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FUZZY WEAKLY PRIME Γ-IDEALS IN Γ-RINGS

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ABSTRACT. In this work, we investigate fuzzy weakly prime Γ-ideal, fuzzy partial weakly prime Γ-ideal and fuzzy semiprime Γ- ideal of a commutative Γ-ring with nonzero identity. We obtained some characterizations of fuzzy weakly prime Γ-ideal, partial weakly prime Γ-ideal and semiprime Γ-ideal of a Γ-ring. Also some properties of these concepts have been studied.

Key Words: Fuzzy Prime Ideal, Fuzzy Weakly Prime Ideal,Fuzzy Weakly Prime Γ-Ideal, Fuzzy Partial Weakly Prime Γ-Ideal, Fuzzy Weakly Semiprime Γ-ideal.

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1. Introduction

L. A. Zadeh [5] introduced the concept of a fuzzy set in 1965, and N. Nobusawa [6] introduced the notion of a Γ-ring which is more general than a ring. Two years after that W. E. Barnes [7] emaciated relatively the conditions in the definition of the Γ-ring in the sense of Nobusawa. In [8], Jun and Lee introduced the concept of a fuzzy Γ-ring and again Jun [9] defined fuzzy prime ideal of a Γ-ring and obtained certain characterizations for a fuzzy ideal to be a fuzzy prime ideal. In [3], T. K. Dutta and Tanusree Chanda studied fuzzy prime ideal of a Γ-ring via its operator rings and obtained a number of characterisations of fuzzy prime ideal of a Γ-ring. As a continuation of the paper [3], in this paper, we assert the notion of a fuzzy weakly prime, fuzzy partial weakly prime

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and fuzzy semiprime Γ-ideals of a Γ-ring. We obtain some characterizations of fuzzy weakly prime, fuzzy partial weakly prime and fuzzy semiprime Γ-ideal.

2. Preliminaries

Definition 2.1. Let R be a ring and $\mu \in$ *LI(R)*. Then μ is called a fuzzy prime if μ is non-constant and for every α, $\beta \in LI(R)$ such that $\alpha \circ \beta \subseteq \mu$ the either $\alpha \subseteq \mu$ or $\beta \subseteq \mu$. [1]

Theorem 2.2. *Let μ is a fuzzy prime ideal of R, then μ_* is a prime ideal of R.* [2]

Definition 2.3. An L-ideal μ of R is called a fuzzy weakly prime if for every L-ideals α, β of R such that $0 \neq \alpha \circ \beta \subseteq \mu$ the either $\alpha \subseteq \mu$ or $\beta \subseteq \mu$. [1]

Theorem 2.4. *Let μ is non-constant fuzzy weakly prime ideal of R, then μ_* is a weakly prime ideal of R.* [1]

Definition 2.5. An element $1 \neq t \in L$ is called a prime element if $a \wedge b \leq t$ implies that either $a \leq t$ or $b \leq t$, for all $a,b \in L$. [1]

Definition 2.6. An element $t \neq 1$ in L is called a weakly prime element if $0 \neq a \wedge b \leq t$ implies that either $a \leq t$ or $b \leq t$, for all $a,b \in L$. [1]

Definition 2.7. Let R and Γ be two additive abelian groups. R is called a Γ- ring if there exists a mapping $f : R \times \Gamma \times R \longrightarrow R$ such that $f(a,\alpha,b) = a\alpha b$, $a,b \in R$, $\alpha \in \Gamma$, satisfying the following conditions for all $a,b,c \in R$ and for all α, β, $\gamma \in \Gamma$

(1) $(a+b)\alpha c = a\alpha c + b\alpha c$
(2) $a(\alpha+\beta)b = a\alpha b + a\beta b$
(3) $a\alpha(b+c) = a\alpha b + a\alpha c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$. [3]

Definition 2.8. A subset S of a Γ- ring R is said to be a Γ-ideal of R if

(1) S is an additive subgroup of R
(2) $r\alpha a \in S$ and $a\alpha r \in S$ for all $r \in R$, $\alpha \in \Gamma$, $a \in S$.

[3]

Proposition 2.9. *Let μ and ν be fuzzy Γ-ideals, then $\mu \cap \nu$ is a fuzzy Γ-ideal.*

Definition 2.10. Let μ and σ be two fuzzy subsets of Γ-ring R. Then the product of μ and σ is denoted by $\mu\Gamma\sigma$ and defined by as follows. [3]

$$(\mu\Gamma\sigma)(x) = \begin{cases} \sup_{x=r\gamma s}[min[\mu(r)\sigma(s)]], & for r, s \in R \text{ and } \gamma \in \Gamma \\ 0 & otherwise \end{cases}$$

Definition 2.11. Let R be a Γ- ring. A non-constant fuzzy ideal μ of R is called a fuzzy prime Γ-ideal of R, if $\alpha \circ \beta \subseteq \mu$ implies that $\alpha \subseteq \mu$ or $\beta \subseteq \mu$ for any α, β fuzzy ideals of R. [3]

Definition 2.12. An element $1 \neq t \in L$ is called a weakly prime Γ -element if $0 \neq a \wedge \gamma \wedge b \leq t$ implies that either $a \leq t$ or $b \leq t$, for all a,$b \in L$ and $\gamma \in R$. [3]

Definition 2.13. Let f be a mapping from Γ-ring R into Γ-ring S and μ be a fuzzy ideal of R. Now μ is said to be an f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all x,$y \in R$. [3]

Definition 2.14. A function f: $R \mapsto S$, where R,S are Γ-rings is said to be a Γ-homomorphism if

$$f(a + b) = f(a) + f(b), f(a\gamma b) = f(a)\gamma f(b)$$

for all a, $b \in R$, $\gamma \in \Gamma$. [3]

Definition 2.15. A fuzzy subset μ of a Γ-ring R is called a fuzzy point if $\mu(x) \in (0, 1]$ for some $x \in R$ and $\mu(y) = 0$ for all $y \in R \setminus x$. If $\mu(x) = t$, then the fuzzy point μ is denoted by x_t. [3]

Definition 2.16. Let μ be a fuzzy Γ-ideal of Γ-ring R. Then μ is said to be a fuzzy semiprime Γ-ideal if $\nu\Gamma\nu \subseteq \mu$, for all Γ-ideals ν implies that $\nu \subseteq \mu$. [10]

Theorem 2.17. *If R is a Γ-ring and μ is a fuzzy ideal of R. Then the following expressions are equivalent :*

(1) *If $0_R \neq x_r\Gamma x_r \subseteq \mu$, then $x_r \subseteq \mu$ where x_r fuzzy point on R and $\alpha \in \Gamma$.*
(2) *μ is a fuzzy semiprime Γ-ideal of R.* [10]

3. Fuzzy Weakly prime Γ-Ideals

Throughout this paper R be a commutative Γ-ring with nonzero identity.

Definition 3.1. Let R be a Γ- ring. A non-constant fuzzy ideal μ of R is called a fuzzy weakly prime Γ- ideal if $0 \neq \alpha\Gamma\beta \subseteq \mu$ implies that $\alpha \subseteq \mu$ or $\beta \subseteq \mu$ for any α, β fuzzy ideals of R. [3]

Definition 3.2. Let μ be a nonempty fuzzy subset of a Γ- ring R. If μ provide the following conditions, then μ is said to be a fuzzy Γ-ideal of R.

(1) $\mu(x\text{-}y) \geq \mu(x) \wedge \mu(y)$
(2) $\mu(x\alpha y) \geq \mu(x)$ and $\mu(x\alpha y) \geq \mu(y)$, for all x, $y \in R$ and all $\alpha \in \Gamma$. [3]

Theorem 3.3. *If R is a commutative Γ-ring and μ is a fuzzy ideal of R. Then the following expressions are equivalent :*

(1) *If $0_R \neq x_r\alpha y_t \subseteq \mu$, then $x_r \subseteq \mu$ or $y_t \subseteq \mu$, where x_r and y_t two fuzzy point on R and $\alpha \in \Gamma$*
(2) *μ is a fuzzy weakly prime Γ-ideal of R*

Proof. (1) $\Rightarrow$ (2) Let $0_R \neq \sigma\Gamma\theta \subseteq \mu$ for the fuzzy ideals σ and θ of R. Assume that $\sigma \nsubseteq \mu$. Then there exists an $x \in R$ such that $\mu(x) < \sigma(x)$. Let $\sigma(x) = a$ for this $x \in R$ and $\theta(y) = b$ for $y \in R$. Now there are two cases : $b = 0$ and $b \neq 0$. If $b = 0$, then automatically $x_a\gamma y_b \subseteq \mu$. If there exist $0 \neq b$ such that $\theta(y) = b \neq 0$, some $y \in R$ and if $z = x\gamma y$ for some $\gamma \in \Gamma$, then $(x_a\gamma y_b)(z) = a \wedge b$. Hence,

$$\begin{aligned} \mu(z) = \mu(x\gamma y) \geq (\sigma\Gamma\theta)(x\gamma y) &= \bigvee \{(\sigma(x) \wedge (\theta(y)\} \\ &\geq \sigma(x) \wedge \theta(y) = a \wedge b = (x_a\gamma y_b\ )(x\gamma y). \end{aligned}$$

Thus $x_a\gamma y_b \subseteq \mu$. In both cases by (1) $x_a \subseteq \mu$ or $y_b \subseteq \mu$. Hence $a \leq \mu(x)$ or $b \leq \mu(y)$. Therefore $\theta \subseteq \mu$, since $\sigma \nsubseteq \mu$. Thus μ is a weakly prime Γ- ideal of R.

(2) $\Rightarrow$ (1) Assume that μ is a fuzzy weakly prime Γ-ideal of Γ-ring R. Also let x_r, y_t be two fuzzy points of Γ-ring R and $0 \neq x_r\Gamma y_t \subseteq \mu$. From this we can say for all $\gamma \in \Gamma$,

$$(x_r\Gamma y_t)(x\gamma y) = \min\ \{r,t\} \leq \mu(x\gamma y)$$

Now, let we define two fuzzy subsets σ and θ as follows.

$$\sigma(a) = \begin{cases} r & a \in <x> \\ 0 & otherwise \end{cases} \quad and \quad \theta(a) = \begin{cases} t & a \in <y> \\ 0 & otherwise \end{cases}$$

$(\sigma\Gamma\theta)\ (a) = \sup_{a=u\gamma v}\ [\min[\sigma(u),\ \theta(v)]] = \min\ \{r,\ t\}$ where $u \in <x>$, $v \in <y>$ and $\gamma \in \Gamma$ or $(\sigma\Gamma\theta)\ (a) = 0$ where $u \notin <x>$, $v \notin <y>$ and $\gamma \in \Gamma$. From this two cases we get $(\sigma\Gamma\theta) \subseteq \mu$. Besides $\sigma \subseteq \mu$ or $\theta \subseteq \mu$ since μ is a fuzzy weakly prime Γ-ideal. Then $x_r \subseteq \mu$ or $y_t \subseteq \mu$ since $x_r \subseteq \sigma$ and $y_t \subseteq \theta$.

Example 3.4. Every fuzzy prime Γ-ideal is a fuzzy weakly prime Γ-ideal. But a fuzzy weakly prime Γ-ideal need not be fuzzy prime Γ-ideal.

Let $R = \mathbb{Z}_6$ and $\Gamma = \mathbb{Z}$. Then μ is defined by

$$\mu(x) = \begin{cases} 0 & x \in \{\overline{0},\ \overline{3}\} \\ 1 & \textit{otherwise} \end{cases} \tag{3.1}$$

for all $t \in L$. Let $\mu_t = \{\overline{0},\ \overline{3}\}$. Since $\{\overline{0},\ \overline{3}\}$ is a fuzzy prime Γ-ideal of R, then μ_t is a fuzzy weakly prime Γ-ideal but μ is not a fuzzy weakly prime Γ-ideal.

For fuzzy point $\overline{3}_{0.6}$, $\overline{1}_{0.5}$ of R.

$$\overline{3}_{0.6} \cdot 5 \cdot \overline{1}_{0.5} = (\ \overline{3}\ \cdot\ 5\ \cdot\ \overline{1}\ )_{0.6\wedge 0.5} = (\overline{3.5.1})_{0.6\wedge 0.5} = \overline{3}_{0.5}$$
$$\overline{3}_{0.5}(3) = 0.5 \leq \mu(3) \Rightarrow \overline{3}_{0.5} \in \mu$$
$$\overline{3}_{0.6}(3) = 0.6 \nleq \mu(3){=}0.5 \Rightarrow \overline{3}_{0.5} \notin \mu$$
$$\overline{1}_{0.5}(1) = 0.5 \nleq \mu(1){=}0 \Rightarrow \overline{1}_{0.5} \notin \mu$$

Hence μ is not a weakly prime Γ-ideal of Γ-ring.

Remark. It is obvious that $\mu \cap \nu$ is a Γ- ideal for all Γ- ideal μ and ν, but it is not a fuzzy weakly prime Γ-ideal.

Theorem 3.5. *Let R be a Γ-ring and I be an ideal of R. Also μ be a fuzzy subset of R defined by*

$$\mu(x) = \begin{cases} 1 & x \in I \\ \alpha & x \notin I \end{cases} \tag{3.2}$$

where $\alpha \in [0,1)$. Then where μ is a fuzzy weakly prime Γ-ideal of Γ-ring R if and only if I is weakly prime Γ-ideal of R.

Proof. ($\Longrightarrow$): Suppose that I be a prime Γ- ideal of R.

$$\mu(x) \wedge \mu(y) = \alpha \Longrightarrow \mu(x\text{-}y) \geq \alpha = \mu(x) \wedge \mu(y).$$
$$\mu(x) \wedge \mu(y) = 1 \Longrightarrow \mu(x) = 1 \text{ and } \mu(y) = 1.$$

Let $x,y \in I$, then x - $y \in I$. Hence $\mu(x$ - $y) = 1$ and $\mu(x$ - $y) \geq \mu(x) \wedge \mu(y)$ for all $x,\ y \in M$.

$$\text{If } \mu(x) \vee \mu(y) = \alpha, \text{ then } \mu(x\gamma y) \geq \mu(x) \vee \mu(y)$$

and hence $\mu(x\gamma y) \geq \mu(x)$ and $\mu(x\gamma y) \geq \mu(y)$.

If $\mu(x) \vee \mu(y) = 1$, then $\mu(x) = 1$ or $\mu(y) = 1$ or both are equal to 1.

Thus $x \in I$ or $y \in I$ or both. So $x\gamma y \in I$. Then $\mu(x\gamma y) = 1 \geq \mu(x)$ and $\mu(x\gamma y) = 1 \geq \mu(y)$. Therefore μ is a fuzzy prime Γ-ideal. Now we take two fuzzy ideals σ, θ of M such that $0 \neq \sigma\Gamma\theta \subseteq \mu$. Let $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Then there exists $a,b \in R$ such that $\sigma(a) > \mu(a)$ and $\theta(b) > \mu(b)$. In this case $\mu(a) = \alpha$ and $\mu(b) = \alpha$. Hence $a,b \notin I$. Also $a\Gamma R\Gamma b \subsetneq I$, since I is weakly prime Γ-ideal [4]. Then there exist $a\gamma_1 r\gamma_2 b \notin I$ where $r \in R$, $\gamma_1,\gamma_2 \in \Gamma$ such that $\mu(a\gamma_1 r\gamma_2 b) = \alpha$. Then,

$$\begin{aligned}\sigma\Gamma\theta(a\gamma_1 r\gamma_2 b) &\geq \sigma(a) \wedge \theta(r\gamma_2 b) \geq \sigma(a) \wedge \theta(b) \\ &> \mu(a) \wedge \mu(b) = \alpha = \mu(a\gamma_1 r\gamma_2 b).\end{aligned}$$

This is a contradiction. Therefore μ is fuzzy a weakly prime Γ-ideal. $(\Longleftarrow)$: Let μ be a fuzzy prime Γ-ideal and P,Q be ideals of Γ-ring R such that $0 \neq P\Gamma Q \subseteq I$. Suppose that $P \subsetneq I$ and $Q \subsetneq I$. Then there exist $p \in P\text{-}I$ and $q \in Q\text{-}I$.

$$\sigma\ (x) = \begin{cases} 1 & x \in P \\ \alpha & x \notin P \end{cases} \quad \textit{and} \quad \theta(x) = \begin{cases} 1 & x \in Q \\ \alpha & x \notin Q \end{cases}$$

two fuzzy subsets are defined as above. Obviously σ, θ are fuzzy ideals of R. $\sigma \subsetneq \mu$ since $\sigma(p) = 1 > \alpha = \mu(p)$. Also $\theta \subsetneq \mu$ since $(q) = 1 > \alpha = \mu(q)$. So there is a contradiction since $\sigma\Gamma\theta \subseteq \mu$. Thus I is weakly prime.

Proposition 3.6. *If μ be a non-constant fuzzy weakly prime Γ-ideal of R such that $\mu \neq 0_R$. Then $\mu(0) = 1$.*

Proof. Let μ be a fuzzy weakly prime Γ-ideal of R. Suppose $\mu(0) < 1$. Since we know that μ is a non-constant, there exist $a \in R$ such that $\mu(a) < \mu(0)$. Let σ, θ be two fuzzy Γ-ideals of R defined by

$$\sigma(x) = \begin{cases} 1 & x \in \mu_* \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\theta(x) = \mu(0)$$

for all $x \in R$. Then $\sigma\gamma\theta \subseteq \mu$. Also $\sigma\gamma\theta \neq 0$, since $\sigma\gamma\theta(0) = \mu(0) \neq 0$. Since $\sigma(0) = 1 > \mu(0)$ and $\theta(\text{a}) = \mu(0) > \mu(\text{a})$, $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Thus this is a contradiction.

Proposition 3.7. *Let μ be a non-constant fuzzy weakly prime Γ-ideal of R. Then μ_* is a weakly prime Γ-ideal R.*

Proof. Let we take $0 \neq x\gamma y \in \mu_*$ for all $\gamma \in \Gamma$. Then $\mu(x\gamma y) = \mu(0) = 1$ and for all $t \in (0,1]$, $t \leq \mu(x\gamma y)$. So, $x_t\gamma y_t \in \mu$ and since μ is a weakly prime Γ-ideal, $x_t \in \mu$ or $y_t \in \mu$. Therefore, $\mu(x) \geq t$ or $\mu(y) \geq t$. For $t = 1$,

$$\mu(x) \geq 1 = \mu(0) \text{ or } \mu(y) \geq 1 = \mu(0)$$

and hence $\mu(x) = \mu(0)$ and $\mu(y) = \mu(0)$. So $x \in \mu_*$ or $y \in \mu_*$. Thus μ_* is a weakly prime Γ-ideal.

Lemma 3.8. *If μ is a non-constant fuzzy weakly prime Γ-ideal of R, then μ_t is a fuzzy weakly prime Γ-ideal.*

Proof. Let $0 \neq x\gamma y \in \mu_t$, for all x, $y \in R$ and $\gamma \in \Gamma$. $x\gamma y \neq 0_t$ since

$$0_t(0) = t \neq 0 = (x\gamma y)_t(0).$$

Also $\mu(x\gamma y) \geq t$ and $(x\gamma y)_t = x_t\gamma y_t \in \mu$ since $x\gamma y \in \mu_t$. We know that μ is fuzzy weakly prime Γ-ideal and so $x \in \mu_t$ or $y \in \mu_t$. Therefore μ_t is a weakly prime Γ-ideal.

Theorem 3.9. *Let R and S be Γ-rings, $f : R \longrightarrow S$ be an epimorphism, μ be an f-invariant and $\mu(R)$ is finite. If μ_* is a fuzzy weakly prime Γ-ideal of R, then $f(\mu_*)$ is a weakly prime Γ-ideal of S.*

Proof. All subsets of $\mu(R)$ are finite since $\mu(R)$ is finite. Also μ has the sup-propery since all subsets of $\mu(R)$ have maximal element. Hence $f(\mu_*) = f(\mu)_*$.[2] Now, we take

$$0 \neq x'\gamma'y' \in f(\mu_*).$$

There exist x,$y \in R$ and $\gamma \in \Gamma$ such that $f(x) = x'$, $f(y) = y'$, $f(\gamma) = \gamma'$ since f is an epimorphism and hence $f(x\gamma y) = x'\gamma'y' \in f(\mu_*) = f(\mu)_*$. Thus,

$$f(\mu)(0) = f(\mu)(f(x\gamma y)) = \bigvee \{\mu(a\sigma b) : f(a\sigma b) = f(x\gamma y)\} = \mu(0).$$

Hence, there exist $z\alpha t \in R$ such that $\mu(0) = \mu(z\alpha t)$ and $f(x\gamma y) = f(z\alpha t)$. Thus we obtained $\mu(z\alpha t) = \mu(x\gamma y) = \mu(0)$ since μ is f-invariant.

Thereby $0 \neq x\gamma y \in \mu_*$. From hence $x \in \mu_*$ or $y \in \mu_*$ since μ_* is a weakly prime Γ-ideal. And then $x' = f(x) \in f(\mu_*)$ or $y' = f(y) \in f(\mu_*)$. Therefore, $f(\mu_*)$ is a weakly prime Γ-ideal of S.

Theorem 3.10. *Let R and S be Γ-rings, S be an integral domain, ν be a fuzzy ideal of S and $f : R \longrightarrow S$ be a homomorphism. If ν_* is a fuzzy weakly prime Γ-ideal of S, then $f^{-1}(\nu)_*$ is a weakly prime Γ-ideal of R.*

Proof. If we can show that $f^{-1}(\nu_*)$ is a weakly prime Γ-ideal, it is enough since

$$f^{-1}(\nu_*) = f^{-1}(\nu)_* \text{ [2]}$$

Let $0 \neq x\gamma y \in f^{-1}(\nu_*)$. Then $f(x\gamma y) \in \nu_*$. From this $f(x)f(\gamma)f(y) \in \nu_*$. There are two cases that need to be considered. If $f(x)f(\gamma)f(y) = 0$, $f(x) = 0$ or $f(y) = 0$. If $f(x)f(\gamma)f(y) \neq 0$, then $f(x) \in \nu_*$ or $f(y) \in \nu_*$ since ν_* is a fuzzy weakly prime Γ-ideal. Therefore $x \in f^{-1}(\nu_*)$ or $y \in f^{-1}(\nu_*)$ and $f^{-1}(\nu_*)$ is a weakly prime Γ-ideal.

4. Fuzzy Partial Weakly Prime Γ-Ideal

Definition 4.1. Let μ be a non-constant fuzzy Γ-ideal of Γ-ring R. Then μ is said to be a fuzzy partial weakly prime Γ-ideal if for $0 \neq x\gamma y$

$$\mu(x\gamma y) = \mu(x) \text{ or } \mu(x\gamma y) = \mu(y),$$

where $x,y \in R$, $\gamma \in \Gamma$.

Theorem 4.2. *Let μ be a nonconstant fuzzy Γ-ideal of R. Then μ is a fuzzy partial weakly prime Γ-ideal if and only if μ_t is a fuzzy weakly prime Γ-ideal of R for all $t \in (0,1]$.*

Proof. Suppose that μ is a fuzzy partial weakly prime Γ-ideal. Take $0 \neq x\gamma y \in \mu_t$. Then $\mu(x\gamma y) \geq t$ and also we get

$$\mu(x\gamma y) = \mu(x) \geq t \text{ or } \mu(x\gamma y) = \mu(y) \geq t,$$

since μ is a fuzzy partial weakly prime Γ-ideal. Thus $x \in \mu_t$ or $y \in \mu_t$. So, μ_t is a fuzzy weakly prime Γ-ideal.

Conversely, assume that μ_t is a fuzzy weakly prime Γ-ideal. Let $\mu(x\gamma y) = t$, for all $x\gamma y \neq 0$. There are two cases to be examined here: $t = 0$ and $t \neq 0$. If $t = 0$, then $\mu(x) \leq \mu(x\gamma y) = t = 0$ and $\mu(x) = 0$ or $\mu(y) \leq \mu(x\gamma y) = t = 0$ and $\mu(y) = 0$, since μ is a fuzzy Γ-ideal. If $t \neq 0$, then $0 \neq x\gamma y \in \mu_t$. Since μ_t is a fuzzy weakly prime Γ-ideal, we get $x \in \mu_t$ or $y \in \mu_t$. Thus

$$\mu(x) \geq t = \mu(x\gamma y) \text{ or } \mu(y) \geq t = \mu(x\gamma y).$$

From here, we obtain $\mu(x) = \mu(x\gamma y)$ or $\mu(y) = \mu(x\gamma y)$ because μ is also a fuzzy Γ-ideal. Thus, μ is a fuzzy partial weakly Γ-ideal.

Theorem 4.3. *Let R, S be a* Γ*-ring and* $f : R \longrightarrow S$ *be an injective ring homomorphism. If* μ *is a fuzzy partial weakly prime* Γ*-ideal of S, then* $f^{-1}(\mu)$ *is a fuzzy partial weakly prime* Γ*-ideal of R.*

Proof. We take $0 \neq x\gamma y \in R$ where $x, y \in R$ and $\gamma \in \Gamma$. Since f is a homomorphism

$$f^{-1}\ (\mu)\ (x\gamma y) = \mu(f(x\gamma y)) = \mu(f(x)\gamma f(y)).$$

f is injective and so $f(x\gamma y) \neq 0$. Also because μ is a fuzzy partial weakly prime Γ-ideal,

$$f^{-1}\ (\mu)\ (x\gamma y) = \mu(f(x\gamma y)) = \mu(f(x)\gamma f(y) = \mu(f(x)) \text{ or}$$

$$f^{-1}\ (\mu)\ (x\gamma y) = \mu(f(x\gamma y)) = \mu(f(x)\gamma f(y)) = \mu(f(y)).$$

So, $f^{-1}\ (\mu)\ (x\gamma y) = f^{-1}\ (\mu)\ (x)$ or $f^{-1}\ (\mu)\ (x\gamma y) = f^{-1}\ (\mu)\ (y)$. Then $f^{-1}(\mu)$ is a fuzzy partial weakly prime Γ-ideal of R.

Theorem 4.4. *Let R and S be two* Γ*-ring and* $f : R \longrightarrow S$ *be a surjective ring homomorphism which is a constant on Kerf. Then f(*μ*) is a fuzzy partial weakly prime* Γ*-ideal of S.*

Proof. Suppose that $0 \neq x\gamma y \in S$ where $x, y \in S$. Since f is an epimorphism, there exist r,t such that $x = f(\mathrm{r})$ and $y = f(\mathrm{t})$. Then,

$$f(\mu)(x\gamma y) = f(\mu)(f(r)\gamma f(t)) = f(\mu)(f(r\gamma t)) = \mu(r\gamma t).$$

If $r\gamma t = 0$, we obtain $f(0) = 0 = f(r\gamma t) = f(\mathrm{r})\gamma f(\mathrm{t}) = x\gamma y$ and this is a contradiction. Then $r\gamma t \neq 0$. Since μ is a fuzzy partial weakly prime Γ-ideal, we get

$$f(\mu)(x\gamma y) = \mu(r\gamma t) = \mu(r) = f(\mu)f(r) = f(\mu)(x) \text{ or}$$

$$f(\mu)(x\gamma y) = \mu(r\gamma t) = \mu(t) = f(\mu)f(t) = f(\mu)(y).$$

Hence $f(\mu)$ is a fuzzy partial weakly prime Γ-ideal.

5. Fuzzy Semiprime Γ-Ideal

Definition 5.1. Let μ be a non-constant fuzzy Γ-ideal of R. Then μ is said to be a fuzzy weakly semiprime Γ-ideal if $0 \neq \nu\Gamma\nu \subseteq \mu$, for all Γ-ideal ν implies that $\nu \subseteq \mu$.

Theorem 5.2. *If R is a Γ-ring and μ is a fuzzy ideal of R. Then the following expressions are equivalent :*

(1) *If $0_R \neq x_r\Gamma x_r \subseteq \mu$, then $x_r \subseteq \mu$ where x_r fuzzy point on R and $\alpha \in \Gamma$.*
(2) *μ is a fuzzy weakly semiprime Γ-ideal of R.*

Proof (1) $\Rightarrow$ (2) Let $0_R \neq \nu\Gamma\nu \subseteq \mu$ for the fuzzy ideal ν of R. Suppose $\nu \nsubseteq \mu$. Then there exist $x \in R$ such that $\mu(x) < \nu(x)$. Let $\nu(x) = r$. If $z = x\gamma x$ for some $\gamma \in \Gamma$, then $x_r\Gamma x_r(z) = r$. So,

$$\mu(z) = \mu(x\gamma x) \geq \nu\Gamma\nu(x\gamma x) \geq \nu(x) = r = x_r\Gamma x_r(z).$$

Hence $x_r\Gamma x_r \subseteq \mu$. From condition 1) $x_r(x) \leq \mu(x)$. From this we obtain

$$r = \nu(x) \leq \mu(x) \text{ and } \nu \subseteq \mu.$$

This is a contradiction. Thus μ is a weakly semiprime Γ- ideal of R.
(2) $\Rightarrow$ (1) Assume that μ is a fuzzy weakly semiprime Γ-ideal of Γ-ring R. Also let x_r be a fuzzy point of Γ-ring R and $0 \neq x_r\Gamma x_r \subseteq \mu$. From this we can say for all $\gamma \in \Gamma$,

$$(x_r\Gamma x_r)(x\gamma y) = r \leq \mu(x\gamma y) \text{ or } (x_r\Gamma x_r)(x\gamma y) = 0 \leq \mu(x\gamma y)$$

Now, let we define a fuzzy subset σ as follows.

$$\sigma\ (a) = \begin{cases} r & a \in <x> \\ 0 & a \notin <x> \end{cases}$$

$(\sigma\Gamma\sigma)\ (a) = \sup_a = u\gamma v\ [\min[\sigma(u), \sigma(v)]] = r$, where $u \in <x>$, $v \in <x>$ and $\gamma \in \Gamma$ or $(\sigma\Gamma\sigma)\ (a) = 0$ where $u \notin <x>$, $v \notin <x>$ and $\gamma \in \Gamma$. From this two cases we get $(\sigma\Gamma\sigma) \subseteq \mu$. Hence $\sigma \subseteq \mu$ since μ is a fuzzy weakly semiprime Γ-ideal. Then $x_r \subseteq \mu$ since $x_r \subseteq \sigma$.

Theorem 5.3. *Every fuzzy weakly prime Γ-ideal of R is a fuzzy weakly semiprime Γ-ideal of R.*

Proof. Let μ be a weakly prime Γ-ideal and $0 \neq \nu\Gamma\nu \subseteq \mu$. Since μ be a weakly prime Γ-ideal $\nu \subseteq \mu$ from the definition of fuzzy weakly prime Γ-ideal.

Theorem 5.4. *Let R be a Γ-ring and I be an ideal of R. Also ν be a fuzzy subset of R defined by*

$$\nu(x) = \begin{cases} 1 & x \in I \\ \alpha & x \notin I \end{cases}$$

where $\alpha \in$ [0,1). Then if I is a weakly semiprime Γ-ideal of R, then ν is a fuzzy weakly semiprime Γ-ideal of R.

Proof.($\Longrightarrow$): Suppose that I be a semiprime Γ- ideal of R.

$$\mu(x) \wedge \mu(y) = \alpha \Longrightarrow \mu(x\text{-}y) \geq \alpha = \mu(x) \wedge \mu(y).$$

$$\mu(x) \wedge \mu(y) = 1 \Longrightarrow \mu(x) = 1 \text{ and } \mu(y) = 1.$$

Let $x,y \in I$, then x - $y \in I$. Hence $\mu(x$ - $y) = 1$ and $\mu(x$ - $y) \geq \mu(x) \wedge \mu(y)$ for all x, $y \in M$.

$$If \mu(x) \vee \mu(y) = \alpha, \text{ then } \mu(x\gamma y) \geq \mu(x) \vee \mu(y)$$

and hence $\mu(x\gamma y) \geq \mu(x)$ and $\mu(x\gamma y) \geq \mu(y)$.
If $\mu(x) \vee \mu(y) = 1$, then $\mu(x) = 1$ or $\mu(y) = 1$ or both are equal to 1. Thus $x \in I$ or $y \in I$ or both. So $x\gamma y \in I$. Then

$$\nu(x\gamma y) = 1 \geq \mu(x) \text{ and } \nu(x\gamma y) = 1 \geq \mu(y).$$

Therefore, μ is a fuzzy prime Γ-ideal. And so μ is a fuzzy semiprime Γ-ideal. Now we take a fuzzy ideal ν of R such that $0 \neq \nu\Gamma\nu \subseteq \mu$. Let $\nu \subsetneq \mu$. Then there exist $a \in R$ such that $\nu(a) > \mu(a)$. In this case $\mu(a) = \alpha$ and $a \notin I$. Also $a\Gamma R\Gamma a \subsetneq I$, since I is a weakly semiprime Γ-ideal of R [4]. Then there exist $a\gamma_1 r\gamma_2 a \notin I$ where $r \in R$, γ_1,$\gamma_2 \in \Gamma$ such that $\mu(a\gamma_1 r\gamma_2 b) = \alpha$. Then

$$\begin{aligned} \nu\Gamma\nu(a\gamma_1 r\gamma_2 b) &\geq \nu(a) \wedge \nu(r\gamma_2 a) \geq \nu(a) \wedge \nu(a) \\ &> \nu(a) \wedge \nu(a) = \alpha = \mu(a\gamma_1 r\gamma_2 a). \end{aligned}$$

This is a contradiction. Therefore μ is a fuzzy weakly semiprime Γ-ideal.

Conclusion

In this study, we introduced the definitions of fuzzy weakly prime Γ-ideal, fuzzy partial weakly prime Γ-ideal and fuzzy weakly semiprime Γ-ideals on the commutative Γ-ring with nonzero identity.In addition to that we obtained some characterizations of fuzzy weakly prime Γ-ideal, fuzzy partial weakly prime Γ-ideal and fuzzy weakly semiprime Γ-ideals and gave example.

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