of H.

References

[1] M. Al Tahan and B. Davvaz, *Algebraic hyperstructures associated to biological inheritance*, Math. Biosci. **285** (2017), 112–118.

[2] P. Bonansinga and P. Corsini, *On semihypergroup and hypergroup homomorphisms* (Italian), Boll. Un. Mat. Ital. B (6) **1**, no. 2 (1982), 717–727.

[3] J. Chvalina and L. Chvalinova, *Transposition hypergroups formed by transformation operators on rings of differentiable functions*, Ital. J. Pure Appl. Math. no. **15** (2004), 93–106.

[4] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. **I**, Math. Surveys no. 7, Amer. Math. Soc. Providence, R.I. 1961 xv+224pp.

[5] P. Corsini, *Sur les semi-hypergroupes*, Atti. Soc. Peloritana Sci. Fis. Mat. Natur. **26**, no. 4 (1980), 363–372.

[6] P. Corsini, *Prolegomena of Hypergroup Theory*, Supplement to Riv. Mat. Pura Appl. Aviani Editore, Tricesimo, 1993. 215 pp. ISBN: 88-7772-025-5.

[7] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics (Dordrecht), 5. Kluwer Academic Publishers, Dordrecht, 2003. xii+322 pp. ISBN: 1-4020-1222-5.

[8] P. Corsini, M. Shabir and T. Mahmood, *Semisimple semihypergroups in terms of hyperideals and fuzzy hyperideals*, Iran. J. Fuzzy Syst. **8**, no. 1 (2011), 95–111.

[9] T. Changphas and B. Davvaz, *Bi-hyperideals and quasi-hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math. no. **35** (2015), 493–508.

[10] C. Darwin, *The Variation of Animals and Plants Under Domestication* (1st American edition), Orange Judd and Co., New York, 1868.

[11] B. Davvaz, *Some results on congruences in semihypergroups*, Bull. Malays. Math. Sci. Soc. (2) **23** (2000), no. 1, 53–58.

[12] B. Davvaz, *Characterizations of sub-semihypergroups by various triangular norms*, Chechoslovak Math. J. **55**, no. 4 (2005), 923–932.

[13] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ 2013. viii+200 pp.

[14] B. Davvaz, *Semihypergroup Theory*, 1st Edition, Elsevier 2016.

[15] B. Davvaz and A. Dehghan-Nezhad, *Chemical examples in hypergroups*, Ratio Matematica **14** (2003), 71–74.

[16] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA 2007.

[17] B. Davvaz, A. D. Nezad and A. Benvidi, *Chain reactions as experimental examples of ternary algebraic hyperstructures*, MATCH Commun. Math. Comput. Chem. **65**, no. 2 (2011), 491–499.

[18] B. Davvaz, A. D. Nezad and M. Mazloum-Ardakani, *Chemical hyperalgebra: redox reactions*, MATCH Commun. Math. Comput. Chem. **71** (2014), 323–331.

[19] B. Davvaz, A. D. Nezhad, *Dismutation reactions as experimental verifcations of ternary algebraic hyperstructures*, MATCH Commun. Math. Comput. Chem. **68** (2012), 551–559.

[20] B. Davvaz, A. D. Nezhad and A. Benvidi, *Chemical hyperalgebra: dismutation reactions*, MATCH Commun. Math.Comput. Chem. **67** (2012), 55–63.

[21] B. Davvaz, A. D. Nezhad and M. Heidari, *Inheritance examples of algebraic hyper-structures*, Inform. Sci. **224** (2013), 180–187.

[22] B. Davvaz and N. S. Poursalavati, *Semihypergroups and S-hypersystems*, Pure Math. Appl. **11** (2000), 43–49.

[23] A. J. F. Griffith, *An Introduction to Genetic Analysis*, 7th edn. New York: W. H. Freeman, 1999.

[24] K. Hila, B. Davvaz and J. Dine, *Study on the structure of Γ-semihypergroups*, Comm. Algebra **40**, no. 8 (2012), 2932–2948.

[25] K. Hila, B. Davvaz and K. Naka, *On quasi-hyperideals in semihypergroups*, Comm. Algebra **39**, no. 11 (2011), 4183–4194.

[26] N. Kehayopulu and M. Tsingelis, *Remark on ordered semigroups*. In: Ljapin, E. S. (edit.), Decompositions and Homomorphic Mappings of Semigroups. Interuniversitary collection of scientific works. St. Petersburg: Obrazovanie (ISBN 5-233-00033-4), pp. 50–55 (1992).

[27] M. Kondo and N. Lekkoksung, *On intra-regular ordered Γ-semihypergroups*, Int. J. Math. Anal. (Ruse) **7**, no. 25–28 (2013), 1379–1386.

[28] S. Lekkoksung, *On weakly semi-prime hyperideals in semihypergroups*, Int. J. Algebra **6**, no. 13–16 (2012), 613–616.

[29] S. Lekkoksung, *On left, right weakly prime hyperideals on semihypergroups*, Int. J. Contemp. Math. Sci. **7**, no. 21–24 (2012), 1193–1197.

[30] S. Lekkoksung, *Fuzzy bi-hyperfilters in semihypergroups*, Int. J. Math. Anal. (Ruse) **7**, no. 33–36 (2013), 1629–1634.

[31] T. Mahmood, *Some contributions to semihypergroups*, Ph.D. Thesis, Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan 2011.

[32] F. Marty, Sur une généralization de la notion de groupe, Actes du Congrès des Math. Scand., Stockholm 1934, p. 45–49.

[33] M. S. Mitrović, *Semilattices of Archimedean Semigroups*, University of Nis, Faculty of Mechanical Engineering, Nis, 2003. xiv+160pp.

[34] J. Mittas, *Hypergroupes canoniques valués et hypervalués*, Math. Balkanica **1** (1971), 181–185.
[35] J. Mittas, *Hypergroups canoniques*, Math. Balkanica **2** (1972), 165–179 .
[36] B. Pibaljommee, K. Wannatong and B. Davvaz, *An investigation of fzzy hyper-ieals of ordered semihypergroups*, Quasigroup Related Systems **23** (2015), 297–308.
[37] M. Petrich, *Introduction to Semigroups*, Charles E. Merrill Publ. Comp. A Bell & Howell Comp. Columbus, Ohio 1973. viii+198pp.
[38] B. A. Pierce, *Genetics, A Conceptual Approach*, MacMillan 2012.
[39] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press Monographs in Mathematics. Hadronic Press, Inc., Palm Harbor, FL, 1994. vi+180 pp. ISBN: 0-911767-76-2.
[40] T. Vougiouklis, *Convolutions on WASS hyperstructures*, Combinatorics (Rome and Montesilvano, 1994). Discrete Math. **174** (1997), no. 1–3, 347-355.
[41] T. Vougiouklis, *Hypermatrix representations of finite H_v-groups*, European J. Combin. **44** (2015), part B, 307-315.
[42] T. Vougiouklis, *On the hyperstructure theory*, Southeast Asian Bull. Math. **40** (2016), no. 4, 603-620.

Niovi Kehayopulu
Department of Mathematics, University of Athens, P.O.Box 16784, Athens, Greece
`Email: nkehayopmath.uoa.gr`