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THE CONNECTION OF HYPER LATTICE IMPLICATION ALGEBRAS AND RELATED HYPER ALGEBRAS

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Abstract. In this paper, we define the concepts of (good) congruences and strong congruences on hyper lattice implication algebras and use them to construct quotient hyper lattice implication algebras. We describe the relations between hyper lattice implication algebra and hyper MV-algebra, hyper K-algebra, (weak) hyper residuated lattices.

Key Words: (Quotient) Hyper lattice implication algebra, Congruence, Hyper MV-algebra, (weak) Hyper residuated lattice.

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1. Introduction

Lattice implication algebras introduced by Y. Xu in 1993 [2] to research the logical system whose propositional value is given in a lattice from the semantic viewpoint. The properties and structure of this important logic algebra have been studied by many researchers. Also, the relations between lattice implication algebra and MV-algebra, BCK-algebra and residuated lattice have been studied by researchers. All of these algebras are related to logic.
The hyperstructure theory was introduced in 1934 by F. Marty [9] at the 8th Congress of Scandinavian Mathematicians. He applied them

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to group, algebraic functions, rational fractions and non-commutative groups. There are extensive applications in many branches of mathematics and applied sciences, such as Euclidian and Non Euclidian geometries, graphs and hyper-graphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, probabilities, information sciences and so on. Some interesting applications of hyperstructures can be find in the books [4, 5].
Since then many researchers have worked on this area. In [10], Mittas et al. applied the hyperstructures to lattices and introduced the concepts of hyper lattices and supper lattices. R. A. Borzooei et al. introduced and study hyper K-algebras in [1] and S. Ghorbani et al. applied the hyperstructure to MV -algebras and introduced the concept of hyper MV -algebra, which is generalization of MV -algebra in [6]. The relations between hyper MV-algebra and hyper K-algebra have been discussed in [6]. S. Rasouli and B. Davvaz proved that a hyper MV-algebra induced a hyper lattice in [11]. Then R. A. Borzooei et al. introduced the (weak) hyper residuated lattice which is the generalization of residuated lattice and the relation between hyper MV-algebras and hyper residuated lattice have been studied in [2, 3, 14]. Recently, in [8] hyper lattice implication algebras have been studied and some related results obtained.
This paper is organized as follows. In Section 2, we recall some definitions which will be needed in this paper. In Section 3, we define the quotient hyper lattice implication algebra by good congruence. We will prove that quotient hyper lattice implication algebra induced by congruence θ is a lattice implication algebra if and only if θ is a strong congruence. In Section 4, we study the relation between hyper lattice implication algebra and hyper MV-algebra, hyper K-algebra, commutative semihypergroup, (weak) hyper residuated lattices.

2. Preliminaries

Definition 2.1.([9]) Let H be a non-empty set and "$\circ$" be a function from $H \times H$ to $P(H)\backslash\{\emptyset\}$. Then " $\circ$ " is called a hyper operation on H. Note that if $\emptyset \neq X, Y \subseteq H$, then by $X \circ Y$ we mean the subset $\cup\{x \circ y : x \in X, y \in Y\}$ of H , $x \circ Y := \{x\} \circ Y$ and $X \circ y := X \circ \{y\}$ for all $x, y \in H$.

Definition 2.2.([10]) Let L be a non-empty set endowed with hyper operations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a hyper lattice if for any

$x, y, z \in L$, the following conditions are satisfied:
(1) $x \in x \wedge x$, $x \in x \vee x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$,
(4) $x \in x \wedge (x \vee y)$, $x \in x \vee (x \wedge y)$.

Definition 2.3.([10]) Let $(S, \leq)$ be a partially order set endowed with hyper operations $\wedge$ and $\vee$. Then $(L, \leq, \wedge, \vee)$ is called a super lattice if for any $x, y, z \in L$, the following conditions are satisfied:
(1) $(L, \wedge, \vee)$ is a hyper lattice,
(2) $x \leq y$ implies $y \in x \vee y$ and $x \in y \wedge x$,
(3) if $y \in x \vee y$ or $x \in y \wedge x$, then $x \leq y$.

Definition 2.4.([8]) A hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a non-empty set L equipped with three hyper operations $\wedge$, $\vee$ and $\rightarrow$, a unary operation $'$ and two constant 0 and 1 which satisfies the following axioms:for all $x, y, z \in L$,
(HLI1) $(L, \vee, \wedge, 0, 1)$ is a hyper lattice such that $0' = 1$ and $1' = 0$,
(HLI2) $1 \in x \rightarrow x$,
(HLI3) $1 \in x \rightarrow 1$,
(HLI4) $(x')' = x$,
(HLI5) $x \rightarrow y = y' \rightarrow x'$,
(HLI6) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(HLI7) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(HLI8) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,
(HLI9) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(HLI10) $1 \in x \rightarrow y$ and $1 \in y \rightarrow x$ implies $x = y$.

Proposition 2.5.([8]) Let $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ be a hyper lattice implication algebra. Then for all $x, y, z \in L$ the following hold:
(1) $1 \in 1 \rightarrow x$ implies $x = 1$,
(2) $x \in 1 \rightarrow x$ and $x' \in x \rightarrow 0$,
(3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y')' \rightarrow z$,
(4) $x \in 1 \rightarrow 0$ implies $x \neq x'$,
(5) $1 \rightarrow 0 = \{0\}$,
(6) $1 \rightarrow (1 \rightarrow x) = 1 \rightarrow x$,
(7) $y \in 1 \rightarrow x$ implies $1 \in y \rightarrow x$,
(8) $1 \rightarrow x = 1 \rightarrow y$ implies $x = y$.

Definition 2.6.([6]) A hyper MV-algebra is a non-empty set M endowed with a hyper operation " $\oplus$ ", a unary operation " $*$ " and a constant 0 satisfying the following axioms:
(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(hMV2) $x \oplus y = y \oplus x$,
(hMV3) $(x^*)^* = x$,
(hMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(hMV5) $0^* \in x \oplus 0^*$,
(hMV6) $0^* \in x \oplus x^*$,
(hMV7) if $x << y$ and $y << x$, then $x = y$,
for all $x, y, z \in M$, where $x << y$ is defined by $0^* \in x^* \oplus y$.
For every $A, B \subseteq M$, we define $A << B$ if and only if there exist $a \in A$ and $b \in B$ such that $a << b$. Also, We define $0^* := 1$ and $A^* = \{a^* : a \in A\}$.

Definition 2.7.([1]) A hyper K-algebra is a non-empty set H endowed with a hyper operation " $\circ$ "and a constant 0 satisfying the following conditions:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y$ and $y < x$ imply $x = y$,
(HK5) $0 < x$,
for all $x, y, z \in H$, where $x < y$ if and only if $0 \in x \circ y$ and for every $X, Y \subseteq H$, if there exist $x \in X$ and $y \in Y$ such that $x < y$, then we define $X < Y$.

Definition 2.8.([1]) Let $< H, \circ, 0 >$ be a hyper K-algebra. If there exists an element $e \in H$ such that $x < e$ for all $x \in H$, then H is called a bounded hyper K-algebra and e is said to be the unit of H.

Definition 2.9.([5]) Let A be a set, $\circ$ be a binary hyper operation on A and $1 \in A$. $(A, \circ, 1)$ is called a commutative semihypergroup with 1 as an identity if it satisfies the following properties: for all $x, y, z \in A$,
(CSHG1) $x \circ (y \circ z) = (x \circ y) \circ z$,
(CSHG2) $x \circ y = y \circ x$,
(CSHG3) $x \in 1 \circ x$.

Definition 2.10.([3]) A weak hyper residuated lattice is a non-empty set L endowed with two binary operations $\wedge, \vee$ and two binary hyper operations $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:
(WHRL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(WHRL2) $(L, \odot, 1)$ is a commutative semihypergroup with 1 as the identity,
(WHRL3) $x \odot y << z$ if and only if $x << y \rightarrow z$,
where $A << B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all non-empty subset A and B of L.

Definition 2.11.([14]) A hyper residuated lattice is a non-empty set L endowed with four binary hyper operations $\wedge$, $\vee$, $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:
(HRL1) $(L, \wedge, \vee, 0, 1)$ is a bounded super lattice,
(HRL2) $(L, \odot, 1)$ is commutative semihypergroup with 1 as an identity,
(HRL3) $x \odot y << z$ if and only if $x << y \rightarrow z$,
where $A << B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all non-empty subset A and B of L.

3. Quotient Hyper Lattice Implication Algebras

In this section, we will study the relation between lattice implication algebras and hyper lattice implication algebras. We know that each lattice implication algebra is a hyper lattice implication algebra. We will show that each hyper lattice implication algebra give us a lattice implication algebra.

Definition 3.1. Let X and Y be non-empty subsets of a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ and θ be an equivalence relation on L. Then
(1) $X\theta Y$ means that there exist $x \in X$ and $y \in Y$ such that $x\theta y$.
(2) $X\overline{\theta}Y$ means that for all $x \in X$ there exists $y \in Y$ such that $x\theta y$ and for all $y \in Y$ there exists $x \in X$ such that $x\theta y$.
(3) $X\overline{\overline{\theta}}Y$ means that for all $x \in X$ and for all $y \in Y$ we have $x\theta y$.

Proposition 3.2. Let θ be an equivalence relation on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ and $X, Y, Z \subseteq L$. Then
(1) if $X\overline{\theta}Y$ and $Y\overline{\theta}Z$, then $X\overline{\theta}Z$,

(2) if $X\bar{\bar{\theta}}Y$ and $Y\bar{\bar{\theta}}Z$, then $X\bar{\bar{\theta}}Z$,
(3) if $X\overline{\theta}Y$ and $Y\theta Z$, then $X\theta Z$,
(4) if $X\theta Y$ and $Y\overline{\theta}Z$, then $X\theta Z$.

Proof. The proof is straightforward.■

Definition 3.3. Let θ be an equivalence relation on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then θ is called a congruence on L, if for all $x, y, z, w \in L$
(1) if $x\theta y$, then $x'\theta y'$,
(2) if $x\theta y$ and $z\theta w$, then $(x \star z)\overline{\theta}(y \star w)$ for all $\star \in \{\wedge, \vee, \rightarrow\}$.

Let θ be an equivalence relation on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. We denote the set of all equivalence classes of θ by L/θ, that is

$$L/\theta = \{[x] : x \in L\}.$$

Theorem 3.4. Let θ be an equivalence relation on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then θ is a congruence on L if and only if the following hyper operations on L/θ are well defined for all $x, y \in L$
$[x]\bar{\star}[y] = \{[t] : t \in x \star y\}$ for all $\star \in \{\wedge, \vee, \rightarrow\}$ and $[x]' = [x']$.

Proof. Let $\star \in \{\wedge, \vee, \rightarrow\}$. We will show that $\bar{\star}$ is well defined on L/θ. Suppose that $[x] = [y]$ and $[z] = [w]$. Then we have $x\theta y$ and $z\theta w$. Since θ is a congruence on L, we have $(x \star z)\overline{\theta}(y \star w)$ by Definition 3.3 part (2). Now, suppose that $t \in x \star z$. By Definition 3.1 part (2), there exists $s \in z \star w$ such that $t\theta s$, that is $[t] = [s]$. Hence $([x]\bar{\star}[z]) \subseteq ([y]\bar{\star}[w])$. Similarly, we can prove $([y]\bar{\star}[w]) \subseteq ([x]\bar{\star}[z])$ and so the hyper operation $\bar{\star}$ is well defined on L/θ.
Conversely, suppose that $\bar{\star}$ is well defined for all $\star \in \{\wedge, \vee, \rightarrow\}$. Let $x\theta y$ and $z\theta w$. If $t \in x \star z$, then

$$[t] \in [x]\bar{\star}[z] = [y]\bar{\star}[w] = \{[s] : s \in y \star w\}.$$

So, there exists $s \in y \star w$ such that $t\theta s$. Similarly, we can prove that for all $t \in y \star w$, there exists $s \in x \star z$ such that $t\theta s$. Hence $(x \star z)\overline{\theta}(y \star w)$. Therefore θ is a congruence on L.■

Proposition 3.5. Let θ be a congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\rightarrow}, ', 0, 1)$ satisfies (HLI1)-(HLI9).

Proof. The proof is straightforward.■

Lemma 3.6. Let $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ be a hyper lattice implication algebra such that $0 = 1$. Then L is trivial.

Proof. Let $x \in L$ be arbitrary. Since $0 = 1$, then $1 \in x \rightarrow 1 = 0 \rightarrow x' = 1 \rightarrow x'$ by (HLI4) and (HLI5). We obtain $x' = 1$ by Proposition 2.5 part (1). Hence $x = 0$.■

If θ is a congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$, then $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\rightarrow}, ', 0, 1)$ may not be a hyper lattice implication algebra in general. Consider the following example.

Example 3.7. Let $L = \{0, b, 1\}$ and consider the following tables:

$\rightarrow$	0	b	1
0	$\{1\}$	$\{b, 1\}$	$\{1\}$
b	$\{0, b\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{0\}$	$\{0, b\}$	$\{1\}$

$'$	0	b	1
	1	b	0

Then $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ is a hyper lattice implication algebra where $x \wedge y = \{x, y\}$ and $x \vee y = \{x, y\}$ for all $x, y \in L$. Define

$$\theta = \{(0, 0), (b, b), (1, 1), (0, 1), (1, 0)\}$$

Then θ is a congruence on L. We obtain $L/\theta = \{[0], [b]\}$ where $[0] = [1]$. Let $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\rightarrow}, ', 0, 1)$ be a hyper lattice implication algebra. By Lemma 3.6, L/θ must be trivial which is a contradiction.

Definition 3.8. Let θ be a congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then θ is called good congruence on L, if for all $x, y \in L$

$$(x \rightarrow y)\theta\{1\} \text{ and } (y \rightarrow x)\theta\{1\} \text{ implies } x\theta y.$$

Proposition 3.9. The relation $\triangle = \{(x, x) : x \in L\}$ is a good congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$.

Proof. It is easy to prove that $\triangle$ is a congruence on L. Suppose that $(x \to y) \triangle \{1\}$ and $(y \to x) \triangle \{1\}$. There exists $s, t \in L$ such that $t \triangle 1$ and $s \triangle 1$. We obtain that $t = s = 1$. Hence $1 \in x \to y$ and $1 \in y \to x$. By (HLI10), we get $x = y$. So $x \triangle y$. Therefore $\triangle$ is a good congruence.$\blacksquare$

Proposition 3.10. Let θ be a good congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$ such that $0\theta 1$. Then $\theta = \nabla$ where $\nabla = \{(x, y) : x, y \in L\}$.

Proof. It is clear that ∇ is a good congruence. Let $x \in L$ be arbitrary. Since $0\theta 1$, $x'\theta x'$ and θ is congruence, we have $(x' \to 0)\bar{\theta}(x' \to 1)$. By (HLI3), we obtain $1 \in (x' \to 1)$. So there exists $t \in (x' \to 0)$ such that $t\theta 1$. Hence we get $(x' \to 0)\theta\{1\}$. Using (HLI5)and (HLI4), we get $(1 \to x)\theta\{1\}$. On he other hand, we have $(x \to 1)\theta\{1\}$ by (HLI3). Since θ is a good congruence, we get that $x\theta 1$. By transitivity of θ, we obtain $x\theta y$ for all $x, y \in L$. Hence $\theta = \nabla$.$\blacksquare$

Theorem 3.11. Let θ be a congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \to, ', 0, 1)$. Then $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\to}, ', 0, 1)$ is a hyper lattice implication algebra if and only if θ is a good congruence on L.

Proof. Suppose that θ is a good congruence on L. By Theorem 3.4, the hyper operations $\bar{\wedge}, \bar{\vee}, \bar{\to}$ are well defined on L/θ and by Proposition 3.5, $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\to}, ', 0, 1)$ satisfies (HLI1)-(HLI9). We will prove that it satisfies (HLI10). Let $[1] \in [x]\bar{\to}[y]$ and $[1] \in [y]\bar{\to}[x]$. Then there exist $t \in x \to y$ and $s \in y \to x$ such that $t\theta 1$ and $s\theta 1$. By Definition 3.1 part (1), we have $(x \to y)\theta\{1\}$ and $(y \to x)\theta\{1\}$. Since θ is a good congruence on L, we get that $x\theta y$, that is $[x] = [y]$.
Conversely, Let $(L/\theta, \bar{\wedge}, \bar{\vee}, \bar{\to}, ', 0, 1)$ be a hyper lattice implication algebra. Suppose that $(x \to y)\theta\{1\}$ and $(y \to x)\theta\{1\}$. Then there exist $t \in x \to y$ and $s \in y \to x$ such that $t\theta 1$ and $s\theta 1$. Hence $[1] \in [x]\bar{\to}[y]$ and $[1] \in [y]\bar{\to}[x]$. By (HLI10), we get $[x] = [y]$, that is $x\theta y$.$\blacksquare$

Example 3.12. Suppose $L = \{0, a, b, 1\}$ and consider the following tables:

$\wedge$	0	a	b	1
0	$\{0\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0\}$	$\{0,a\}$
b	$\{0\}$	$\{0\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{b,1\}$

$\vee$	0	a	b	1
0	$\{0,a\}$	$\{a\}$	$\{b,1\}$	$\{1\}$
a	$\{a\}$	$\{a\}$	$\{1\}$	$\{1\}$
b	$\{b,1\}$	$\{1\}$	$\{b,1\}$	$\{1,b\}$
1	$\{1\}$	$\{1\}$	$\{1,b\}$	$\{1\}$

$\rightarrow$	0	a	b	1
0	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$
a	$\{b\}$	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$
b	$\{0,a\}$	$\{0,a\}$	$\{b,1\}$	$\{b,1\}$
1	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{b,1\}$

$'$	0	a	b	1
	1	b	a	0

Then $(L,\wedge,\vee,\rightarrow,', 0,1)$ is a hyper lattice implication algebra. Define

$$\theta = \triangle \cup \{(0,a),(a,0),(b,1),(1,b)\}.$$

Then θ is a good congruence on a hyper lattice implication algebra L. By Proposition 3.11, $(L/\theta,\bar{\wedge},\bar{\vee},\bar{\rightarrow},', 0,1)$ is a hyper lattice implication algebra and we have:

$\bar{\wedge}$	$[0]$	$[1]$
$[0]$	$\{[0]\}$	$\{[0]\}$
$[1]$	$\{[0]\}$	$\{[1]\}$

$\bar{\vee}$	$[0]$	$[1]$
$[0]$	$\{[0]\}$	$\{[1]\}$
$[1]$	$\{[1]\}$	$\{[1]\}$

$\bar{\rightarrow}$	$[0]$	$[1]$
$[0]$	$\{[1]\}$	$\{[1]\}$
$[1]$	$\{[0]\}$	$\{[1]\}$

$'$	$[0]$	$[1]$
	$[1]$	$[0]$

where $[0]=\{0,a\}=[a]$ and $[b]=\{b,1\}=[1]$.

Definition 3.13. Let θ be an equivalence relation on a hyper lattice implication algebra $(L,\wedge,\vee,\rightarrow,', 0,1)$. Then θ is called strong congruence on L, if for all $x,y,z,w\in L$
(1) if $x\theta y$, then $x'\theta y'$,
(2) if $x\theta y$ and $z\theta w$, then $(x\star z)\overline{\overline{\theta}}(y\star w)$ for all $\star\in\{\wedge,\vee,\rightarrow\}$.

Remark 3.14. (1) The relation $\bigtriangledown=\{(x,y): x,y\in L\}$ is a strong congruence on a hyper lattice implication algebra $(L,\wedge,\vee,\rightarrow,', 0,1)$.
(2) Each strong congruence on a hyper lattice implication algebra is a congruence.
(3) The congruence on a hyper lattice implication algebra $(L,\wedge,\vee,\rightarrow$

$,', 0, 1)$ may not be strong congruence in general. Consider Example 3.12. It is easily prove that $\triangle$ is not strong congruence.
(4) In Example 3.12, the congruence θ is a strong congruence.

Theorem 3.15. Let θ be an equivalence relation on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then θ is a strong congruence on L if and only if the following hyper operations on L/θ are well defined $[x]\bar{\bar{\star}}\,[y] = [z]$ for all $z \in x \star y$ where $\star \in \{\wedge, \vee, \rightarrow\}$.

Proof. Suppose that $\bar{\bar{\star}}$ is well defined for all $\star \in \{\wedge, \vee, \rightarrow\}$. Let $x_1 \theta x_2$ and $y_1 \theta y_2$. We will prove that $(x_1 \star y_1)\bar{\bar{\theta}}(x_2 \star y_2)$. For all $z \in x_1 \star y_1$ and for all $s \in x_2 \star y_2$, we have

$$[t] = [x_1]\bar{\bar{\star}}\,[y_1] = [x_2]\bar{\bar{\star}}\,[y_2] = [s]$$

that is $t\theta s$. Hence θ is a strong congruence on L.
Conversely, let $\star \in \{\wedge, \vee, \rightarrow\}$. We will prove $\bar{\bar{\star}}$ is well defined on L/θ. Suppose that $[x_1] = [x_2]$ and $[y_1] = [y_2]$. Then we have $x_1 \theta x_2$ and $y_1 \theta y_2$. Since θ is a strong congruence on L, we have $(x_1 \star y_1)\bar{\bar{\theta}}(x_2 \star y_2)$. Now, suppose that $z \in (x_1 \star y_1)$. Let $z_1 \in [z]$ be arbitrary. Then $z_1 \theta z$. By definition 3.1 part (3), we have $z\theta s$ for all $s \in x_2 \star y_2$. So $z_1 \theta s$, that is $[z] \subseteq [s]$ for all $s \in x_2 \star y_2$. Similarly, we can prove $[s] \subseteq [z]$ for all $z \in x_1 \star y_1$ and for all $s \in x_2 \star y_2$. Hence $[x_1]\bar{\bar{\star}}\,[y_1] = [x_2]\bar{\bar{\star}}\,[y_2]$.■

Proposition 3.16. Let θ be a good strong congruence on a hyper lattice implication algebra $(L, \wedge, \vee, \rightarrow, ', 0, 1)$. Then $(L/\theta, \bar{\bar{\wedge}}, \bar{\bar{\vee}}, \bar{\bar{\rightarrow}}, ', 0, 1)$ is a lattice implication algebra.

Proof. Since $(x \star y)\bar{\bar{\theta}}(x \star y)$, then $x\bar{\bar{\star}}\,y = \{[z] : z \in x \star y\} = \{[z]\}$, that is $x\bar{\bar{\star}}\,y$ has exactly one element. Hence $(L/\theta, \bar{\bar{\wedge}}, \bar{\bar{\vee}}, \bar{\bar{\rightarrow}}, ', 0, 1)$ is a lattice implication algebra.

4. Connections with Related Hyper Algebras

In the following theorems, we study the relations between hyper implication algebras and hyper MV-algebras.

Theorem 4.1 Let $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ be a hyper lattice implication algebra. Define the hyper operation $\oplus$ on L as follows: for any $x, y \in L$,

$$x \oplus y = x' \rightarrow y.$$

Then $(L, \oplus, ', 0)$ is a hyper MV-algebra.

Proof. We will prove $(L, \oplus, *, 0)$ satisfies (HMV1)-(HMV7). Let $x, y, z \in L$.
By (HLI5), (HLI4) and (HLI7), we obtain that

$$\begin{aligned}(x \oplus y) \oplus z &= (x' \rightarrow y)' \rightarrow z \\ &= z' \rightarrow (x' \rightarrow y) \\ &= x' \rightarrow (y' \rightarrow z) \\ &= x \oplus (y \oplus z),\end{aligned}$$

hence (HMV1) holds. The other conditions can be proved similarly.■

Corollary 4.2. Let $(L, \wedge, \vee, \rightarrow, ', 0, 1)$ be a hyper lattice implication algebra. Define the hyper operation $\circ$ on L as follows: for any $x, y \in L$,

$$x \circ y = (x \rightarrow y)'.$$

Then $(L, \circ, 0)$ is a is a bounded hyper K-algebra.

Proof. It follows from the above Theorem and Proposition 3.11 in [6].■

Theorem 4.3. Let $(M, \oplus, ^*, 0)$ be a hyper MV -algebra. Define the binary hyper operations $\rightarrow$ on M as follows: for any $x, y \in M$.

$$x \rightarrow y = x^* \oplus y.$$

Then $(M, \rightarrow, ^*, 0, 1)$ satisfies conditions (HLI2)-(HLI7) and (HLI10).

Proof. Suppose that $x, y, z \in M$.
(HLI2) By (hMV6) and (hMV32), we have

$$1 = 0^* \in x \oplus x^* = x^* \oplus x = x \rightarrow x.$$

(HLI3) It follows from (hMV5) and (hMV2).
(HLI5) By (hMV3) and (hMV3), we get

$$x \rightarrow y = x^* \oplus y = x^* \oplus (y^*)^* = (y^*)^* \oplus x^* = y^* \rightarrow x^* = y' \rightarrow x'.$$

(HLI6) Using (hMV6), we have

$$(x \rightarrow y) \rightarrow y = (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x = (y \rightarrow x) \rightarrow x.$$

(HLI7) By (hMV1) and (hMV2), we obtain $x \rightarrow (y \rightarrow z) = (x^* \oplus y^*) \oplus z = (y^* \oplus x^*) \oplus z = y^* \oplus (x^* \oplus z) = y \rightarrow (x \rightarrow z)$.

(HLI10) It follows from(hMV7).■

Proposition 4.4. Let $(M, \oplus, ^*, 0)$ be a hyper MV -algebra. Define three binary hyper operations $\vee, \wedge, \to$ on M as follows: for any $x, y \in M$,

$$x \vee y = \{x, y\},$$
$$x \wedge y = \{x, y\},$$
$$x \to y = x^* \oplus y.$$

Then $(M, \wedge, \vee, \to, ^*, 0, 1)$ is a hyper lattice implication algebra.

Proof. It follows from Theorem 4.3 and Proposition 3.6 in [8].■

Proposition 4.5. Let $(M, \oplus, ^*, 0)$ be a hyper MV -algebra. Define three binary hyper operations $\vee, \wedge, \to$ on M as follows: for any $x, y \in M$,

$$x \vee y = (x^* \oplus y)^* \oplus y,$$
$$(x \wedge y)' = (x' \vee y'),$$
$$x \to y = x^* \oplus y.$$

If the hyper operations satisfies conditions (HLI8) and (HLI9), then $(L, \wedge, \vee, \to, ^*, 0, 1)$ is a hyper lattice implication algebra.

Proof. By Proposition 3.8 in [8], we know $(L, \wedge, \vee)$ is a hyper lattice and by Proposition 3.6 in [8], $(L, \to, ^*, 0, 1)$ satisfies conditions (HLI2)-(HLI7) and (HLI10). Since the hyper operations satisfies conditions (HLI8) and (HLI9), then $(L, \wedge, \vee, \to, ^*, 0, 1)$ is a hyper lattice implication algebra.■

Remark: In the above theorem, $(M, \wedge, \vee, \to, ^*, 0, 1)$ may not be a hyper lattice implication algebra in general. See the following example.

Example 4.6. Let $M = \{0, b, 1\}$. Consider the following tables:

$\oplus$	0	b	1
0	$\{0\}$	$\{0, b\}$	$\{1\}$
b	$\{0, b\}$	$\{0, b, 1\}$	$\{0, b, 1\}$
1	$\{1\}$	$\{0, b, 1\}$	$\{1\}$

$'$	0	b	1
	1	b	0

Then $(M, \to, \oplus, 0, 1)$ is a hyper MV-algebra. By Proposition 4,5, we define hyper operations $\to$, $\vee$ and $\wedge$ as the following tables.

$\to$	0	b	1
0	$\{0\}$	$\{0,b,1\}$	$\{1\}$
b	$\{0,b\}$	$\{0,b,1\}$	$\{0,b,1\}$
1	$\{0\}$	$\{0,b\}$	$\{1\}$

$\vee$	0	b	1
0	$\{0\}$	$\{0,b,1\}$	$\{1\}$
b	$\{0,b,1\}$	$\{0,b,1\}$	$\{0,b,1\}$
1	$\{1\}$	$\{0,b,1\}$	$\{1\}$

$\wedge$	0	b	1
0	$\{0\}$	$\{0,b,1\}$	$\{0\}$
b	$\{0,b,1\}$	$\{0,b,1\}$	$\{0,b,1\}$
1	$\{0\}$	$\{0,b,1\}$	$\{1\}$

Then $(M,\wedge,\vee,\to,',0,1)$ is not hyper lattice implication algebra because $(0\vee 1)\to b=\{0,b\}$ but $(0\to b)\vee(1\to b)=L$.

Corollary 4.7. Let $(H,\circ,0)$ is a is a bounded hyper K-algebra with a unite element e such that satisfying the following conditions:
(i) $e\circ(e\circ x)=\{x\}$,
(ii) $e\circ((e\circ(x\circ y))\circ y)=e\circ((e\circ(y\circ x))\circ x)$.
Then $e\circ x$ has only one element. We define a unary operation $'$ on H such that if $e\circ x=\{a\}$, then $x'=a$.
(1) Define three binary hyperoperations $\vee,\wedge,\to$ as follows: for any $x,y\in H$,

$$x\to y=(x\circ y)',$$
$$x\vee y=\{x,y\},$$
$$x\wedge y=\{x,y\}.$$

Then $(H,\wedge,\vee,\to,',0,1)$ is a hyper lattice implication algebra.
(2) Define three binary hyperoperations $\vee,\wedge,\to$ as follows: for any $x,y\in H$,

$$x\to y=(x\circ y)',$$
$$x\vee y=((x\circ y)'\circ y)',$$
$$(x\wedge y)'=(x'\vee y').$$

If $(H,\wedge,\vee,\to,',0,1)$ satisfies conditions (HLI8) and (HLI9), then $(H,\wedge,\vee,\to,',0,1)$ is a hyper lattice implication algebra.

Proof. (1) It follows from Theorem 3.13 in [6] and Proposition 4.4.
(2) It can be obtained by Theorem 3.13 in [6] and Proposition 4.5.■

Theorem 4.8. Let $(L,\wedge,\vee,\to,',0,1)$ be a hyper lattice implication algebra. Define binary operation $\odot$ as follows: for any $x,y\in L$,

$$x\odot y=(x\to y')'.$$

Then $(L, \odot, 1)$ is a commutative semihypergroup with 1 as an identity.

Proof. Suppose that $x, y, z \in L$. Then
(CSHG1) $x' \in 1 \to x'$. So $x = (x')' \in (1 \to x')' = 1 \odot x$.
(CSHG2) $x \odot y = (x \to y')' = (y \to x')' = y \odot x$,
(CSHG3) $x \odot (y \odot z) = (x \to (y \to z')')' = (x \to (z \to y'))' = (z \to (x \to y'))' = ((z \to y') \to z')' = (x \odot y) \odot z$.
Hence $(L, \odot, 1)$ is a commutative semihypergroup with 1 as an identity.■

Lemma 4.9. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra. Then $(x \odot y) \to z = x \to (y \to z)$, for all $x, y, z \in L$.

Proof. Suppose that $x, y, z \in L$. By (HLI5), (HLI4), (HLI7) and again (HLI5) , we have

$$\begin{aligned}(x \odot y) \to z &= (x \to y')' \to z = z' \to (x \to y') \\ &= x \to (z' \to y') = x \to (y \to z).\blacksquare\end{aligned}$$

Theorem 4.10. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra. If $(L, \wedge, \vee, 0, 1)$ is a bounded lattice such that $1 \in x \to y$ implies $x \leq y$, then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a weak hyper residuated lattice.

Proof. By assumption and Theorem 4.8, we need to show that $x \odot y << z$ if and only if $x << y \to z$. Suppose that $x \odot y << z$. So there exists $t \in x \odot y$ such that $t \leq z$, that is $t = t \wedge z$. By (HLI8), we get that

$$1 \in (t \to z) \vee (z \to z) = (t \wedge z) \to z = t \to z.$$

By Lemma 4.9, $1 \in t \to z \subseteq (x \odot y) \to z = x \to (y \to z)$. So there exists $s \in y \to z$ such that $1 \in x \to s$. By assumption $x \leq s$, that is $x << y \to z$. Similarly, we can prove that $x << y \to z$ implies $x \odot y << z$.■

Corollary 4.11. Let $(L, \wedge, \vee, \to, ', 0, 1)$ be a hyper lattice implication algebra. If $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and $\mid L \mid \leq 4$, then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a weak hyper residuated lattice.

Proof. We will show that $1 \in x \to y$ implies $x \leq y$. Suppose that there exist $x, y \in L$ such that $1 \in x \to y$ but $x \not\leq y$. First we will show that $y \in L - \{0\}$. If $y = 0$, then $1 \in x \to 0 = 1 \to x'$ by (HLI5). By Proposition 2.5 part (1), we have $x' = 1$, that is $x = 0$. Hence $x = y$ which is contradiction. Also, we have $x \in L - \{1\}$. If $x = 1$, then

$1 \in 1 \to y$. Hence $y = 1$, that is $x = y$ which is a contradiction. Hence if $\mid L \mid \leq 3$, then $1 \in x \to y$ implies $x \leq y$ for all $x, y \in L$.
Now, let $L = \{0, a, b, 1\}$ and let there exist $x \in L - \{1\}$ and $y \in L - \{0\}$ such that $1 \in x \to y$ but $x \not\leq y$. If $y \leq x$, then $1 \in y \to x$. By assumption and (HLI10), we get that $x = y$ which is a contradiction. Hence $y \not\leq x$. If $y \not\leq x$, then $x \vee y = 1$. By (HLI9)

$$1 \in (x \to y) \wedge (x \to y) = (x \vee y) \to y = 1 \to y.$$

By Proposition 2.5 part (1), $y = 1$ which is a contradiction. Hence $1 \in x \to y$ implies $x \leq y$ for all $x, y \in L$.Therefore $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a weak hyper residuated lattice by Theorem 4.10.■

Example 4.12. Let $L = \{0, b, 1\}$ where $0 < b < 1$. Then $(L, \wedge, \vee, 0, 1)$ is a bounded lattice where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Consider the following tables:

$\to$	0	b	1
0	$\{1\}$	$\{b, 1\}$	$\{1\}$
b	$\{0, b\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{0\}$	$\{0, b\}$	$\{1\}$

$'$	0	b	1
	1	b	0

Then $(L, \wedge, \vee, \to,', 0, 1)$ be a hyper lattice implication algebra. Define hyperoperations $\odot$as the following table:

$\odot$	0	b	1
0	$\{0\}$	$\{0, b\}$	$\{0\}$
b	$\{0, b\}$	$\{0, b\}$	$\{b, 1\}$
1	$\{0\}$	$\{b, 1\}$	$\{1\}$

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a weak hyper residuated lattice by corollary 4.9.

Theorem 4.13. Let $(L, \wedge, \vee, \to,', 0, 1)$ be a hyper lattice implication algebra. If $(L, \wedge, \vee, 0, 1)$ is a super lattice such that $x \leq y$ if and only if $1 \in x \to y$, then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a hyper residuated lattice.

Proof. We have

$$\begin{aligned} & x \odot y << z && \text{if and only if} \\ & t << z \text{ for some } t \in x \odot y && \text{if and only if} \\ & 1 \in t \to z \text{ for some } t \in x \odot y && \text{if and only if} \\ & 1 \in (x \odot y) \to z && \text{if and only if} \end{aligned}$$

$$\begin{array}{ll} 1 \in x \to (y \to z) & \text{if and only if} \\ 1 \in s \to z \text{ for some } s \in y \to z & \text{if and only if} \\ s \leq z \text{ for some } s \in y \to z & \text{if and only if} \\ x << y \to z. \blacksquare & \end{array}$$

Theorem 4.14. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a hyper residuated lattice such that
(2) $(x \to 0) \to 0 = \{x\}$,
(3) $((x \odot (y \to 0)) \to 0) \odot (y \to 0) = ((y \odot (x \to 0)) \to 0) \odot (x \to 0)$.
Define a unary operation $'$ and three binary hyperoperations $\rightsquigarrow$, $\wedge$, $\vee$ on L as follows: for any $x, y \in L$,

$$\begin{aligned} x' &\in x \to 0, \\ x \rightsquigarrow y &= (x \odot y')', \\ x \wedge y &= \{x, y\}, \\ x \vee y &= \{x, y\}. \end{aligned}$$

Then $(L, \wedge, \vee, \rightsquigarrow, ', 0, 1)$ is a hyper lattice implication algebra.

Proof. It follows from Theorem 5.3 in [2] and Proposition 4.4.$\blacksquare$

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