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SOFT SET THEORETIC APPROACH TO HYPERSTRUCTURES

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Abstract. In this paper, we have achieved hyperoperations using soft set theory. We have showed that any given set H is a hyperstructure with **And** and **Or** operations in soft set theory. Furthermore, for any given algebraic structure $(H, *)$, we have created hyperstructure using soft set operation which mentioned by Molodtsov [8] and binary operation $*$.

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1. Introduction and Preliminaries

Hyperstructure theory was initiated by Marty in 1934 [7]. He defined the concept of hypergroups as a generalization of groups. Many scientists have developed this theory for years. In [4], there are so many applications of hyperstructures such as geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intellegence and probabilistic theory. As described in [4, 7], we define the concept of hyperoperation as follows:

Definition 1.1. [4, 7] Let H be a non-empty set and $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. A *n-hyperoperation* on H is a map $f : H^n \to \mathcal{P}^*(H)$. The number n is called the *arity* of f.

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For instance, if $n = 1$, then f is called *unary hyperoperation*, if $n = 2$, then f is called *binary hyperoperation* etc.

Immediately after the definition of hyperoperation, the definition of hyperstructure is given as follows:

Definition 1.2. [4] A set H, endowed with a hyperoperation, is called a *hyperstructure* (or a *multi-valued algebra*).

Generally, the hyperoperation is denoted by "$\circ$" and the image of the pair (a, b) is denoted by $a \circ b$ and called the *hyperproduct* of a and b. Moreover, if A and B are non-empty subsets of H, then

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b.$$

As in classical theory of algebraic structure, hyperstructures are named according to the properties provided by the hyperoperation. $(H, \circ)$ is called *hypergroupoid* where $\circ$ is a binary hyperoperation. If $\circ$ is an associative hyperoperation, i.e. $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in H$, then we call that $(H, \circ)$ is a *semihypergroup*. A hypergroupoid $(H, \circ)$ is called *quasihypergroup* which satisfies the *reproductive law*:

$$\forall a \in H, H \circ a = a \circ H = H.$$

We call that $(H, \circ)$ is a *hypergroup* if it is a semihypergroup which is also a quasihypergroup. In addition to these, let $(H, \circ)$ be a hypergroupoid. An element $e \in H$ is called an *identity* or *unit* if

$$\forall a \in H, a \in a \circ e \cap e \circ a$$

and let $(H, \circ)$ be a hypergroup, endowed with at least an identity. An element $a' \in H$ is called an *inverse* of $a \in H$ if there is an identity $e \in H$ such that

$$e \in a \circ a' \cap a' \circ a.$$

We say that two binary hyperoperations "$\circ_1$", "$\circ_2$" on the same set H are *mutually associative* if $\forall x, y, z \in H$, we have $(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z)$ and $(x \circ_2 y) \circ_1 z = x \circ_2 (y \circ_1 z)$. We also say that the pair $((H, \circ_1), (H, \circ_2)$ is mutually associative. A semihypergroup $(H, \circ)$ is called *simplifiable on the left* if $x \circ a \cap x \circ b \neq \varnothing \Rightarrow a = b$ for all $x, a, b \in H$. Similarly, simplifiability on the right can be defined [4].

On the other hand, soft set theory which put forward a general mathematical tool for dealing with uncertainty, fuzziness in the real life is founded by Molodtsov in 1999 [8]. He built the fundamental results of this theory and applied in analysis, game theory, and probability theory. In [2, 6], set theoretic operations on soft sets was investigated such as soft subset, soft union, soft intersection, soft complement etc. Molodtsov defined the concept of soft set as follows:

Definition 1.3. [8] Let H be an initial universe, E be the set of all possible parameters which are attributes, characteristic or properties of the objects in H. For $A \subseteq E$, a pair (F, A) is called a *soft set* over H where $F : A \to \mathcal{P}(H)$ is a set-valued function.

In other words, a soft set is a parametrized family of subsets of the set H[8]. As mentioned in [6], a soft set (F, A) can be viewed

$$(F, A) = \{a = F(a) \mid a \in A\}$$

where the symbol "$a = F(a)$" indicates that the approximation for $a \in A$ is $F(a)$. Also, the definition of soft subset is given as $(F, A)\widetilde{\subset}(G, B)$ such that $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$ in [2]. We recommend the readers to read [2] and [6] for detailed information about set-theoretic operations among soft sets. Soft set-theoretic operation between two soft sets was stated as

$$(H, A \times B) = (F, A) * (G, B) \tag{1.1}$$

by Molodtsov in [8]. In [6], this operation has been described two different concept. These two operations are called **And** and **Or**, respectively, and definitions are given by;

Definition 1.4. [6] Let (F, A) and (G, B) be soft sets over the initial universe H where $A, B \subseteq E$. Then,

(1) $(F, A)\mathbf{And}(G, B)$ is defined by $(F, A)\mathbf{And}(G, B) = (H, A \times B)$ where $H((a, b)) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$,
(2) $(F, A)\mathbf{Or}(G, B)$ is defined by $(F, A)\mathbf{Or}(G, B) = (H, A\times B)$ where $H((a, b)) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$.

Of course, $(F, A)\mathbf{And}(G, B)$ and $(F, A)\mathbf{Or}(G, B)$ are soft sets over the inital universe H. To give an example for soft sets any given universe and operations between them:

Example 1.5. Let $H = \{a, b, c, d, e\}$ be the initial universe, $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the parameters set. Then

$$(F, A) = \{1 = \{a, c\}, 3 = \{b, c, d\}, 7 = \varnothing, 8 = H\}$$

is a soft set over H where $A \subseteq E$. Moreover, if we take the soft set

$$(G, B) = \{2 = \{d\}, 5 = \{a, b, c, d\}\},$$

then we obtain that

$$\begin{aligned}(F,A)\mathbf{And}(G,B) = \{&(1,2)=\varnothing,(1,5)=\{a,c\},\\ &(3,2)=\{d\},(3,5)=\{b,c,d\},\\ &(7,2)=\varnothing,(7,5)=\varnothing,\\ &(8,2)=\{d\},(8,5)=\{a,b,c,d\}\}\end{aligned}$$

and

$$\begin{aligned}(F,A)\mathbf{Or}(G,B) = \{&(1,2)=\{a,c,d\},(1,5)=\{a,b,c,d\},\\ &(3,2)=\{b,c,d\},(3,5)=\{a,b,c,d\},\\ &(7,2)=\{d\},(7,5)=\{a,b,c,d\},\\ &(8,2)=H,(8,5)=H\}.\end{aligned}$$

2. Results

In this section, we obtain two hyperoperation on any given set using the operations **And** and **Or** between soft sets, and examine the provided properties.

Let H be a initial universe, of course non-empty set, and $\mathcal{P}(H)$ be power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) - \{\varnothing\}$ as we mentioned above. We can define the soft set $\alpha : H \to \mathcal{P}^*(H)$ over H and call that the soft set is *non-empty approximated*. At that case, we gain a unary hyperoperation on H, id est the soft set (α, H) is a unary hyperoperation on H. In the meantime, if we take $(\alpha, H)\mathbf{And}(\beta, H)$ and $(\alpha, H)\mathbf{Or}(\beta, H)$, then we obtain two binary hyperoperation on H from Definition 1.4. From now on, each soft set over given initial universe H will be adopted non-empty approximated. Let write

$$(\alpha, H)\mathbf{And}(\beta, H) = (\mathbf{And}, H \times H) \tag{2.1}$$

such that $\mathbf{And} : H \times H \to \mathcal{P}^*(H)$, and $a\mathbf{And}b = \alpha(a) \cap \beta(b) \neq \varnothing$. If we will draw attention, we assume that two soft sets which is subjected to **And** operation is non-empty approximated soft set.

On the other hand, let

$$(\alpha, H)\mathbf{Or}(\beta, H) = (\mathbf{Or}, H \times H) \tag{2.2}$$

such that $\mathbf{Or} : H \times H \to \mathcal{P}^*(H)$, and $a\mathbf{Or}b = \alpha(a) \cup \beta(b)$. Here, we call that α and β are *left part* and *right part* of the operations **And** and **Or**, respectively. Due to these configurations, we obtain following theorem.

Theorem 2.1. *The operations* **And** *and* **Or** *are binary hyperoperations on H. So $(H, \mathbf{And})$ and $(H, \mathbf{Or})$ are hypergoupoids.*

Proof. It is obvious from Equations 2.1 and 2.2. □

Let (α, H), (β, H) and (γ, H) be soft sets over H. Since

$$(\mathbf{And}, H \times (H \times H)) = (\mathbf{And}, (H \times H) \times H)$$

and

$$(\mathbf{Or}, H \times (H \times H)) = (\mathbf{Or}, (H \times H) \times H),$$

we obtain following theorem.

Theorem 2.2. **And** *and* **Or** *are associative. So* $(H, \mathbf{And})$ *and* $(H, \mathbf{Or})$ *are semihypergroups.*

In [2, 6], Ali *et al.* and Maji *et al.* defined the concept of *absolute soft set* which considered as universal set in soft set theory. They call (F, A) is *absolute soft set* over H, if $F(a) = H$ for all $a \in A$. Using this definition, we can obtain following theorem;

Theorem 2.3. *If* (α, H) *and* (β, H) *are absolute soft sets over* H*. Then* **And** *satisfies reproductive law. And so,* $(H, \mathbf{And})$ *is a quasihypergroup.*

Similarly,

Theorem 2.4. *If either of* (α, H) *and* (β, H) *absolute soft set over* H*, then* **Or** *satisfies reproductive law and* $(H, \mathbf{Or})$ *is a quasihypergroup.*

Proofs of Theorem 2.3 and Theorem 2.4 are obvious from definition of absolute soft set.

In [5], Kim and Min investigated the concept of full soft set. So, (F, A) is called *full soft set* over H if $\bigcup_{a \in A} F(a) = H$ for all $a \in A$.

Theorem 2.5. *If either* (α, H) *or* (β, H) *is a full soft set, then* **Or** *satisfies reproductive law, and* $(H, \mathbf{Or})$ *is a quasihypergroup.*

Proof. Suppose that (β, H) is a full soft set. Then for all $a \in H$ we have $\bigcup_{a\in H}\beta(a) = H$. For arbitrary element $a \in H$,

$$\begin{aligned} a\mathbf{Or}H &= \bigcup_{b\in H} a\mathbf{Or}b \\ &= \bigcup_{b\in H} a\mathbf{Or}b \\ &= \bigcup_{b\in H} \alpha(a)\cup\beta(b) \\ &= \alpha(a)\cup\left(\bigcup_{b\in H}\beta(b)\right) \\ &= \alpha(a)\cup H = H \end{aligned}$$

Moreover, since the operation **Or** is commutative then we gain $H\mathbf{Or}a = H$ in the similar way. Thus **Or** satisfies reproductive law. □

Corollary 2.6. $(H, \mathbf{And})$ *is a hypergroup if and only if all part of the operation* **And** *is a absolute soft set.*

Corollary 2.7. $(H, \mathbf{Or})$ *is a hypergroup if and only if any part of the operation* **Or** *is a absolute soft set.*

Corollary 2.8. $(H, \mathbf{Or})$ *is a hypergroup if and only if any part of the operation* **Or** *is a full soft set.*

Proofs of Corollary 2.6, Corollary 2.7 and Corollary 2.8 are straightforward from definition of hypergroup, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 2.5.

Example 2.9. Let $H = \{a, b, c\}$ be the initial universe. Define the soft sets

$$(\alpha, H) = \{a = \{a\}, b = \{b\}, c = \{c\}\}$$

and

$$(\beta, H) = \{a = \{a = \{a, b\}, b = \{a, b\}, c = H\}$$

where $\alpha : H \to \mathcal{P}^*(H)$ and $\beta : H \to \mathcal{P}^*(H)$. It is obtained from here that

$$\begin{aligned} (\alpha, H)\mathbf{Or}(\beta, H) &= \{(a, a) = \{a, b\}, (a, b) = \{a, b\}, (a, c) = H \\ &\quad (b, a) = \{a, b\}, (b, b) = \{a, b\}, (b, c) = H \\ &\quad (c, a) = H, (c, b) = H, (c, c) = H\} \end{aligned}$$

Accordingly, we obtain following table;

Or	a	b	c
a	$\{a,b\}$	$\{a,b\}$	H
b	$\{a,b\}$	$\{a,b\}$	H
c	H	H	H

Table 1. Table of **Or** operation on H

In addition to this, since

$$\bigcup_{x\in H} \alpha(x) = \alpha(a) \cup \alpha(b) \cup \alpha(c) = \{a\} \cup \{b\} \cup \{c\} = H$$

and

$$\bigcup_{x\in H} \beta(x) = \beta(a) \cup \beta(b) \cup \beta(c) = \{a,b\} \cup \{a,b\} \cup H = H,$$

(α, H) and (β, H) are full soft sets over H.

Thereby, $(H, \mathbf{Or})$ is a hypergroup from Corollary 2.8.

Note that, identity of the hypergroup $(H, \mathbf{Or})$ is c.

Theorem 2.10. *Let (α, H), (β, H) and (γ, H) be soft sets over H. If three of them are absolute soft sets, then* **And** *and* **Or** *operations are mutually associative.*

Proof. From definition of mutually associativity and the operations **And** and **Or**, we have

$$(x\mathbf{And}y)\mathbf{Or}z = (\alpha(x) \cap \beta(y)) \cup \gamma(z) = (\alpha(x) \cup \gamma(z)) \cap (\beta(y) \cup \gamma(z)) \tag{2.3}$$

and

$$x\mathbf{And}(y\mathbf{Or}z) = \alpha(x) \cap (\beta(y) \cup \gamma(z)) = (\alpha(x) \cap \beta(y)) \cup (\alpha(x) \cap \gamma(z)) \tag{2.4}$$

for all $x, y, z \in H$.

Since (α, H), (β, H) and (γ, H) are absolute soft sets, The Equations 2.3 and 2.4 are equal. Thus, **And** and **Or** are mutually associative. □

We can invent a hyperoperation using any algebraic structure and soft set theory. Let's see how we do it. Let H be a non-empty set and $*$ be an binary operation on H. We know that $(H, *)$ is a *magma* (or *groupoid*) as mentioned in [3]. Herefrom, suppose that (α, H) and (β, H)

are soft sets which are non-empty approximated. From Equation 1.1 as Molodtsov's stated in [8], we can invent the soft set

$$(\star, H \times H) = (\alpha, H) \star (\beta, H) \tag{2.5}$$

over $(H, *)$ such that $\star(x, y) = \alpha(x) * \beta(y)$ for each $x, y \in H$. Therefore, we have achieved a hyperoperation on H with this invention, and we denote $x \star y = \alpha(x) * \beta(y) \in \mathcal{P}^*(H)$. This operation is called a *hyperoperation induced by* $*$. Whence, $(H, \star)$ will be a hypergroupoid.

If we have a semigroup i.e. set with associative binary relation $*$, then we obtain following obvious lemma.

Lemma 2.11. *If $A, B, C \subseteq H$, then $A * (B * C) = (A * B) * C$.*

Theorem 2.12. *If $(H, *)$ is a semigroup, then $(H, \star)$ is a semihypergroup.*

Proof. From Equation 2.5 and Lemma 2.11, the proof is straightforward. □

Theorem 2.13. *Let $(H, *)$ be a semigroup, (α, H) and (β, H) be soft sets over H and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation as mentioned in Equation 2.5. If all parts of $\star$ are absolute soft sets, then $\star$ satisfies reproductive law.*

Proof. Let $a \in H$ be an arbitrary element. Since (α, H) and (β, H) are absolute soft sets over H, then we have

$$\begin{aligned} a \star H &= \bigcup_{b \in H} a \star b \\ &= \bigcup_{b \in H} \alpha(a) * \beta(b) \\ &= \bigcup (H * H) = H \end{aligned}$$

In similar way, we obtain $H \star a = H$ for any $a \in H$. Thus $\star$ is a hyperoperation that satisfies reproductive law. □

As a result of the Theorem 2.13;

Corollary 2.14. *Let $(H, *)$ be a semigroup, (α, H) and (β, H) be soft sets over H and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation as mentioned in Equation 2.5. If all parts of $\star$ are absolute soft sets, then $(H, \star)$ is a hypergroup.*

3. Conclusion

In this paper, we have achieved a hyperoperation using soft set-theoretic operations **And** and **Or** which are defined in [6]. We have studied properties of them. In this manner, we have got some hyperstructures with these operations. With all that, we have obtained more hyperoperation from a binary operation defined on a set using soft set-theoretic operation which is given in [8] by Molodtsov.

In future, after the characterization of this article, the hyperoperations derived by soft group theory [1] can be investigated. Because soft set theory and hyperstructures can be applied to many areas such as graph theory, coding theory etc., properties and affects in these areas can be examined.

The author hopes that this article is shed light on to working scientists in these areas.

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