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ON WELL-POSEDNESS OF GENERALIZED EQUILIBRIUM PROBLEMS INVOLVING α-MONOTONE BIFUNCTION

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Abstract. The aim of this paper is to establish some uniqueness and well-posedness results for a general inequality of equilibrium problems type involving α-monotone bifunction, whose solution is sought in a subset K of a Banach space X. Some metric characterizations and sufficient conditions for these types of well-posedness are obtained. Moreover, we prove that the well-posedness of generalized equilibrium problems is equivalent to the existence and uniqueness of its solution.

Key Words: Equilibrium problems, Well-posed optimization problems, Monotonicity, Metric characterizations.

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1. Introduction

Well-posedness plays a crucial role in the stability analysis and numerical methods for optimization theory and nonlinear operator equations. The concept of well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied by Levitin and Polyak [17] and by Tykhonov [29], respectively, which has been known as the Levitin-Polyak and Tykhonov well-posedness, respectively.

There are many papers in the literature that deal with a generalization of Tykhonov well-posedness relating with optimization problems

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with more than one solution. This requires the existence and the convergence of some subsequences of every minimizing sequence towards a solution. For more details, we refer readers to (see [9, 2, 14, 33, 34]). Moreover, we notice that several problems of well-posedness have been generalized to nonconvex variational inequalities, saddle point problems, fixed-point problems, mathematical programming, Nash equilibrium problems, optimization problems with variational inequalities constrains, optimization problems with Nash equilibrium constrains (see [2, 5, 7, 8, 19, 18, 22, 23, 24, 25, 6]). Fang et al. [11] investigated the well-posedness of equilibrium problems; Kimura et al.[15] studied the parametric well-posedness for vector equilibrium problems; Bianchi et al. [3] introduced and studied two types of well-posedness for vector equilibrium problems; SJ and MH [?] investigated the Levitin-Polyak well-posedness of vector equilibrium problems with variable domination structures; Salamon [28] analyzed the Hadamard well-posedness of parametric vector equilibrium problems; Peng et al. [26] investigated several types of Levitin-Polyak well-posedness of generalized vector equilibrium problems. Long et al. [20] and Zaslavski [30] introduced the notions of generalized Levitin-Polyak well-posedness for explicit constrained EPs and generic well-posedness for EPs, respectively. Most of these works considered the perturbation of the parameters in the vector-valued case.

The main purpose aim of this work is to give a new contribution in this area. In particular, we establish some concepts of well-posedness by parametric for a class of generalized equilibrium problems with perturbations which includes in special case the classical equilibrium problems. Under suitable conditions, we further prove that the well-posedness of generalized equilibrium problems is equivalent to the existence and uniqueness of its solution. The distinguishing feature of our work lies in "ask F not to be monotone (as in most papers dealing with equilibrium problems in well-posed), but to be α-monotone (which is rather a weak condition compared to monotonicity)".

In order to achieve the above aim, the study is divided into the following sections. In Section 2, we recall necessary denitions and refer to some results. In Section 3, we establish and generalize the concepts of well-posedness for equilibrium problems to generalized equilibrium problems (EP_{Ψ}). We also derive some metric characterizations of well-posedness. In Section 4, we present a new concepts of well-posedness for optimization problems with constraints described by parametric generalized equilibrium problems. Additionally, Under suitable conditions, we prove that the well-posedness of generalized of equilibrium problems is equivalent to the existence and uniqueness of its solution.

2. Preliminaries

In this paper, unless stated otherwise we assume that E and X are two Banach spaces and K is a nonempty convex, closed subset of a Banach space X and X^* is topological dual space of X, while $\| \, . \, \|$ denote the norm in X^*.

For the convenience of the reader, we recall some definitions and results that need to be imposed in order to prove our main results. We recalled the following generalized equilibrium problem [13] (for short, (EP_Ψ)), in fact to find a $x \in K$ such that

$$F(x,y) + \Psi(x,y) \geq 0 \;\forall y \in K. \tag{2.1}$$

in which $F, \Psi : K \times K \to \mathbb{R}$ are two bifunctions and F is a generalized equilibrium function with $F(x,x) = 0 \;\forall x \in K$.

In what follows, we introduce the formulation of optimization problems with equilibrium constraint.

Let $h : T \times K \to \mathbb{R}$ and $F : T \times K \times K \to \mathbb{R}$ be two functions, in which $T \subset E$ is a nonempty set. The optimization problem with generalized equilibrium constraint (denoted by (OPGPEC)) is formulated as follows:

$$\min h(t,u) \; s.t. \; (t,u) \in T \times K \; and \; u \in S(t),$$

where $S(t)$ is the solution set of the parametric generalized equilibrium problem $(EP_\Psi(t))$ defined by, $u \in S(t)$ if and only if

$$F(t,u,v) + \Psi(u,v) \geq 0, \forall v \in K. \tag{2.2}$$

Instead of writing $\{(EP_\Psi(t)) : t \in T\}$ for the family of generalized equilibrium problems i.e., the parametric problem, we will simply write (EP_Ψ) in the sequel.

In order to highlight the generality of the problem (EP_Ψ) we recall below some special cases, as below:

(i) If $\Psi \equiv 0$ then problem (2.2) is reduces to the parametric equilibrium problem (for short, EP(t)), in fact finding $x \in K$ such that $F(t,x,y) \geq 0 \;\forall y \in K$ (see [1, 11]).

(ii) If $\Psi \equiv 0$ and $F(t,x,y) = -h(t,x,x-y)$ $\forall y \in K$ then problem (2.2) is reduces to the parametric quasivariational inequality (for short, QVI (t)) see [31].

(iii) If $\Psi(x,y) \equiv \Psi(x) - \Psi(y) \forall y \in K$ and $F(t,x,y) = \langle F(x), y-x\rangle$ then problem (2.2) is reduces to the mixed variational inequalities (for short, (MVI))see [10].

In recent years, some of authors have proposed many essential generalizations of monotonicity. We shall use a kind of generalized monotonicity, so called α- monotone bifunction.

Definition 2.1. [13] Let $\alpha : K \times K \to \mathbb{R}$ be a real-valued function. A bifunction $F : K \times K \to \mathbb{R}$ is called α- *monotone* if

$$F(x,y) + F(y,x) + \alpha(x,y) \leq 0 \ \forall x,y \in K. \tag{2.3}$$

Definition 2.2. [34] A real-valued function G, defined on a convex subset K of X, is said to be *hemicontinuous*, if

$$\lim_{t\to 0^+} G(tx + (1-t)y) = G(y) \ \forall x,y \in K. \tag{2.4}$$

Throughout this paper, we assume that for every $r \in [0,1]$

$$\lim_{r\to 0} \frac{\alpha(x,x_r)}{r} = 0 \tag{2.5}$$

$$\alpha(x,y) \leq \lim_{r\to 0} \frac{r-1}{r}\Big[\psi(x,x) + \alpha(x,x)\Big] \tag{2.6}$$

Definition 2.3. Let X be a Banach space. A mapping $\Lambda : X \to \mathbb{R}$ is said to be

(i) *lower semicontinuous* (for short, (l.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \leq \liminf_n \Lambda(x_n) \tag{2.7}$$

(ii) *upper semicontinuous* (for short ,(u.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \geq \limsup_n \Lambda(x_n) \tag{2.8}$$

for any sequence x_n of X such that $x_n \to x_0$

Let us recall that the concepts of noncompactness measure and Hausdorff metric.

Definition 2.4. [16] Let M, N be nonempty subsets of E. The Hausdorff metric $H(\cdot,\cdot)$ between N and M is defined by

$$H(N,M)=\max\{e(N,M),e(M,N)\},$$

where $e(N,M)=\sup_{a\in N} d(a,M)$ with $d(a,M)=\inf_{b\in M}\|a-b\|$.

Definition 2.5. [16] Assume that A is a nonempty subset of X. The measure of noncompactness β of the set A is defined by

$$\beta(A)=\inf\{\epsilon>0 : A\subset \bigcup_{j=1}^{n} A_j,\ diam A_j<\epsilon, j=1,...,n\},$$

where diam means the diameter of a set.

We close this section with theorem that will play a key role in the proof of our main results.

Theorem 2.6. [13] Suppose that $F: K\times K\to\mathbb{R}$, is α-monotone bifunction, hemicontinuous in the first argument and convex in the second argument. Let $\Psi,\alpha : K\times K\to\mathbb{R}$ be convex in the second argument, then generalized equilibrium problem (EP_Ψ) is equivalent to the following problem:
Find $x\in K$ such that

$$F(y,x)+\alpha(x,y)\le\Psi(x,y)\ \forall y\in K. \tag{2.9}$$

3. Well-posed of (EP_Ψ) with metric characterizations

In this section we establish some concepts of well-posed for generalized equilibrium problem (EP_Ψ). To start our analysis, through the results of this section, we give some conditions under which the equilibrium problem is strongly well-posed in the generalized sense.

Definition 3.1. A sequence $\{(t_n,x_n)\}\subset T\times K$ is said to be an approximating sequence for (EP_Ψ) if there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n\to 0$ as $n\to\infty$ such that

$$F(t,x_n,y)+\Psi(x_n,y)\ge -\epsilon_n\|y-x_n\|\ \forall n\in\mathbb{N}, y\in K. \tag{3.1}$$

Definition 3.2. The problem (EP_Ψ) is said to be strongly (resp., weakly) well-posed (resp., strongly (resp., weakly) well-posed in the generalized sense) if (EP_Ψ) has a unique solution x, and for every sequence $\{x_n\}$ with $x_n\to x$, every approximating sequence for (EP_Ψ) converges strongly (resp., weakly) to the unique solution (resp., if (EP_Ψ) has a nonempty solution set S(t), and every approximate solution sequence has a subsequence which strongly (resp., weakly) converges to some point of S(t)).

In what follows, we shall establish some characterizations of well-posedness for (EP_Ψ). For any $\epsilon > 0$ we define two sets:

$$\Gamma(\epsilon) := \Big\{(t,x) \in K : F(t,x,y) + \Psi(x,y) \geq -\epsilon\|y-x\| \ \forall y \in K\Big\}.$$

and

$$\Lambda(\epsilon) := \Big\{(t,x) \in K : F(t,y,x) + \alpha(x,y) \leq \Psi(x,y) + \epsilon\|y-x\| \ \forall y \in K\Big\}.$$

Lemma 3.3. *Let K be a nonempty convex, closed subset of a Banach space X. Suppose that $F : T \times K \times K \to \mathbb{R}$ and $\Psi, \alpha : K \times K \to \mathbb{R}$ be three functions. Let the following conditions hold:*

(i) $F(t,x,x) = 0$ *,* $\forall t \in T, x \in K$*,*
(ii) $F(t,\cdot,\cdot)$ *is* α*-monotone bifunction and hemicontinuous* $\forall t \in T$*,*
(iii) $F(t,x,\cdot)$ *is convex* $\forall t \in T, x \in K$*,*
(iv) $\Psi(x,\cdot)$ *and* $\alpha(x,\cdot)$ *are convex* $\forall x \in K$*.*

Then, $\Gamma(\epsilon) = \Lambda(\epsilon)$ *for all* $\epsilon > 0$*.*

Proof. Suppose that $(t,x) \in \Gamma(\epsilon)$. There exists $(t,x) \in T \times K$ such that

$$F(t,x,y) + \Psi(x,y) \geq -\epsilon\|y-x\| \ \forall y \in K. \tag{3.2}$$

Since $F(t,\cdot,\cdot)$ is α-monotone bifunction

$$F(t,y,x) + \alpha(x,y) \leq -F(t,x,y) \ \forall x,y \in K,$$

so

$$\begin{aligned} F(t,y,x) + \alpha(x,y) &\leq -F(t,x,y) \\ &\leq \Psi(x,y) + \epsilon\|y-x\| \end{aligned} \tag{3.3}$$

Therefore, $(t,x) \in \Lambda(\epsilon)$. Conversely, assume that $(t,x) \in \Lambda(\epsilon)$ and fix $y \in K$.

Letting $x_\lambda = x - \lambda(x-y), \lambda \in]0,1[$ then $x_\lambda \in K$, since K is a convex. Then

$$\begin{aligned} F(t,x_\lambda,x) + \alpha(x,x_\lambda) - \Psi(x,x_\lambda) &\leq \epsilon\|x_\lambda - x\| \\ &= \lambda\epsilon\|y-x\|. \end{aligned} \tag{3.4}$$

Taking into account $F(t,x,\cdot)$ is convex

$$0 = F(t,x_\lambda,x_\lambda) \leq F(t,x_\lambda,x) - \lambda\Big[F(t,x_\lambda,x) - F(t,x_\lambda,y)\Big],$$

so,

$$\lambda\Big[F(t,x_\lambda,x)-F(t,x_\lambda,y)\Big]\le F(t,x_\lambda,x). \tag{3.5}$$

Since $\Psi(x,\cdot)$ and $\alpha(x,\cdot)$ are convex $\forall x\in K$.

$$\alpha(x,x_\lambda)\le\alpha(x,x)-\lambda\Big[\alpha(x,x)-\alpha(x,y)\Big] \tag{3.6}$$

$$\Psi(x,x_\lambda)\le\Psi(x,x)-\lambda\Big[\Psi(x,x)-\Psi(x,y)\Big] \tag{3.7}$$

Then, from (3.4),(3.5),(3.6) and (3.7),

$$\begin{aligned}&\lambda\big[F(t,x_\lambda,x)-F(t,x_\lambda,y)+\alpha(x,x)-\alpha(x,y)+\Psi(x,x)-\Psi(x,y)\big]\\&\le F(t,x_\lambda,x)+\alpha(x,x)-\alpha(x,x_\lambda)+\Psi(x,x)-\Psi(x,x_\lambda)\\&\le\lambda\epsilon\|y-x\|-2\alpha(x,x_\lambda)+\alpha(x,x)+\Psi(x,x).\end{aligned}$$

Since $F(t,\cdot,\cdot)$ is hemicontinuous,

$$\begin{aligned}\lambda\big[-F(t,x,y)-\Psi(x,y)-\epsilon\|y-x\|\big] \quad\le\quad & -2\alpha(x,x_\lambda)+\lambda\alpha(x,y)\\&+(1-\lambda)\big[\Psi(x,x)+\alpha(x,x)\big],\end{aligned}$$

so

$$\begin{aligned}F(t,x,y)+\Psi(x,y)+\epsilon\|y-x\| \quad\ge\quad & \frac{2\alpha(x,x_\lambda)}{\lambda}-\alpha(x,y)\\&+\frac{(\lambda-1)}{\lambda}\Big[\Psi(x,x)+\alpha(x,x)\Big].\end{aligned}$$

From (2.5) and (2.6),

$$F(t,x,y)+\Psi(x,y)\ge-\epsilon\|y-x\|\ \forall y\in K. \tag{3.8}$$

Hence, $(t,x)\in\Gamma(\epsilon)$. Therefore, $\Gamma(\epsilon)=\Lambda(\epsilon)$ for all $\epsilon>0$. □

Lemma 3.4. *Let K be a nonempty convex, closed subset of a Banach space X. Suppose that $F:T\times K\times K\to\mathbb{R}$ and $\Psi,\alpha:K\times K\to\mathbb{R}$ satisfy in the following conditions:*

(i) $F(\cdot,y,\cdot)$ and $\alpha(\cdot,y)$ are l.s.c $\forall y\in K$,
(ii) $\Psi(\cdot,y)$ is u.s.c $\forall y\in K$.

Then $\Lambda(\epsilon)$ is closed in $T\times K$ for any $\epsilon>0$.

Proof. Assume that $\{(t_n, x_n)\} \subset \Lambda(\epsilon)$ is a sequence in which $(t_n, x_n) \to (t, x)$ in $T \times K$. Then

$$F(t_n, y, x_n) + \alpha(x_n, y) \leq \Psi(x_n, y) + \epsilon \|y - x_n\| \; \forall y \in K.$$

Since $\alpha(\cdot, y)$, $F(\cdot, y, \cdot)$ are l.s.c and $\Psi(\cdot, y)$ is u.s.c, then

$$\begin{aligned} F(t, y, x) + \alpha(x, y) &\leq \liminf_n \big[F(t_n, y, x_n) + \alpha(x_n, y)\big] \\ &\leq \limsup_n \big[\Psi(x_n, y) + \epsilon\|y - x_n\|\big] \\ &\leq \Psi(x, y) + \epsilon\|y - x\|. \end{aligned}$$

which implies that $(t, x) \in \Lambda(\epsilon)$. Therefore, $\Lambda(\epsilon)$ is closed in $T \times K$ for all $\epsilon > 0$. □

Theorem 3.5. Assume that K is a nonempty convex, closed subset of a Banach space X. Let $F : T \times K \times K \to \mathbb{R}$ and $\Psi, \alpha : K \times K \to \mathbb{R}$ be three functions. If (EP_Ψ) is strongly well-posed, then

$$\Gamma(\epsilon) \neq \emptyset \; \forall \epsilon > 0, \; \lim_{\epsilon \to 0} diam(\Gamma(\epsilon)) = 0. \tag{3.9}$$

Moreover, if the following assumptions hold:

(i) $F(\cdot, x, \cdot)$ *is l.s.c* $\forall x \in K$,
(ii) $F(t, x, \cdot)$ *is convex* $\forall (t, x) \in T \times K$,
(iii) $F(t, \cdot, \cdot)$ *is* α*-monotone bifunction, hemicontinuous,* $\forall t \in T$,
(iv) $F(t, x, x) = 0$, $\forall t \in T, x \in K$,
(v) $\alpha(\cdot, y)$ *and* $\Psi(\cdot, y)$ *are u.s.c* $\forall y \in K$,
(vi) $\Psi(x, \cdot)$ *and* $\alpha(x, \cdot)$ *are convex* $\forall x \in K$.

Then the converse holds.

Proof. Assume that (EP_Ψ) is strongly well-posed. Then, (EP_Ψ) admit a unique solution $(t, x) \in T \times K$, i.e.,

$$F(t, x, y) + \Psi(x, y) \geq 0 \; \forall y \in K.$$

Clearly, $\Gamma(\epsilon) \neq \emptyset$ for any $\epsilon > 0$. By contradiction, assume that $\lim_{\epsilon_n \to 0} diam(\Gamma(\epsilon_n)) > p > 0$, for some sequence $\{\epsilon_n\} > 0$. We could find two sequences $\{(t_n, x_n)\}$ and $\{(t_n, y_n)\}$ satisfying $(t_n, x_n) \in \Gamma(\epsilon)$, $(t_n, y_n) \in \Gamma(\epsilon)$, and

$$\|(t_n, x_n) - (t_n, y_n)\| > p \; \forall n \in \mathbb{N}. \tag{3.10}$$

Since $\{(t_n, x_n)\}$ and $\{(t_n, y_n)\}$ are approximating sequence for (EP_Ψ). By the well-posedness of (EP_Ψ), they have to converge strongly to the unique solution of (EP_Ψ) a contradiction to 3.10.

Conversely, suppose that condition 3.9 holds. Let $\{(t_n, x_n)\}$ be an approximating sequence for (EP_Ψ). Then, there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ in which

$$F(t_n, x_n, y) + \Psi(x_n, y) \geq -\epsilon_n \|y - x_n\| \;\forall n \in \mathbb{N}, y \in K. \tag{3.11}$$

This yields that $\{(t_n, x_n)\} \in \Gamma(\epsilon_n)$. It follows from 3.9 that $\{(t_n, x_n)\}$ is a Cauchy sequence and so it converges strongly to a point $(t, x) \in T \times K$. It follows from 3.11, α-monotonicity and since $\alpha(\cdot, y)$, $F(\cdot, y, \cdot)$ are l.s.c and $\Psi(\cdot, y)$ is u.s.c, that

$$\begin{aligned} 0 &= \liminf_n \epsilon_n \|y - x_n\| \\ &\geq \liminf_n \big[- F(t_n, x_n, y) - \Psi(x_n, y) \big] \\ &\geq \liminf_n \big[F(t_n, y, x_n) + \alpha(x_n, y) - \Psi(x_n, y) \big] \\ &\geq F(t, y, x) + \alpha(x, y) - \Psi(x, y). \end{aligned}$$

This fact together with Theorem 2.6, (t, x) solves (EP_Ψ). The uniqueness follows immediately from 3.9. □

Remark 3.6. Not that the diameter of Γ does not tend to zero, if (EP_Ψ) has more than one solutions. In the next result we consider the Kuratowski non compactness measure of approximating solution set instead of the diameter.

Theorem 3.7. Assume that T and K are nonempty, closed and convex subsets of real Banach spaces E and X respectively. If (EP_Ψ) is strongly well-posed in the generalized sense, then

$$\Gamma(\epsilon) \neq \emptyset \;\forall \epsilon > 0, \; \lim_{\epsilon \to 0} \beta(\Gamma(\epsilon)) = 0. \tag{3.12}$$

Moreover, if the following assumptions hold:

(1) $F(\cdot, x, \cdot)$ *is l.s.c* $\forall x \in K$,
(2) $F(t, x, \cdot)$ *is convex* $\forall (t, x) \in T \times K$,
(3) $F(t, \cdot, \cdot)$ *is* α*-monotone bifunction, hemicontinuous,* $\forall t \in T$,
(4) $F(t, x, x) = 0$, $\forall t \in T, x \in K$,
(5) $\alpha(\cdot, y)$ *and* $\Psi(\cdot, y)$ *are u.s.c* $\forall y \in K$,
(6) $\Psi(x, \cdot)$ *and* $\alpha(x, \cdot)$ *are convex* $\forall x \in K$.

Then the converse holds.

Proof. Let (EP_Ψ) be strongly well-posed in the generalized sense. Then the solution set S of (EP_Ψ) is a nonempty. This indicates that, for any $\epsilon > 0$, $\Gamma(\epsilon) \neq \emptyset$ since $S \subset \Gamma(\epsilon)$. Moreover, we claim here that the solution set S of (EP_Ψ) is compact. Indeed, for any sequence $\{(t_n, x_n)\}$ in S, $\{(t_n, x_n)\}$ is an approximating sequence for (EP_Ψ). Thus there exists a converging subsequence to some point of S. This implies that S is compact. Now, we show that $\lim_{\epsilon\to 0}(\beta\Gamma(\epsilon)) \to 0$. It follows from $S \subset \Gamma(\epsilon)$ that

$$H(\Gamma(\epsilon), S) = \max\{(e\Gamma(\epsilon), S), (e\Gamma(S, (\epsilon))\}.$$

Since the solution set S is compact, one can have

$$\beta(\Gamma(\epsilon)) \leq 2H(\Gamma(\epsilon), S) + \beta(S) = 2e(\Gamma(\epsilon), S),$$

where $\beta(S) = 0$, since S is compact. To prove $\lim_{\epsilon\to 0}\beta(\Gamma(\epsilon)) = 0$. It is sufficient to show that $e(\Gamma(\epsilon), S) \to 0$ as $\epsilon \to 0$. If not, there exists a constant $c > 0$ and $\epsilon_n \to 0$, and $\{(t_n, x_n)\} \subset \Gamma(\epsilon_n)$ in which

$$(t_n, x_n) \notin S + B_{\frac{c}{2}}(0), \ \forall n \in \mathbb{N}. \tag{3.13}$$

where $B_{\frac{c}{2}}(0)$ is an open ball with center 0 and radius $\frac{c}{2}$. However, $\{(t_n, x_n)\} \subset \Gamma(\epsilon_n)$, is an approximating sequence for (EP_Ψ). It follows the generalized well-posedness of (EP_Ψ) that there exists a subsequence converges to some point of $(t, x) \in S$, which contradicts 3.13.

Conversely, suppose that 3.12 holds. By Lemma 3.3 and Lemma 3.4, $\Gamma(\epsilon)$ is nonempty and closed for all $\epsilon > 0$. By the Kuratowski theorem [16], we can obtain

$$H(\Gamma(\epsilon), S) \to 0 \ as \ \epsilon \to 0, \tag{3.14}$$

where $S = \bigcap_{\epsilon>0} \Gamma(\epsilon)$ is a nonempty and compact. Let $\{(t_n, x_n)\} \subset K$ be any approximate solution sequence for (EP_Ψ). Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$F(t_n, x_n, y) + \Psi(x_n, y) \geq -\epsilon_n\|y - x_n\| \ \forall n \in \mathbb{N}, y \in K.$$

This means that $(t_n, x_n) \in \Gamma(\epsilon_n)$. This together with 3.12 indicates that

$$d((t_n, x_n), S) \leq e(\Gamma(\epsilon), S) \to 0.$$

Since S is compact, it follows that there exists $(\bar{t_n}, \bar{x_n}) \in S$ in which

$$\|(t_n, x_n) - (\bar{t_n}, \bar{x_n})\| = d((t_n, x_n), S) \to 0.$$

Again, by the compactness of the solution set S, the sequence $(\bar{t_n}, \bar{x_n})$ has a subsequence $\{(\bar{t_{n_k}}, \bar{x_{n_k}})\}$ converging strongly to $\{(\bar{t_n}, \bar{x_n})\} \in S$. Therefore, the corresponding $\{(t_{n_k}, x_{n_k})\}$ subsequence of $\{(t_n, x_n)\}$ converges strongly to $\{(\bar{t_n}, \bar{x_n})\}$. Hence (EP_Ψ) is well-posed in the generalized sense.

□

4. Well-posedness for optimization problems with generalized parametric equilibrium constraints

In this section, let us introduce the formulation of optimization problems with equilibrium constraint. The optimization problem with generalized equilibrium constraint (denoted by (OPGPEC)) is formulated as follows:

$$\min h(t,u) \ s.t. \ (t,u) \in T \times K,$$

where $u \in S(t)$, T is a nonempty closed subset of a parametric normed space, $h : T \times K \to \mathbb{R}$, $F : T \times K \times K \to \mathbb{R}$, and $S(t)$ is the solution set of the parametric generalized equilibrium problem $(EP_\Psi(t))$, defined by, $u \in S(t)$ if and only if

$$F(t,u,v) + \Psi(u,v) \geq 0, v \in K.$$

Definition 4.1. A sequence $\{(t_n, u_n)\} \subset T \times K$ is said to be an approximating sequence for $(OPGPEC)$ if

(i) there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$F(t_n, u_n, v) + \Psi(u_n, v) \geq -\epsilon_n \|v - u_n\| \ \forall n \in \mathbb{N}, v \in K.$$

(ii)

$$\limsup_n h(t_n, u_n) \leq \inf_{r \in T, v \in S(r)} h(r,v).$$

Definition 4.2. $(OPGPEC)$ is said to be strongly (resp., weakly) well-posed (resp., strongly (resp., weakly) well-posed in the generalized sense) if $(OPGPEC)$ has a unique solution x and for every approximating sequence for $(OPGPEC)$ converges strongly (resp., weakly) to the unique solution (resp., if $S \neq \emptyset$ and every approximate solution sequence has a subsequence which strongly (resp., weakly) converges to some point of S).

The set of approximating solutions of $(OPGPEC)$ is defined by

$$\eta(\epsilon,\delta) := \begin{cases} (t,u) \in T \times K : h(t,u) \leq \inf_{r\in T, v\in S(r)} h(r,v) + \delta \quad and \\ F(t,u,v) + \Psi(u,v) \geq -\epsilon_n \|v-u\| \ \forall v \in K \end{cases}$$

Theorem 4.3. *Assume that K is a nonempty convex, closed subset of a Banach space X. Let $F : T \times K \times K \to \mathbb{R}$, $h : T \times K \to \mathbb{R}$ and $\Psi, \alpha : K \times K \to \mathbb{R}$ be four functions. If (OPGPEC) is strongly well-posed, then*

$$\eta(\epsilon,\delta) \neq \emptyset \ \forall \epsilon, \delta > 0, \ diam\ \eta(\epsilon,\delta) \to 0, \tag{4.1}$$

where $(\epsilon,\delta) \to (0,0)$. Moreover, if the following assumptions hold:

(i) $F(\cdot, x, \cdot)$ is l.s.c $\forall x \in K$,
(ii) $F(t, x, \cdot)$ is convex $\forall (t,x) \in T \times K$,
(iii) $F(t,\cdot,\cdot)$ is α-monotone bifunction, hemicontinuous, $\forall t \in T$,
(iv) $F(t,x,x) = 0$, $\forall t \in T, x \in K$,
(v) $\alpha(\cdot, y)$ and $\Psi(\cdot, y)$ are u.s.c $\forall y \in K$,
(vi) $\Psi(x,\cdot)$ and $\alpha(x,\cdot)$ are convex $\forall x \in K$,
(vii) h is l.s.c.

Then the converse holds.

Proof. Assume that $(OPGPEC)$ is strongly well-posed. Then $(OPGPEC)$ admits a unique solution $(t,x) \in T \times K$, i.e.,

$$\begin{cases} h(t,x) = \inf_{r\in T, y\in S(r)} h(r,y) \ and \\ F(t,x,y) + \Psi(x,y) \geq 0 \ \forall y \in K. \end{cases}$$

Obviosly, $\eta(\epsilon,\delta) \neq \emptyset$ for any $\epsilon, \delta > 0$, since $(t,x) \in \eta(\epsilon,\delta)$ for any $\epsilon, \delta > 0$. If diam $\eta(\epsilon,\delta) \nrightarrow 0$ as $\epsilon \to 0$, $\delta \to 0$ then there exists a constant $p > 0$ and a nonnegative sequences $\{\epsilon_n\}$ and $\{\delta_n\}$ with $\epsilon_n \to 0$, $\delta_n \to 0$ as $n \to 0$ and (t_n, x_n), $(t_n, y_n) \in \eta(\epsilon_n, \delta_n)$ in which

$$\|(t_n, x_n) - (t_n, y_n)\| > p \ \forall n \in \mathbb{N}. \tag{4.2}$$

Since $(t_n, x_n), (t_n, y_n) \in \eta(\epsilon_n, \delta_n) \ \forall n \in \mathbb{N}$, so both $\{(t_n, x_n)\}$ and $\{(t_n, y_n)\}$ are approximating sequence for $(OPGPEC)$. By the well-posedness of $(OPGPEC)$, they have to converge strongly to the unique solution of $(OPGPEC)$ a contradiction to 4.2.

Conversely, suppose that condition 4.1 holds. Let $\{(t_n, x_n)\}$ be an approximating sequence for $(OPGPEC)$. Then, there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ in which

$$(4.3)\qquad \begin{cases} \limsup_n h(t_n, x_n) \leq \inf_{r\in T, y\in S(r)} h(r,y) \text{ and} \\ F(t_n, x_n, y) + \Psi(x_n, y) \geq -\epsilon_n \|y - x_n\| \ \forall n \in \mathbb{N},\ y \in K. \end{cases}$$

This yields that $(t_n, x_n) \in \eta(\epsilon_n, \delta_n)$. It follows from 4.1, that $\{(t_n, x_n)\}$ is a Cauchy sequence and so it converges strongly to a point $(t, x) \in T \times K$. It follows from 4.3 and assumptions (i), (iii) and (v) that

$$\begin{aligned} 0 &= \liminf_n \epsilon_n \|y - x_n\| \\ &\geq \liminf_n \big[-F(t_n, x_n, y) - \Psi(x_n, y) \big] \\ &\geq \liminf_n \big[F(t_n, y, x_n) + \alpha(x_n, y) - \Psi(x_n, y) \big] \\ &\geq F(t, y, x) + \alpha(x, y) - \Psi(x, y). \end{aligned}$$

Also, one can note from 4.3 and assumption (vii) that

$$\begin{aligned} \inf_{r\in T, y\in S(r)} h(r,y) &\geq \limsup_n\ h(t_n, x_n) \\ &\geq \liminf_n\ h(t_n, x_n) \\ &\geq h(t,x), \end{aligned}$$

So, by Theorem 2.6 (t, x) solves $(OPGPEC)$. The uniqueness follows immediately from 4.1. Therefore, we complete the proof. □

By a similar proof as that of Theorem 3.7, we can obtain the following result for the well-posedness of $(OPGPEC)$.

Theorem 4.4. *Assume that T and K are nonempty, closed and convex subsets of real Banach spaces E and X respectively. If $(OPGPEC)$ is strongly well-posed in the generalized sense, then*

$$(4.4)\qquad \eta(\epsilon, \delta) \neq \emptyset \ \forall \epsilon, \delta > 0, \quad \lim_{(\epsilon,\delta)\to(0,0)} \beta(\eta(\epsilon, \delta)) = 0.$$

Moreover, if the following assumptions hold:

(i) $F(\cdot, x, \cdot)$ *is l.s.c* $\forall x \in K$,
(ii) $F(t, x, \cdot)$ *is convex* $\forall (t, x) \in T \times K$,
(iii) $F(t, \cdot, \cdot)$ *is* α*-monotone bifunction, hemicontinuous,* $\forall t \in T$,
(iv) $F(t, x, x) = 0$, $\forall t \in T, x \in K$,
(v) $\alpha(\cdot, y)$ *and* $\Psi(\cdot, y)$ *are u.s.c* $\forall y \in K$,
(vi) $\Psi(x, \cdot)$ *and* $\alpha(x, \cdot)$ *are convex* $\forall x \in K$,
(vii) h *is l.s.c.*

Then the converse holds.

To investigate the uniqueness of solutions to $(OPGPEC)$, we show that under suitable conditions, in the next result the well-posedness of $(OPGPEC)$ is equivalent to the existence and uniqueness of solutions.

Theorem 4.5. *Let T and K be nonempty, closed and convex subsets of finite dimensional Banach spaces E and X respectively. Let $F : T \times K \times K \to \mathbb{R}$, $\Psi, \alpha : K \times K \to \mathbb{R}$ and $h : T \times K \to \mathbb{R}$ be four functions. Suppose that*

(i) $F(\cdot, x, \cdot)$ *is l.s.c and convex* $\forall x \in K$,
(ii) $F(t, \cdot, \cdot)$ *is* α*-monotone bifunction, hemicontinuous,* $\forall t \in T$,
(iii) $F(t, x, x) = 0$, $\forall t \in T, x \in K$,
(iv) $\alpha(\cdot, y)$ *and* $\Psi(\cdot, y)$ *are u.s.c* $\forall y \in K$,
(v) $\Psi(x, \cdot)$ *and* $\alpha(x, \cdot)$ *are convex* $\forall x \in K$,
(vi) h *is convex and l.s.c.*

Then, $(OPGPEC)$ is strongly well-posed if and only if it has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that $(OPGPEC)$ has a unique solution (t^*, x^*). It follows that

$$(4.5) \qquad \begin{cases} h(t^*, x^*) = \inf\limits_{(r,y)\in T\times K, y\in S(r)} h(r, y), \\ F(t^*, x^*, y) + \Psi(x^*, y) \geq 0 \ \forall y \in K \end{cases}$$

Let $\{(t_n, x_n)\} \subset T \times K$ be an approximating sequence for $(OPGPEC)$. Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$(4.6) \qquad \begin{cases} h(t_n, x_n) = \inf\limits_{(r,y)\in T\times K, y\in S(r)} h(r, y), \\ F(t_n, x_n, y) + \Psi(x_n, y) \geq -\epsilon_n \|y - x_n\| \ \forall y \in K, \forall n \in \mathbb{N}. \end{cases}$$

We claim that $\{(t_n, x_n)\}$ is bounded. If not without loss of generality, one can assume that $\|(t_n, x_n)\| \to +\infty$. Let $r_n = \frac{1}{\|(t_n,x_n)-(t^*,x^*)\|}$ and

$$(u_n, v_n) = r_n(t_n, x_n)+(1-r_n)(t^*, x^*) = (r_n t_n+(1-r_n)t^*, r_n x_n+(1-r_n)x^*).$$

without loss of generality, one can assume that $r_n \in]0, 1[$ and $(u_n, v_n) \to (u, v)$ with $(u, v) \neq (t^*, x^*)$ since $E \times X$ is finite-dimensional. Taking into account the closedness and convexity of T and K, one has $(u_n, v_n) \in T \times K$. Thus, by assupmtion (vi), 4.5 and 4.6, for any $(u, v) \in T \times K$, we have

$$(4.7)\quad \begin{aligned} h(t^*, x^*) &= \limsup_n \ r_n h(t_n, x_n) + \limsup_n (1-r_n) h(t^*, x^*) \\ &\geq \limsup_n \ \big[r_n h(t_n, x_n) + (1-r_n) h(t^*, x^*)\big] \\ &\geq \limsup_n \ h(u_n, v_n) \\ &\geq \liminf_n \ h(u_n, v_n) \\ &\geq h(u, v). \end{aligned}$$

Moreover, it follows from conditions (i), (ii), 4.5 and 4.6 that

$$(4.8)\quad \begin{aligned} 0 &= \liminf_n r_n \epsilon_n \|y - x_n\| \\ &\geq \liminf_n - r_n \big[F(t_n, x_n, y) + \Psi(x_n, y)\big] - (1-r_n)[F(t^*, x^*, y) + \Psi(x^*, y)\big] \\ &\geq \liminf_n \big[- r_n F(t_n, x_n, y) - (1-r_n) F(t^*, x^*, y) - \Psi(x^* + r_n(x_n - x^*), y)\big] \\ &\geq \liminf_n \big[- F(u_n, v_n, y) - \Psi(v_n, y)\big] \\ &\geq \liminf_n \big[F(u_n, y, v_n) + \alpha(v_n, y) - \Psi(v_n, y)\big] \\ &\geq F(u, y, v) + \alpha(v, y) - \Psi(v, y). \end{aligned}$$

Applying Theorem 2.6 implies that

$$F(u, v, y) + \Psi(v, y) \geq 0, \forall y \in K.$$

Hence, from 4.7 and 4.8, (u, v) solves $(OPGPEC)$, which is a contradiction. So $\{(t_n, x_n)\}$ is bounded. Let $\{(t_{n_i}, x_{n_i})\}$ be any subsequence of $\{(t_n, x_n)\}$ in which $(t_{n_i}, x_{n_i}) \to (t_0, x_0)$ as $i \to \infty$. It follows that

$$(4.9)\quad \begin{aligned} 0 &= \liminf_i \epsilon_{n_i} \|y - x_{n_i}\| \\ &\geq \liminf_i \ \big[- F(t_{n_i}, x_{n_i}, y) - \Psi(x_{n_i}, y)\big] \\ &\geq \liminf_i \big[F(t_{n_i}, y, x_{n_i}) + \alpha(x_{n_i}, y) - \Psi(x_{n_i}, y)\big] \\ &\geq F(t_0, y, x_0) + \alpha(x_0, y) - \Psi(x_0, y). \end{aligned}$$

$\forall y \in K$. Applying Theorem 2.6 implies that

$$(4.10)\quad F(t_0, x_0, y) + \Psi(x_0, y) \geq 0, \forall y \in K.$$

It follows from 4.6 and lower semicontinuity of h that

$$(4.11)\quad \begin{aligned} \inf_{(r,v) \in T \times K, v \in S(r)} h(r, v) &\geq \limsup_i \ h(t_{n_i}, u_{n_i}) \\ &\geq \liminf_i \ h(t_{n_i}, u_{n_i}) \\ &\geq h(t_0, u_0). \end{aligned}$$

From 4.10 and 4.11, (t_0, u_0) solves $(OPGPEC)$. Taking into account the uniqueness of the solution, we have $(t_0, u_0) = (t^*, u^*)$. Hence, (t_n, u_n) converges to (t^*, u^*). Therefore, $(OPGPEC)$ is stronge well-posed. $\square$

Theorem 4.6. *Let T and K be nonempty, closed and convex subsets of finite dimensional Banach spaces E and X respectively. Let , $F : T \times K \times K \to \mathbb{R}$, $\Psi, \alpha : K \times K \to \mathbb{R}$ and $h : T \times K \to \mathbb{R}$ be four functions. If there exists some $\delta > 0$ such that $\eta(\delta, \delta)$ is nonempty and bounded and suppose that the following assumptions hold:*

(i) $F(\cdot, x, \cdot)$ *is l.s.c and convex* $\forall x \in K$,
(ii) $F(t, \cdot, \cdot)$ *is α-monotone bifunction, hemicontinuous,* $\forall t \in T$,
(iii) $F(t, x, x) = 0$, $\forall t \in T, x \in K$,
(iv) $\alpha(\cdot, y)$ *and* $\Psi(\cdot, y)$ *are u.s.c* $\forall y \in K$,
(v) $\Psi(x, \cdot)$ *and* $\alpha(x, \cdot)$ *are convex* $\forall x \in K$,
(vi) *h is convex and l.s.c.*

Then, $(OPGPEC)$ is strongly well-posed in the generalized sense.

Proof. Let $\{(t_n, x_n)\} \subset T \times K$ be an approximating sequence for $(OPGPEC)$. Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to 0$ such that

$$(4.12) \quad \begin{cases} h(t_n, x_n) = \inf\limits_{(r,y) \in T \times K, y \in S(r)} h(r, y), \\ F(t_n, x_n, y) + \Psi(x_n, y) \geq -\epsilon \|y - x_n\| \ \forall y \in K, \forall n \in \mathbb{N}. \end{cases}$$

Since $\eta(\delta, \delta)$ is a nonempty and bounded. Then there exists n_0 such that $(t_n, x_n) \in \eta(\delta, \delta)$ for all $n \geq n_0$. Taking into account the boundedness of $\eta(\delta, \delta)$, there exists some subsequence $\{(t_{n_i}, x_{n_i})\}$ of $\{(t_n, x_n)\}$ in which $(t_{n_i}, x_{n_i}) \to (t_0, x_0)$ as $i \to \infty$. Consequently, As proved in Theorem 4.5, (t_0, x_0) solves $(OPGPEC)$. Then, $(OPGPEC)$ is strongly well-posed in the generalized sense. $\square$

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